Bull. Korean Math. Soc. **46** (2009), No. 5, pp. 873–884
DOI 10.4134/BKMS.2009.46.5.873

ON QB-IDEALS OF EXCHANGE RINGS

Huanyin Chen

Abstract. We characterize QB-ideals of exchange rings by means of quasi-invertible elements and annihilators. Further, we prove that every 2×2 matrix over such ideals of a regular ring admits a diagonal reduction by quasi-inverse matrices. Prime exchange QB-rings are studied as well.

1. Introduction

In [1], Ara et al. discovered a new class of rings, the QB-rings, so as to study directly infinite rings. A ring R is a QB-ring if $aR + bR = R$ with $a, b \in R$ implies that $a + by \in R_q^{-1}$ for a $y \in R$, where $R_q^{-1} = \{u \in R \mid \exists v \in R$ such that $(1 - uv) \perp (1 - vu)\}$. Further, they extended QB-rings to rings without unit. An ideal I of a ring R is a QB-ideal in case $xa - x - a + b = 0$ with $x, a, b \in I$ implies that there exists $y \in I$ such that $1 - (a - yb) \in R_q^{-1}$ (cf. [3-6]).

A ring R is an exchange ring if for every right R-module A and any two decompositions $A = M \oplus N = \bigoplus_{i\in I} A_i$, where $M_R \cong R$ and the index set I is finite, then there exist submodules $A_i' \subseteq A_i$ such that $A = M \oplus (\bigoplus_{i\in I} A_i')$. Many authors have investigated exchange rings with some kind of weak stable range conditions so as to study problems related partial cancellation properties of modules. For general theory of exchange rings, we refer the reader to [9].

We establish, in this article, several equivalent conditions for an ideal of an exchange ring to be a QB-ideal. We show that an ideal I of an exchange ring R is a QB-ideal if and only if for any $x \in 1 + I$, $x = xyx$ implies that there exists a $u \in R_q^{-1}$ such that $x = xyu = uyx$ if and only if for any $x \in 1 + I$, $x = xyx$ implies that there exists a $u \in R_q^{-1}$ such that $xyu = uyx$. These extend the corresponding results on weakly stable rings (cf. [10]). Further, we characterize such ideals by means of annihilators for regular rings. The $m \times m$ matrix $A = (a_{ij}), 1 \le i, j \le n$, is said to be diagonal if $a_{ij} = 0$ for all $i \ne j$. The $m \times m$ matrix A admits diagonal reduction if there exist $P, Q \in M_n(R)_q^{-1}$ such that PAQ is a diagonal matrix. We will prove that every 2×2 matrix over

Received July 14, 2008.
2000 *Mathematics Subject Classification.* 16E50, 19B10.
Key words and phrases. QB-ideal, exchange ring, annihilator, diagonal reduction.

QB-ideal of regular rings admits diagonal reduction. Prime exchange QB-rings are studied as well.

Throughout, all rings are associative with identity. An element $x \in R$ is regular provided that $x = xyx$ for a $y \in R$. $GL_2(R)$ denotes the 2-dimensional general linear groups of a ring R. The notation $u \perp v$ means that $uRv = 0 = vRu$.

2. Element-wise characterizations

By the symmetry of QB-ideals, we see that I is a QB-ideal of R if and only if $aR + bR = R$ with $a \in 1 + I, b \in R$ implies that there exists $y \in R$ such that $a + by \in R_q^{-1}$ if and only if $Ra + Rb = R$ with $a \in 1 + I, b \in R$ implies that there exists $z \in R$ such that $a + zb \in R_q^{-1}$. We begin with a simple fact.

Lemma 2.1. *Let I be an ideal of a ring R. Then the following are equivalent*:

(1) *I is a QB-ideal.*
(2) *Whenever $aR + bR = R$ with $a \in 1 + I, b \in I$, there exists $y \in R$ such that $a + by \in R_q^{-1}$.*
(3) *Whenever $Ra + Rb = R$ with $a \in 1 + I, b \in I$, there exists $z \in R$ such that $a + zb \in R_q^{-1}$.*

Proof. (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (1) Suppose that $ax + b = 1$ with $a \in 1 + I, x, b \in R$. Then $a(x + b) + (1 - a)b = 1$; hence, $aR + (1 - a)bR = R$. So we have $y \in R$ such that $a + (1 - a)by \in R_q^{-1}$. As is known, we have $z \in R$ such that $x + \big(1 + z(1 - a)\big) = x + b + z(1 - a)b \in R_q^{-1}$. Therefore I is a QB-ideal of R.

(1) $\Leftrightarrow$ (3) is proved by symmetry. □

Lemma 2.2. *Let I be an ideal of an exchange ring R. Then the following are equivalent*:

(1) *I is a QB-ideal.*
(2) *For any regular $x \in 1 + I$, there exists $u \in R_q^{-1}$ such that $x = xux$.*

Proof. It is an immediate consequence of [6, Lemma 2.1]. □

Theorem 2.3. *Let I be an ideal of an exchange ring R. Then the following are equivalent*:

(1) *I is a QB-ideal.*
(2) *For any regular $x \in 1 + I$, there exists $u \in R_q^{-1}$ such that $ux \in 1 + I$ is an idempotent.*
(3) *For any regular $x \in 1 + I$, there exists $u \in R_q^{-1}$ such that $xu \in 1 + I$ is an idempotent.*

Proof. (1) $\Rightarrow$ (2) Given any regular $x \in 1 + I$, by Lemma 2.2, there exists a $u \in R_q^{-1}$ such that $x = xux$. Clearly, $u \in 1 + I$. Set $e = ux$. Then $e \in 1 + I$ is an idempotent.

(2) $\Rightarrow$ (1) Suppose that $ax+b=1$ with $a\in 1+I, x\in R$ and $b\in I$. Since R is an exchange ring, there exists an idempotent $e\in R$ such that $e=bs$ and $1-e=(1-b)t$ for some $s,t\in R$. Analogously to Lemma 2.2, we claim that $(1-e)a\in 1+I$ is regular. Hence there exists $u\in R_q^{-1}$ such that $u(1-e)a=f$ is an idempotent of R. So $fxt+ue=u$, whence $f(x+ue)+(1-f)ue=u$. Let $g=(1-f)uev(1-f)$. Similarly to Lemma 2.2, we show that $w(a+by)w=w$, where $y=s\big(v(1-f)-a\big)$ and $w=\big(1+fuev(1-f)\big)u\in R_q^{-1}$. Therefore $a+by\in R_q^{-1}$. It follows from Lemma 2.1 that I is a QB-ideal of R.

(1) $\Leftrightarrow$ (3) is symmetric. □

Theorem 2.4. *Let I be an ideal of an exchange ring R. Then the following are equivalent*:

(1) *I is a QB-ideal.*
(2) *For any $x\in 1+I$, $x=xyx$ implies that there exists a $u\in R_q^{-1}$ such that $x=xyu$.*
(3) *For any $x\in 1+I$, $x=xyx$ implies that there exists a $u\in R_q^{-1}$ such that $x=uyx$.*

Proof. (1) $\Rightarrow$ (2) Suppose that $x=xyx$ with $x\in 1+I$. Since $xy+(1-xy)=1$, we have that $x+(1-xy)z\in R_q^{-1}$ for some $z\in R$. Hence $x=xy\big(x+(1-xy)z\big)=xyu$, where $u:=x+(1-xy)z\in R_q^{-1}$.

(2) $\Rightarrow$ (1) Suppose that $x=xyx$ with $x\in 1+I$. Then there exists a $u\in R_q^{-1}$ such that $x=xyu$. This implies that there exist two ideals I and J of R such that $IJ=0=JI$ and $\overline{u}\in R/I$ is right invertible and $\overline{u}\in R/J$ is left invertible. Let $e=xy$. Then $e\in R$ is an idempotent. Since $xy+(1-xy)=1$, we have that $euy+(1-xy)=1$, and so $euy(1-e)+(1-xy)(1-e)=1-e$. Hence, $e+(1-xy)(1-e)=1-euy(1-e)\in U(R)$. Thus, we get $\overline{x+(1-xy)(1-e)u}=\overline{\big(1-euy(1-e)\big)u}\in R/I$ is right invertible. Likewise, $\overline{x+(1-xy)(1-e)u}\in R/J$ is left invertible. Thus, $x+(1-xy)(1-e)u\in R_q^{-1}$. This implies that we have an element $z\in R$ such that $w:=y+z(1-xy)\in R_q^{-1}$. Therefore $x=x\big(y+z(1-xy)\big)x=xwx$, and that I is a QB-ideal.

(1) $\Leftrightarrow$ (3) Applying (1) $\Leftrightarrow$ (2) to the opposite ring R^{op}, we complete the proof. □

Corollary 2.5. *Let I be an ideal of an exchange ring R. Then the following are equivalent*:

(1) *I is a QB-ideal.*
(2) *For every regular $x\in 1+I$, there exist an idempotent $e\in 1+I$ and a $u\in R_q^{-1}$ such that $x=eu$.*
(3) *For every regular $x\in 1+I$, there exist an idempotent $e\in 1+I$ and a $u\in R_q^{-1}$ such that $x=ue$.*
(4) *For every regular $x\in 1+I$, there exist an idempotent $e\in 1+I$ and a $u\in R_q^{-1}$ such that $x=eu$ or ue.*

Proof. (1) $\Rightarrow$ (2) is clear by Theorem 2.4.

(2) $\Rightarrow$ (4) is trivial.

(4) $\Rightarrow$ (1) Given any regular $x \in 1+I$, we have an element $y \in 1+I$ such that $x = xyx$ and $y = yxy$. By assumption, there is an idempotent $e \in 1+I$ and a $u \in R_q^{-1}$ such that $y = eu$ or ue. Assume that $y = eu$. Since $yx + (1-yx) = 1$, $eux + (1-yx) = 1$. As in the proof of Theorem 2.4, we have that $y + (1-yx)(1-e)u = \big(1 - eux(1-e)\big)u \in R_q^{-1}$. This implies that $x = xyx = xwx$, where $w := \big(1-eux(1-e)\big)u \in R_q^{-1}$. Assume that $y = ue$. Obviously, $xy + (1-xy) = 1$. Similarly, there exists a $z \in R$ such that $w := y + z(1-xy) \in R_q^{-1}$. It is easy to verify $x = xyx = xwx$. Lemma 2.2 applies.

(1) $\Leftrightarrow$ (3) Applying (1) $\Leftrightarrow$ (2) to the opposite ring R^{op}, the proof is true. $\square$

Corollary 2.6. *Let I be an ideal of an exchange ring R. Then the following are equivalent*:

(1) *I is a QB-ideal.*
(2) *For any $a, b \in 1+I$, $aR = bR$ implies that there exists a $u \in R_q^{-1}$ such that $b = au$.*
(3) *For any $a, b \in 1+I$, $Ra = Rb$ implies that there exists a $u \in R_q^{-1}$ such that $b = ua$.*

Proof. (1) $\Rightarrow$ (2) For any $a, b \in 1+I$, $aR = bR$ implies that $ax = b$ and $by = a$ for some $x, y \in R$. Further, $x, y \in 1+I$. Since $xy + (1-xy) = 1$, there exists some $z \in R$ such that $u := x + (1-xy)z \in R_q^{-1}$. Therefore $b = ax = a\big(x + (1-xy)z\big) = au$, as asserted.

(2) $\Rightarrow$ (1) Given any regular $x \in 1+I$, there exists a $y \in 1+I$ such that $x = xyx$. Hence, $xyR = xR$. By assumption, we have a $u \in R_q^{-1}$ such that $x = xyu$. According to Corollary 2.5, I is a QB-ideal.

(1) $\Leftrightarrow$ (3) is symmetric. $\square$

The following is an extension of the corresponding result on weakly stable rings (cf. [10, Theorem 3.6]).

Theorem 2.7. *Let I be an ideal of an exchange ring R. Then the following are equivalent*:

(1) *I is a QB-ideal.*
(2) *For any $x \in 1+I$, $x = xyx$ implies that there exists a $u \in R_q^{-1}$ such that $x = xyu = uyx$.*
(3) *For any $x \in 1+I$, $x = xyx$ implies that there exists a $u \in R_q^{-1}$ such that $xyu = uyx$.*

Proof. (1) $\Rightarrow$ (2) Given any $x = xyx$ and $x \in 1+I$, then we have $x = xzx, z = zxz$, where $z = yxy$. Since I is a QB-ideal, it follows by Lemma 2.2 that there exists a $v \in R_q^{-1}$ such that $z = zvz$. Let $u = (1-xz-vz)v(1-zx-zv)$. One easily checks that $(1-xz-vz)^2 = 1 = (1-zx-zv)^2$. Hence $u \in R_q^{-1}$. Clearly, $xzu = -xzv(1-zx-zv) = -xzv + xzx + xzv = xzx = x$ and $uzx =$

$(1-xz-vz)v(-zvzx) = -(1-xz-vz)vzx = -vzx+xzx+vzx = xzx = x$. Thus, $x = xzu = x(yxy)u = xyu$ and $x = uzx = u(yxy)x = uyx$. As a result, we see that $x = xyu = uyx$.

(2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (1) Given $x = xyx$ and $x \in 1+I$, there exists a $u \in R_q^{-1}$ such that $xyu = uyx$. Thus, we can find some ideals I and J such that $IJ = 0 = JI$, and that $uv \equiv 1 \pmod{I}$ or $vu \equiv 1 \pmod{J}$. Construct two maps

$$\varphi : xR \oplus (1-xy)R \to yxR \oplus (1-yx)R;$$
$$\varphi\big(xr + (1-xy)s\big) = yxr + (1-yx)v(1-xy)s \text{ for any } r, s \in R$$

and

$$\phi : yR \oplus (1-yx)R \to xR \oplus (1-xy)R,$$
$$\phi\big(yr + (1-yx)s\big) = xyr + u(1-yx)s \text{ for any } r, s \in R.$$

One easily checks that $x\varphi(1)x = x\varphi(x) = xyx = x$. Furthermore, we see that

$$\begin{aligned} 1-\phi(1)\varphi(1) &= 1-\phi\big(\varphi(1)\big) \\ &= 1-\phi\big(yxy+(1-yx)v(1-xy)\big) \\ &= 1-\big(xyxy+u(1-yx)v(1-xy)\big) \\ &= (1-xy)(1-uv)(1-xy). \end{aligned}$$

Likewise, we have that $1-\varphi(1)\phi(1) = (1-yx)(1-vu)(1-yx)$. Thus, $\varphi(1)\phi(1) \equiv 1 \pmod{I}$ or $\phi(1)\varphi(1) \equiv 1 \pmod{J}$. Hence, $\varphi(1) \in R_q^{-1}$, and so we complete the proof. □

Corollary 2.8. *Let I be an ideal of an exchange ring R. Then the following are equivalent*:

(1) *I is a QB-ideal.*

(2) *For any idempotents $e, f \in 1+I$, $eR \cong fR$ implies that there exists a $u \in R_q^{-1}$ such that $eu = uf$.*

Proof. It is an immediate consequence of Theorem 2.7. □

3. Annihilators

An ideal I of a ring R is regular provided that for any $x \in I$, there exists a $y \in I$ such that $x = xyx$. A ring R is regular in case R as an ideal of itself is regular. As is well known, every regular ring is an exchange ring. In [5], the author investigate QB-ideals of a regular ring from the view point of substitution of modules. The main purpose of this section is to study QB-ideals of such rings by means of annihilators. We always use $r(x)(\ell(x))$ to denote the right (left) annihilator of $x \in R$.

Theorem 3.1. *Let I be an ideal of a regular ring R. Then the following are equivalent*:

(1) *I is a QB-ring.*

(2) *For any $a, b \in 1+I$, $r(a) = r(b)$, there exists $u \in R_q^{-1}$ such that $a = ub$.*

(3) *For any $a, b \in 1+I$, $\ell(a) = \ell(b)$, there exists $u \in R_q^{-1}$ such that $a = bu$.*

Proof. (1) $\Rightarrow$ (2) Suppose that $r(a) = r(b)$, where $a, b \in 1 + I$. Since R is regular, we have $x \in R$ such that $a = axa$. Hence $a(1 - xa) = 0$, and then $1 - xa \in r(a) = r(b)$. This infers that $b(1 - xa) = 0$. So $b = bxa \in Ra$; whence $Rb \subseteq Ra$. Likewise, we get $Rb \subseteq Ra$. So $Ra = Rb$. According to Corollary 2.6, there exists a $u \in R_q^{-1}$ such that $a = ub$.

(2) $\Rightarrow$ (1) Given any $a \in 1 + I$, then we have $x \in 1 + I$ such that $a = axa$. If $r \in r(a)$, then $ar = 0$; hence $xar = 0$. This means that $r \in r(xa)$. So we get $r(a) \subseteq r(xa)$. If $t \in r(xa)$, then $xat = 0$; hence, $at = axat = 0$; hence, $t \in r(a)$. This shows that $r(xa) \subseteq r(a)$. Thus $r(a) = r(xa)$. By hypothesis, we have a $u \in R_q^{-1}$ such that $a = uxa$. One easily checks that $xa \in 1 + I$ is an idempotent. According to Theorem 2.3, I is a QB-ideal.

(1) $\Leftrightarrow$ (3) is symmetric. $\square$

Corollary 3.2. *Let R be a regular QB-ring. If $aR \cap r(a) = 0$, then there exist an idempotent $e \in R$ and a $u \in R_q^{-1}$ such that $a = e + u$.*

Proof. Since R is regular, we have $x \in R$ such that $a = axa$; hence, $R = aR \oplus (1-ax)R$. Let $\psi : R \to R$ given by $\psi\big(ar+(1-ax)s\big) = (1-ax)s$ for any $r, s \in R$. Then $\psi^2 = \psi$. This means that $\psi(1) \in R$ is an idempotent. Let $u = a - \psi(1)$. Assume that $t \in r(u)$. Then $ut = 0$, so $at = \psi(t) \in aR \cap (1 - ax)R = 0$. This infers that $t \in \mathrm{Ker}\psi \cap r(a) = aR \cap r(a) = 0$. That is, $r(u) = 0 = r(1)$. It follows from Theorem 3.1 that $u \in R_q^{-1}$. Let $e = \psi(1)$. Then $a = u + e$ with idempotent $e \in R$, as required. $\square$

Corollary 3.3. *Let R be a regular QB-ring. If $aR \cong bR$, then there exist $u, v \in R_q^{-1}$ such that $a = ubv$.*

Proof. Since $\psi : aR \cong bR$, one easily checks that $Ra = R\psi(a)$ and $\psi(a)R = bR$. Hence $r(a) = r\big(\psi(a)\big)$ and $\ell\big(\psi(a)\big) = \ell(a)$. In view of Theorem 3.1, we can find $u, v \in R_q^{-1}$ such that $a = u\psi(a)$ and $\psi(a) = bv$. Therefore $a = ubv$, as asserted. $\square$

It is well known that a regular ring R has *a.c.c* on right annihilators if and only if it has finitely many minimal prime ideals. A ring R is unit-regular in case for any $x \in R$, there exists a $u \in U(R)$ such that $x = xux$. We now observe the following.

Corollary 3.4. *Let R be a regular ring having a.c.c. on right annihilators. Then R is a QB-ring if and only if it is unit-regular.*

Proof. Clearly, every unit-regular ring is a QB-ring. Conversely, assume now that R is a QB-ring. Given any $a \in R$, then there is a chain $r(a) \subseteq r(a^2) \subseteq \cdots$. Since R is a regular ring having *a.c.c.* on right annihilators, there exists a positive integer n such that $r(a^n) = r(a^{n+1})$. By virtue of Theorem 3.1, we have some $u \in R_q^{-1}$ such that $a^n = ua^{n+1}$. Thus R is strongly π-regular. According to [9, Theorem 30.9], R has stable range one. Therefore R is unit-regular by [9, Theorem 30.6]. $\square$

Following [1], an element $a \in R$ is said to be quasi-invertible provided that $a \in R_q^{-1}$. Let I be a right ideal of a ring R. We say that every quasi-invertible element lifts modulo I in case $(1-ab)R(1-ba) \bigcap (1-ba)R(1-ab) \subseteq I$ for some $b \in R$ implies that $a \equiv x \pmod{I}$ for some $x \in R_q^{-1}$.

Lemma 3.5. *Let I be an ideal of a regular ring R. Then the following are equivalent*:

(1) *I is a QB-ideal.*
(2) *Every quasi-invertible element in $1+I$ lifts modulo any right ideal of R.*

Proof. (1) $\Rightarrow$ (2) Suppose that $(1-ab)R(1-ba) \bigcap (1-ba)R(1-ab) \subseteq I$ for some $b \in R$, where $a \in 1+I$. Since I is a QB-ideal, it follows from $ab + (1-ab) = 1$ that $v = b + y(1-ab) \in R_q^{-1}$ for a $y \in R$. Let u be a quasi-inverse of v such that $u = uvu$, and let $w = u + a(1-vu) + (1-uv)a$. By [1, Theorem 2.3], $w \in R_q^{-1}$. For any $x \in R$, we denote $x + I$ by $\pi(x)$. Similarly to [1, Proposition 7.1], we have $\pi(v)\pi(a)\pi(v) = \pi\big((b+y(1-ab))a(b+y(1-ab))\big) = \pi\big(ba(b+y(1-ab))\big) = \pi\big(b+y(1-ab)\big) = \pi(v)$. Clearly, $\pi(w) = \pi(u) + \pi(a)\big(\pi(1) - \pi(v)\pi(u)\big) + \big(\pi(1) - \pi(u)\pi(v)\big)\pi(a)$. Since $(1-uv)R(1-vu) = 0$, we get $(1-uv)a(1-vu) = 0$. This deduces that $\pi(a) - \pi(a)\pi(v)\pi(u) - \pi(u)\pi(v)\pi(a) + \pi(u) = 0$; whence, $\pi(a) = \pi(u) + \pi(a)\big(\pi(1) - \pi(v)\pi(u)\big) + \big(\pi(1) - \pi(u)\pi(v)\big)\pi(a)$. So we get $\pi(a) = \pi(w)$, as desired.

(2) $\Rightarrow$ (1) Suppose that $ax + b = 1$ in R, where $a \in 1+I, x, b \in R$. If $bR = R$, then $bc = 1$ for a $c \in R$. Hence, $a + bc(1-a) = 1 \in R_q^{-1}$. Now assume that $bR \neq R$. Clearly, $ax \equiv 1 \pmod{bR}$; hence, $a \in 1+I$ is a quasi-invertible element modulo the right ideal bR. So we have a $u \in R_q^{-1}$ such that $a \equiv u \pmod{bR}$. This infers that $a - u = by$ for some $y \in R$. That is, $a + b(-y) = u \in R_q^{-1}$, as required. $\square$

If $x \in R$ is regular, then there exists some $y \in R$ such that $x = xyx$ and $y = yxy$. We say such y is a reflexive inverse of x, and denote it by x^+. Now we characterize QB-ideals by means of reflexive inverses.

Theorem 3.6. *Let I be an ideal of a regular ring R. Then the following are equivalent*:

(1) *I is a QB-ideal.*
(2) *For each $a \in 1+I$, there exist a $u \in R_q^{-1}$ and an element $x \in \ell(a^+)$ such that $a = u + x$.*
(3) *For each $a \in 1+I$, there exist a $u \in R_q^{-1}$ and an element $x \in r(a^+)$ such that $a = u + x$.*

Proof. (1) $\Rightarrow$ (2) Let $a \in 1+I$. Since R is regular, we have $a^+ \in R$ such that $a = aa^+a$ and $a^+ = a^+aa^+$. Let $I = \ell(a^+)$. Then $1 - a^+a \in I$. That is, $a \in R$ is a quasi-invertible element modulo I. By virtue of Lemma 3.5, we can find a $u \in R_q^{-1}$ such that $a - u \in I$, as required.

(2) $\Rightarrow$ (1) For each $a \in 1+I$, there exist a $u \in R_q^{-1}$ and an element $x \in \ell(a^+)$ such that $a = u + x$. Hence $a - u \in \ell(a^+)$, and then $(a-u)a^+ = 0$. This infers that $aa^+ = ua^+$, so $a = aa^+a = ua^+a$. Clearly, $a^+a \in 1+I$ is an idempotent. According to Corollary 2.5, I is a QB-ideal.

(1) $\Leftrightarrow$ (3) is clear by symmetry. □

Corollary 3.7. *Let I be an ideal of a regular ring R. Then the following are equivalent*:

(1) *I is a QB-ideal.*
(2) *For each $a \in 1+I$, there exist a $u \in R_q^{-1}$ and an element $x \in \ell(a)$ such that $a^+ = u + x$.*
(3) *For each $a \in 1+I$, there exist a $u \in R_q^{-1}$ and an element $x \in r(a)$ such that $a^+ = u + x$.*

Proof. For any $a \in 1+I$, there exists a $a^+ \in R$ such that $a = aa^+a, a^+ = a^+aa^+$. Clearly, $a^+ \in 1+I$, and that $a \in R$ can be seen as $(a^+)^+ \in R$. Applying Theorem 3.6, we complete the proof. □

So far, one mainly studied diagonal reduction only for matrices over a ring (cf. [9]). We now consider the matrices over QB-ideals of a regular ring.

Lemma 3.8. *Every 2×2 triangular matrix over a regular ideal of a ring admits a diagonal reduction.*

Proof. Let I be a regular ideal of a ring R, and let $A = \left(\begin{smallmatrix} a & 0 \\ c & b \end{smallmatrix}\right) \in M_2(I)$. As I is regular, there exists an idempotent $e \in I$ such that $aR = eR$. Write $a = ec$ and $e = ad$. Set $U = \left(\begin{smallmatrix} d & 1-dc \\ -1 & c \end{smallmatrix}\right)$. Then $U \in GL_2(R)$. Further, $AU = \left(\begin{smallmatrix} e & 0 \\ t & s \end{smallmatrix}\right)$. Write $s = ss's$ and $s' = s'ss'$. By adding to the first column of AU its 2-th column right multiplied by $-s't$, we may assume that $ss't = 0$. Set

$$V = \begin{pmatrix} s & ss'-1 \\ 1+s's & s' \end{pmatrix}.$$

Then $V \in GL_2(R)$. Furthermore,

$$VAU = \begin{pmatrix} se-t & 0 \\ (1+s's)e & s's \end{pmatrix}.$$

Thus, we may assume that $VAU = \left(\begin{smallmatrix} y & 0 \\ e & f \end{smallmatrix}\right)$, where $f = s's \in R$ is an idempotent. Write $(1-f)e = (1-f)eh(1-f)e$. Set $z = y\big(1 - h(1-f)e\big)$. By elementary transformations, VAU can be reduced to the form $\left(\begin{smallmatrix} z & 0 \\ e & f \end{smallmatrix}\right)$. One easily checks that

$$\begin{pmatrix} z & 0 \\ e & f \end{pmatrix} B_{21}(-e)B_{12}\big(h(1-f)\big)B_{21}\big(-(1-f)e\big)B_{12}\big(-h(1-f)\big) = \mathrm{diag}(*,*),$$

and therefore we complete the proof. □

As is well known, every square matrix over a unit-regular ring admits a diagonal reduction. For 2×2 matrices over a QB-ideal of regular rings, we can derive the following.

Theorem 3.9. *Let I be a QB-ideal of a regular ring R. Then for any $A \in M_2(I)$, there exist $U \in GL_2(R), V \in M_2(R)_q^{-1}$ such that $UAV = \mathrm{diag}(e_1, e_2)$ for some $e_1, e_2 \in I$.*

Proof. Let $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_2(I)$. Since I is a QB-ideal of a regular ring R, as in the proof of Theorem 2.3, we can find $w_1, w_2 \in R_q^{-1}$ such that $aw_1 = f, bw_2 = e$ are idempotents. Let $Q_1 = \left(\begin{smallmatrix} 1 & f-1 \\ -f & 1 \end{smallmatrix}\right)$. Then $(a,b)\mathrm{diag}(w_1,w_2)Q_1 = \big((1-e)f, e\big)$. As $(1-e)f \in I$, we have a $t \in R$ such that $(1-e)f = (1-e)ft(1-e)f$. It follows from $(1-e)ft + \big(1-(1-e)ft\big) = 1$ that $(1-e)fte + \big(1-(1-e)ft\big)e = e$; hence, $1-e+\big(1-(1-e)ft\big)e = 1-(1-e)fte$. Accordingly, $(1-e)f + \big(1-(1-e)ft\big)ef = \big(1-(1-e)fte\big)f$. Clearly,

$$\big((1-e)f + (1-(1-e)ft)ef\big)t + \big(1-(1-e)ft\big)(1-eft) = 1.$$

Thus,

$$\big(1-(1-e)fte\big)ft + \big(1-(1-e)ft\big)(1-eft) = 1.$$

As a result,

$$ft\big(1-(1-e)fte\big) + \big(1+(1-e)fte\big)\big(1-(1-e)ft\big)(1-eft)\big(1-(1-e)fte\big) = 1.$$

By the proceeding discussion, we have a $z \in R$ such that

$$f + \big(1+(1-e)fte\big)\big(1-(1-e)ft\big)z \in U(R).$$

That is,

$$\big(1-(1-e)fte\big)f + \big(1-(1-e)ft\big)z \in U(R),$$

and so $v := (1-e)f + \big(1-(1-e)ft\big)(ef+z) \in U(R)$. Hence, $(1-e)f = (1-e)ftv$ such that $f_1 = (1-e)fu$ is an idempotent, where $u = v^{-1}$. Set $g = f_1(1-e)$. Let $Q_2 = \mathrm{diag}(u,1), Q_3 = \mathrm{diag}(1-f_1e, 1+f_1e), Q_4 = \left(\begin{smallmatrix} 1-g & 1 \\ -g & 1 \end{smallmatrix}\right)$. Then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mathrm{diag}(w_1,w_2)Q_1Q_2Q_3Q_4 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} * & 0 \\ ** & * \end{pmatrix}.$$

One easily checks that $Q_1, Q_2, Q_3, Q_4 \in E_2(R)$. According to Lemma 3.8, we complete the proof. □

Corollary 3.10. *Let R be a regular QB-ring. Then for any $A \in M_2(R)$, there exist $U \in GL_2(R), V \in M_2(R)_q^{-1}$ such that $UAV = \mathrm{diag}(e_1, e_2)$ for some $e_1, e_2 \in R$.*

Proof. It is an immediate consequence of Theorem 3.9. □

The class of regular QB-ring is very large. Let V be an infinite-dimensional vector space over a field F, let $Q = \mathrm{End}_F(V)$, and let $J = \{x \in Q \mid \dim_F(xV) < \infty\}$. Set $R = \{(x,y) \in Q \times Q \mid x - y \in J\}$. Then R is a regular QB-ring.

4. Prime rings

A ring R is prime provided that for any ideals I and J, $IJ = 0$ implies that either $I = 0$ or $J = 0$. For example, every regular ring satisfying the comparability axiom is a prime exchange ring. Every prime factor of exchange rings is a prime exchange ring. The aim of this section is to investigate QB-ideals of prime exchange rings. We use R_l^{-1}(resp. R_r^{-1}) to denote the set of all left (resp. right) invertible elements of R.

Lemma 4.1. *Let I be an ideal of a prime exchange ring R. Then the following are equivalent*:

(1) *I is a QB-ideal.*
(2) *For any regular $x \in 1 + I$, there exists right or left invertible $u \in R$ such that $x = xux$.*

Proof. Since R is a prime ring, we see that $R_q^{-1} = R_l^{-1} \cup R_r^{-1}$. Therefore we get the result by Lemma 2.2. □

Let I be a QB-ideal of a prime exchange ring, and let $A \in FP(I)$. If B and C are any right R-modules such that $A \oplus B \cong A \oplus C$, we note that $B \lesssim^{\oplus} C$ or $C \lesssim^{\oplus} B$ (cf. [10]).

Lemma 4.2. *Let I be an ideal of a prime exchange ring R. Then R is a QB-ring if and only if*

(1) *R/I is a QB-ring.*
(2) *$(R/I)_q^{-1} = (R_q^{-1} + I)/I$.*
(3) *I is a QB-ideal.*

Proof. One direction is clear by [1, Theorem 7.2]. Conversely, assume now that (1), (2) and (3) hold. Since R is a prime ring, $R_q^{-1} = R_l^{-1} \cup R_r^{-1}$. Suppose that p and q is a pair of defect idempotents of R. Then we have right or left invertible $u \in R$ such that $p = 1 - uv$ and $q = 1 - vu$, where $uv = 1$ or $vu = 1$. Hence $p = 1$ or $q = 1$, and then $(1 - p)R(1 - q) = 0$. Therefore $I + R_q^{-1} \subseteq \mathrm{cl}(R_q^{-1})$ by [1, Lemma 1.5]. Using [1, Theorem 7.2], we complete the proof. □

Theorem 4.3. *Let I be an ideal of a prime exchange ring R. Then R is a QB-ring if and only if*

(1) *R/I is a QB-ring.*
(2) *$(R/I)_q^{-1} = (R/I)_l^{-1} \cup (R/I)_r^{-1}$.*
(3) *For any idempotents $e \in 1+I, f \in R$, $eR \cong fR$ implies that $(1-e)R \lesssim^{\oplus} (1 - f)R$ or $(1 - f)R \lesssim^{\oplus} (1 - e)R$.*

Proof. Suppose that R is a QB-ring. Then $(R/I)_q^{-1} = (R_q^{-1} + I)/I$ by Lemma 4.2. Since R is a prime ring, $R_q^{-1} = R_l^{-1} \cup R_r^{-1}$. Thus $(R/I)_q^{-1} = (R_l^{-1} \cup R_r^{-1} + I)/I \subseteq (R_l^{-1} + I)/I \cup (R_r^{-1} + I)/I \subseteq (R/I)_l^{-1} \cup (R/I)_r^{-1}$. Clearly, $(R/I)_l^{-1} \cup (R/I)_r^{-1} \subseteq (R/I)_q^{-1}$. Therefore we have $(R/I)_q^{-1} = (R/I)_l^{-1} \cup$

$(R/I)_r^{-1}$. Suppose that $eR \cong fR$ with idempotents $e, f \in R$. Since R is a prime ring, one easily checks that $(1-e)R \lesssim^{\oplus} (1-f)R$ or $(1-f)R \lesssim^{\oplus} (1-e)R$.

Conversely, assume now that (1), (2) and (3) hold. Given any regular $x \in 1+I$, there exists $y \in 1+I$ such that $x = xyx$. Hence, $xy \in 1+I$. Since $xyR \cong yxR$, we deduce that $(1-xy)R \lesssim^{\oplus} (1-yx)R$ or $(1-yx)R \lesssim^{\oplus} (1-xy)R$. As in the proof of Theorem 2.7, $x = xux$ for a $u \in R_q^{-1}$. In view of Lemma 2.2, I is a QB-ideal.

Clearly, $(R_q^{-1}+I)/I \subseteq (R/I)_q^{-1}$. One the other hand, $(R/I)_q^{-1} = (R/I)_l^{-1} \cup (R/I)_r^{-1}$. We only prove that one-sided invertible elements lift modulo I. Assume that $\overline{xy} = \overline{1}$ in R/I. Since R is an exchange ring, one easily finds some $a, b \in R$ such that $a = aba, b = bab, \overline{a} = \overline{x}$ and $\overline{b} = \overline{y}$. Hence, $\overline{a}\overline{b} = \overline{1}$. So $1 - ab \in I$, i.e., $ab \in 1+I$. Since $abR \cong baR$, we have either $(1-ab)R \lesssim^{\oplus} (1-ba)R$ or $(1-ba)R \lesssim^{\oplus} (1-ab)R$. If $(1-ab)R \lesssim^{\oplus} (1-ba)R$, then we can find $s \in (1-ab)R(1-ba), t \in (1-ba)R(1-ab)R$ such that $1-ab = st$. Clearly, $at = sb = 0$; hence, $(a+s)(t+b) = ab + st = 1$. That is, $a+s \in R$ is right invertible. Obviously, we have $s \in (1-ab)R(1-ba) \subseteq I$, and then $\overline{x} = \overline{a} = \overline{a+s}$. That is, x can be lifted by a right invertible element modulo I. If $(1-ba)R \lesssim^{\oplus} (1-ab)R$, then we have $s \in (1-ba)R(1-ab), t \in (1-ab)R(1-ba)R$ such that $1-ba = st$. Obviously, $bt = sa = 0$; hence, $(b+s)(a+t) = ba+st = 1$. Also we have $t \in (1-ab)R(1-ba) \subseteq I$, so $\overline{x} = \overline{a} = \overline{a+t}$. That is, x can be lifted by a left invertible element modulo I. Analogously, x can be lifted modulo I in case $\overline{x}$ is left invertible. Therefore

$$(R/I)_q^{-1} = (R/I)_l^{-1} \cup (R/I)_r^{-1} \subseteq \big((R_l^{-1}+I)/I\big) \cup \big((R_r^{-1}+I)/I\big) \subseteq (R_q^{-1}+I)/I.$$

Accordingly, $(R/I)_q^{-1} = (R_q^{-1}+I)/I$. It follows from Lemma 4.2 that R is a QB-ring. □

Corollary 4.4. *Let I be an ideal of a prime exchange ring R. Then R is a QB-ring if and only if*

(1) *R/I is a QB-ring.*
(2) $(R/I)_q^{-1} = (R/I)_l^{-1} \cup (R/I)_r^{-1}$.
(3) *For any idempotents $e \in 1+I, f \in R$, $eR \cong fR$ implies that $eu = uf$ for a $u \in R_q^{-1}$.*

Proof. One direction is obvious by Theorem 4.3 and Corollary 2.6. Conversely, assume that (1), (2) and (3) hold. Then $eu = uf$ for a $u \in R_q^{-1}$. Since R is a prime ring, we deduce that either $uv = 1$ or $vu = 1$. If $uv = 1$, then $1 - e = u(1-f)v = ab$, where $a = (1-e)u(1-f)$ and $b = (1-f)v(1-e)$; hence, $(1-e)R \lesssim^{\oplus} (1-f)R$. If $vu = 1$, then we have $f = veu$. Similarly, we get $(1-f)R \lesssim^{\oplus} (1-e)R$. It follows from Theorem 4.3 that R is a QB-ring. □

Acknowledgements. The author would like to thank the referee for her/his helpful comments which lead to the new version of this paper.

References

[1] P. Ara, G. K. Pedersen, and F. Perera, *An infinite analogue of rings with stable rank one*, J. Algebra **230** (2000), no. 2, 608–655.

[2] ———, *Extensions and pullbacks in QB-rings*, Algebr. Represent. Theory **8** (2005), no. 1, 75–97.

[3] H. Chen, *On exchange QB-rings*, Comm. Algebra **31** (2003), no. 2, 831–841.

[4] ———, *On QB_∞-rings*, Comm. Algebra **34** (2006), no. 6, 2057–2068.

[5] ———, *On regular QB-ideals*, Taiwanese J. Math. **10** (2006), no. 5, 1261–1269.

[6] ———, *Extensions of exchange QB-rings*, Algebra Colloq. **13** (2006), no. 4, 667–674.

[7] D. Lu, Q. Li, and W. Tong, *Comparability, stability, and completions of ideals*, Comm. Algebra **32** (2004), no. 7, 2617–2634.

[8] F. Perera, *Lifting units modulo exchange ideals and C^*-algebras with real rank zero*, J. Reine Angew. Math. **522** (2000), 51–62.

[9] A. A. Tuganbaev, *Rings Close to Regular*, Kluwer Academic Publishers, Dordrecht, Boston, London, 2002.

[10] J. Wei, *Unit-regularity and stable range conditions*, Comm. Algebra **33** (2005), no. 6, 1937–1946.

Department of Mathematics
Hangzhou Normal University
Hangzhou 310036, P. R. China
E-mail address: huanyinchen@yahoo.cn