Bull. Korean Math. Soc. **46** (2009), No. 5, pp. 867–871
DOI 10.4134/BKMS.2009.46.5.867

INDEPENDENTLY GENERATED MODULES

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Abstract. A module M over a ring R is said to satisfy (P) if every generating set of M contains an independent generating set. The following results are proved;

(1) Let $\tau = (\mathbb{T}_\tau, \mathbb{F}_\tau)$ be a hereditary torsion theory such that $\mathbb{T}_\tau \neq$ Mod-R. Then every τ-torsionfree R-module satisfies (P) if and only if $S = R/\tau(R)$ is a division ring.

(2) Let $\mathcal{K}$ be a hereditary pre-torsion class of modules. Then every module in $\mathcal{K}$ satisfies (P) if and only if either $\mathcal{K} = \{0\}$ or $S = R/\operatorname{Soc}_{\mathcal{K}}(R)$ is a division ring, where $\operatorname{Soc}_{\mathcal{K}}(R) = \cap\{I \leq R_R : R/I \in \mathcal{K}\}$.

For a right R-module M, a subset X of M is said to be a *generating set* of M if $M = \Sigma_{x\in X} xR$; and a *minimal generating set* of M is any generating set Y of M such that no proper subset of Y can generate M. A generating set X of M is called an *independent generating set* if $\Sigma_{x\in X} xR = \oplus_{x\in X} xR$. Clearly, every independent generating set of M is a minimal generating set, but the converse is not true in general. For example, the set $\{2, 3\}$ is a minimal generating set of $\mathbb{Z}_{\mathbb{Z}}$ but not an independent generating set.

It is well-known that every generating set of a right vector space over a division ring contains a minimal generating set (or a basis). This motivated various interests in characterizing the rings R such that every module in a certain class of right R-modules contains a minimal generating set, or every generating set of each module in a certain class of right R-modules contains a minimal generating set (see, for example, [2], [8], [9], [11]).

In [2, Theorem 2.3], the authors proved that R is a division ring if and only if every R-module has a basis if and only if every irredundant subset of an R-module is independent. This result can be considered in a more general context of a torsion theory. For an R-module M, M is said to satisfy (P) if every

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Received July 13, 2008.
2000 *Mathematics Subject Classification.* 16D10.
Key words and phrases. generated set for modules, basis, (non)-singular modules, division ring, torsion theory.
The content of this paper is a part of a thesis written by Tufan Özdin under the supervision of M. Tamer Koşan (Gebze Institute of Technology).

generating set of M contains an independent generating set. For a hereditary torsion theory $\tau = (\mathbb{T}_\tau, \mathbb{F}_\tau)$, the paper is concerned with the following questions:

(1) When does every τ-torsion module satisfy (P)?
(2) When does every τ-torsionfree module satisfy (P)?

Throughout this paper R denotes an associative ring with unit and M is a right unitary R-module. For a module M, the notions "$\leq$", "Soc(M)", "$x^\perp$", and "$Z(M)$" denote the submodule, the socle, the right annihilator of an element x, and the singular submodule of M, respectively. Moreover, $Z_2(M)$ is defined by $Z(M/Z(M)) = Z_2(M)/Z(M)$. If $M = Z_2(M)$, we say that M is Goldie torsion. If $Z(M) = 0$, then M is called nonsingular. A module is called *quasi-cyclic* if each of its finitely generated submodules is contained in a cyclic submodule (see [7]). According to Bass [3], for a sequence $\{a_n : n = 1, 2, \ldots\}$ of elements of R, let F the free R-module with basis $x_1, x_2, \ldots$, G the submodule of F generated by the set $\{x_n - a_n x_{n+1} : n = 1, 2, \ldots\}$, and $[F, \{a_n\}, G]$ the quotient module F/G. Note that $[F, \{a_n\}, G]$ is a quasi-cyclic module.

We will refer to [1], [4] and [6] for all undefined notions used in the text.

We begin with the following easy but useful lemma.

Lemma 1. *Let xR, yR be nonzero cyclic R-modules with $x^\perp \neq y^\perp$, and let $M = xR \oplus xR \oplus yR \oplus yR$. Then there is a submodule N of M such that N does not satisfy (P).*

Proof. Without loss of generality, we may assume that $x^\perp \not\subseteq y^\perp$. Let $u = (x, x, y, 0)$ and $v = (0, x, y, 0)$, and let N be the submodule of M generated by $\{u, v\}$. Since $u \notin vR$ and $v \notin uR$, $\{u, v\}$ is a minimal generating set of N. But $\{u, v\}$ is not an independent generating set of N since $0 \neq (0, 0, yx^\perp, 0) \subseteq uR \cap vR$. Therefore, the generating set $\{u, v\}$ does not contain any independent generating sets of N. □

Theorem 2. *Let $\tau = (\mathbb{T}_\tau, \mathbb{F}_\tau)$ be a hereditary torsion theory such that $\mathbb{T}_\tau \neq$ Mod-R. The following are equivalent for a ring R :*

(1) *Every R-module satisfies (P).*
(2) *$\tau(R) = 0$ and every τ-torsionfree module satisfies (P).*
(3) *R is a division ring.*

Proof. (3) ⇒ (1). It is well-known.

(1) ⇒ (2). Suppose $0 \neq a \in \tau(R)$. If $a^\perp = 0$, then $R_R \cong aR \in \mathbb{T}_\tau$. Thus, $R \in T_\tau$, implying Mod-$R = \mathbb{T}_\tau$. This contradicts the assumption on τ. Hence, $ab = 0$ for some $0 \neq b \in R$. Therefore, by Lemma 1, the module $aR \oplus aR \oplus R \oplus R$ has a submodule N such that N does not satisfies (P). This contradiction shows that $\tau(R_R) = 0$.

(2) ⇒ (3). Suppose R satisfies (2). First we claim that, for any τ-torsionfree module M with $x \in M$ and $r \in R$, $xr = 0$ implies that $x = 0$ or $r = 0$.

For, if not, by Lemma 1, the τ-torsionfree module $xR \oplus xR \oplus R \oplus R$ has a submodule N such that N does not satisfies (P). This is a contradiction. In particular, our claim implies that R is a domain. Suppose R is not a division ring. Then $aR \neq R$ for some $0 \neq a \in R$. Let $a_n = a$ for $n = 1, 2, \ldots$, let F be the free R-module with basis $\{x_n : n = 1, 2, \ldots\}$, and G the submodule of F generated by the set $\{x_n - x_{n+1}a_n : n = 1, 2, \ldots\}$. Set $H = F/G$. If $\overline{x_1} = x_1 + G \in \tau(H_R)$, then, since R is not in $\mathbb{T}_\tau$, $\overline{x_1}c = \bar{0}$ for some nonzero element $c \in R$. But it is straightforward to check that this is impossible. Therefore, $\overline{x_1} \notin \tau(H)$. So $H/\tau(H)$ is a nonzero τ-torsionfree module. Note that $\{\overline{x_n} + \tau(H) : n = 1, 2, \ldots\}$ is a generating set of $H/\tau(H)$. By (2), there is a nonempty set $\mathbf{L}$ of positive integers such that $\{\overline{x_n} + \tau(H) : n \in \mathbf{L}\}$ is an independent generating set of $H/\tau(H)$. Let m be the least number in $\mathbf{L}$. Note that $\overline{x_m} + \tau(H) = [\overline{x_{m+k}} + \tau(H)]a^k$ for $k = 1, 2, \ldots$. It must be $\mathbf{L} = \{m\}$, i.e., $H/\tau(H)$ is generated by $\overline{x_m} + \tau(H)$. Therefore, $\overline{x_{m+1}} + \tau(H) = [\overline{x_m} + \tau(H)]r$ for some $r \in R$, i.e., $[\overline{x_{m+1}} + \tau(H)](1 - ar) = \bar{\bar{0}}$ $(= \bar{0} + \tau(H))$. Now by the claim above, $\overline{x_{m+1}} + \tau(H) = \bar{\bar{0}}$ or $1 - ar = 0$. Since $\overline{x_1} \notin \tau(H)$, we have $\overline{x_{m+1}} \notin \tau(H)$, and thus $1 - ar = 0$, i.e., $aR = R$. This is a contradiction. □

Applying Theorem 2 to the Goldie torsion theory τ yields the next corollary.

Corollary 3. *The ring R is a division ring if and only if R is right non-singular and every non-singular R-module satisfies (P).*

Let $S = R/\tau(R)$ be the factor ring and $\gamma : R \longrightarrow S$ be the canonical ring homomorphism. Then γ induces a hereditary torsion theory $\sigma = \gamma_{\#}(\tau)$ on Mod-S defined by the condition that an S-module N is a σ-torsion S-module if and only if N_R is a τ-torsion module (see [5, p. 433]).

Theorem 4. *Let $\tau = (\mathbb{T}_\tau, \mathbb{F}_\tau)$ be a hereditary torsion theory such that $\mathbb{T}_\tau \neq$ Mod-R. Then every τ-torsionfree R-module satisfies (P) if and only if $S = R/\tau(R)$ is a division ring.*

Proof. " $\Rightarrow$". Since R is not in $\mathbb{T}_\tau$ and $\tau(R/\tau(R)) = 0$, S is nonzero and $\sigma(S) = 0$. Let N_S be a σ-torsionfree module with a generating set Y. Then N is a τ-torsionfree R-module with a generating set Y. By the assumption, $N_R = \oplus_{x \in X} xR$ for a subset X of Y. It follows that $N_S = \oplus_{x \in X} xS$. By Theorem 1, S is a division ring.

" $\Leftarrow$". Let N be a τ-torsionfree R-module with a generating set Y. Since $N\tau(R) \subseteq \tau(N)$, we see $N\tau(R) = 0$. Thus, N is an S-module and hence is a σ-torsionfree module with a generating set Y. Since S is a division ring, by Theorem 2, we have $N_S = \oplus_{x \in X} xS$ for a subset X of Y. Thus, $N_R = \oplus_{x \in X} xR$. □

When τ is the Goldie torsion theory, Theorem 4 gives the next consequence.

Corollary 5. *Every nonsingular R-module satisfies (P) if and only if $R = Z_2(R)$ or $R/Z_2(R)$ is a division ring.*

Let $\mathcal{K}$ be a hereditary pre-torsion class of modules and $\mathrm{Soc}_{\mathcal{K}}(R) = \cap\{I : I \in H_{\mathcal{K}}(R)\}$, where $H_{\mathcal{K}}(R) = \{I \subseteq R_R : R/I \in \mathcal{K}\}$. The notation is taken from [4]. By the proof of Theorem 2.5 in [12], $\mathrm{Soc}_{\mathcal{K}}(R)$ is a two-sided ideal of R. If $\mathcal{K} = \{$ singular R-modules $\}$, then $\mathrm{Soc}_{\mathcal{K}}(R)$ is just the socle of R.

Theorem 6. *Let $\mathcal{K}$ be a hereditary pre-torsion class of modules. Then every module in $\mathcal{K}$ satisfies (P) if and only if either $\mathcal{K} = \{0\}$ or $S = R/\mathrm{Soc}_{\mathcal{K}}(R)$ is a division ring.*

Proof. " $\Rightarrow$". If $0 \neq R/I_i \in \mathcal{K}$ for $i = 1, 2$, then Lemma 1 implies that $I_1 = I_2$. So, either $\mathrm{Soc}_{\mathcal{K}}(R) = R$ or $\mathrm{Soc}_{\mathcal{K}}(R)$ is a maximal right ideal of R. Therefore, $\mathcal{K} = \{0\}$ or S is a division ring.

" $\Leftarrow$". If $\mathcal{K} = \{0\}$, then the claim follows. Suppose that S is a division ring and $\mathcal{K} \neq \{0\}$. This shows that $H_{\mathcal{K}}(R) = \{\mathrm{Soc}_{\mathcal{K}}(R), R\}$. Then, for any module $M \in \mathcal{K}$ with a generating set Y, $M \cdot \mathrm{Soc}_{\mathcal{K}}(R) = 0$ and thus M is an S-module with a generating set Y. By Theorem 2, $M_S = \oplus_{x \in X} xS$ for a subset X of Y. It follows that $M_R = \oplus_{x \in X} xR$. □

Letting $\mathcal{K}$ be the class of the singular right R-modules in Theorem 6, one obtains the next corollary.

Corollary 7. *Every singular R-module satisfies (P) if and only if either R is a semisimple ring or $R/\mathrm{Soc}(R)$ is a division ring.*

From now on, $\mathcal{K}$ is a hereditary pre-torsion class and $\tau_{\mathcal{K}} = (\mathbb{T}_{\mathcal{K}}, \mathbb{F}_{\mathcal{K}})$ is the torsion theory generated by $\mathcal{K}$, i.e., $\mathbb{F}_{\mathcal{K}} = \{F \in \text{Mod-}R : \mathrm{Hom}(C, F) = 0$ for all $C \in \mathcal{K}\}$ and $\mathbb{T}_{\mathcal{K}} = \{T \in \text{Mod-}R : \mathrm{Hom}(T, F) = 0$ for all $F \in \mathcal{K}\}$. By [10, Proposition 3.3], $\tau_{\mathcal{K}}$ is a hereditary torsion theory.

Theorem 8. *Every module in $\mathbb{T}_{\mathcal{K}}$ satisfies (P) if and only if either*

(1) *$\mathcal{K} = \{0\}$ or*
(2) *$\mathcal{K} = \mathbb{T}_{\mathcal{K}}$ and $R/\mathrm{Soc}_{\mathcal{K}}(R)$ is a division ring.*

Proof. Note that $\mathcal{K} = \{0\}$ if and only if $\mathbb{T}_{\mathcal{K}} = \{0\}$. Thus the sufficiency follows from Theorem 6. For the necessity, by Theorem 6, it suffices to show that $\mathcal{K} = \mathbb{T}_{\mathcal{K}}$. If the equality does not hold, then there exists a module $M \in \mathbb{T}_{\mathcal{K}}$ but $M \notin \mathcal{K}$. Therefore, there is a cyclic submodule xR of M such that $xR \notin \mathcal{K}$. Since $\mathcal{K} \neq \{0\}$, there is a nonzero cyclic module $yR \in \mathcal{K}$. Then $x^{\perp} \neq y^{\perp}$. By Lemma 1, this contradicts the assumption. So $\mathcal{K} = \mathbb{T}_{\mathcal{K}}$. □

Let $\mathcal{K}$ be the class of the singular right R-modules. Applying Theorem 8 to $\mathcal{K}$ yields the next corollary.

Corollary 9. *Every Goldie torsion module satisfies (P) if and only if either R a semisimple ring or R is a right non-singular ring with $R/\mathrm{Soc}(R)$ being a division ring.*

Acknowledgment. We thank to Prof. Yiqiang Zhou (Memorial University, Canada) for introducing us to this problem, and for his valuable comments and suggestions.

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