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# PAIR OF (GENERALIZED-)DERIVATIONS ON RINGS AND BANACH ALGEBRAS

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Abstract. Let $n$ be a fixed positive integer, $\mathcal{R}$ be a $2n!$-torsion free prime ring and $\mu, \nu$ be a pair of generalized derivations on $\mathcal{R}$. If $\langle \mu^2(x) + \nu(x), x^n \rangle = 0$ for all $x \in \mathcal{R}$, then $\mu$ and $\nu$ are either left multipliers or right multipliers. Let $n$ be a fixed positive integer, $\mathcal{R}$ be a noncommutative $2n!$-torsion free prime ring with the center $\mathcal{C}_{\mathcal{R}}$ and $d, g$ be a pair of derivations on $\mathcal{R}$. If $\langle d^2(x) + g(x), x^n \rangle \in \mathcal{C}_{\mathcal{R}}$ for all $x \in \mathcal{R}$, then $d = g = 0$. Then we apply these purely algebraic techniques to obtain several range inclusion results of pair of (generalized-)derivations on a Banach algebra.

## 1. Introduction

Let $\mathcal{R}$ be a ring with the center $\mathcal{C}_{\mathcal{R}}$. A mapping $f : \mathcal{R} \longrightarrow \mathcal{R}$ is said to be *centralizing* if $[f(x), x] \in \mathcal{C}_{\mathcal{R}}$ for all $x \in \mathcal{R}$. In the special case of when $[f(x), x] = 0$ for all $x \in \mathcal{R}$, the mapping $f$ is called *commuting*. A mapping $f : \mathcal{R} \longrightarrow \mathcal{R}$ is said to be *central* if $f(x) \in \mathcal{C}_{\mathcal{R}}$ for all $x \in \mathcal{R}$. Obviously, every central mapping is commuting, but not conversely in general. A mapping $f$ of a ring $\mathcal{R}$ is said to be *skew-centralizing* if $f(x)x + xf(x) \in \mathcal{C}_{\mathcal{R}}$ for all $x \in \mathcal{R}$. In particular, if $f(x)x + xf(x) = 0$ for all $x \in \mathcal{R}$, then it is called *skew-commuting*. The study of (skew-)centralizing and (skew-)commuting mappings was initiated by a well known theorem of Posner which states that the existence of a nonzero centralizing derivation on a prime ring $\mathcal{R}$ implies that $\mathcal{R}$ is commutative [14]. This theorem has been extended by many people in different ways. One interesting topic of all related works is to study the skew-centralizing mappings or skew-commuting mappings involving pair of (generalized-)derivations on (semi-)prime rings and Banach algebras. Various results with respect to pair of (generalized-)derivations are obtained, see [1], [4], [10], [13], [20], [21], [22].

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Let $\mathcal{A}$ be an associative algebra, a linear mapping $\mu : \mathcal{A} \longrightarrow \mathcal{A}$ is called a *generalized derivation* of $\mathcal{A}$ if there exists a derivation $d$ of $\mathcal{A}$ such that

$$\mu(xy) = \mu(x)y + xd(y)$$

for all $x, y \in \mathcal{A}$. $d$ is called an *associated derivation* of the generalized derivation $\mu$. Obviously, the following mapping

$$\mu : \mathcal{R} \longrightarrow \mathcal{R}, \quad x \longmapsto ax - xb$$

is a generalized derivation of $\mathcal{R}$, where $a$ and $b$ are fixed elements in $\mathcal{R}$. Indeed, for all $x, y \in \mathcal{R}$,

$$\mu(xy) = axy - xyb = (ax - xb)y + x(by - yb) = \mu(x)y + xd(y),$$

where $d$ is an inner derivation of $\mathcal{R}$ induced by the element $b$. Such generalized derivations are called *generalized inner derivations*. It is easy to check that if the associated derivation $d$ of a generalized derivation $\mu$ is inner, then $\mu$ is also inner. Moreover, all derivations of $\mathcal{R}$ and all right or left multipliers mappings of $\mathcal{R}$ are also generalized derivations of $\mathcal{R}$.

The main objective of this paper is to consider some special skew-centralizing mappings and some special skew-commuting mappings, which are involved a pair of (generalized-)derivations on (semi-)prime rings. In addition, we use purely algebraic techniques to study the range inclusion problem of pair of (generalized-)derivations on a Banach algebra.

## 2. Preliminaries

Throughout this paper $\mathcal{R}$ always denotes an *associative ring* with the center $\mathcal{C}_{\mathcal{R}}$ and $\mathcal{A}$ always denotes a *Banach algebra* which is a complex normed algebra and its underlying vector space is a Banach space. The *Jacobson radical* of $\mathcal{A}$ is the intersection of all primitive ideals of $\mathcal{A}$ and is denoted by $rad(\mathcal{A})$. Let $\mathcal{I}$ be any closed ideal of the Banach algebra $\mathcal{A}$. Then $Q_{\mathcal{I}}$ denotes the canonical quotient mapping from $\mathcal{A}$ onto $\mathcal{A}/\mathcal{I}$. A ring $\mathcal{R}$ is said to be *n-torsion free* if $nx = 0$ implies that $x = 0$ for all $x \in \mathcal{R}$. As usual, we denote the commutator $xy - yx$ by $[x, y]$ and denote the skew commutator $xy + yx$ by $\langle x, y\rangle$. Recall that a ring $\mathcal{R}$ is said to be *prime* if the product of any two nonzero ideals of $\mathcal{R}$ is nonzero. Equivalently, $a\mathcal{R}b = 0$ with $a, b \in \mathcal{R}$ implies that $a = 0$ or $b = 0$. A ring $\mathcal{R}$ is called *semiprime* if it has no nonzero nilpotent ideals. Equivalently, $a\mathcal{R}a = 0$ with $a \in \mathcal{R}$ implies that $a = 0$.

## 3. Generalized derivations on (semi-)prime rings

In this section we will consider pair of (generalized-)derivations on a (semi-)prime ring. These results will play important roles when we discuss the range inclusion problem of pair of (generalized-)derivations on a Banach algebra in the next section.

For the proof of our main result of this section, we need some basic facts. From now on $\mathcal{R}$ always denotes a (semi-)prime ring and $\mathcal{U}$ always denotes the *left Utumi quotient ring* of $\mathcal{R}$. $\mathcal{U}$ can be characterized as a ring satisfying the following properties:

(1) $\mathcal{R}$ is a subring of $\mathcal{U}$.

(2) For each $q \in \mathcal{U}$, there exists a dense left ideal $\mathcal{I}_q$ of $\mathcal{R}$ such that $\mathcal{I}_q q \subseteq \mathcal{R}$.

(3) If $q \in \mathcal{U}$ and $\mathcal{I}q = 0$ for some dense left ideal $\mathcal{I}$ of $\mathcal{R}$, then $q = 0$.

(4) If $\phi : \mathcal{I} \to \mathcal{R}$ is a left $\mathcal{R}$-module mapping from a dense left ideal $\mathcal{I}$ of $\mathcal{R}$ into $\mathcal{R}$, then there exists an element $q \in \mathcal{U}$ such that $\phi(i) = iq$ for all $i \in \mathcal{I}$.

Up to isomorphisms, $\mathcal{U}$ is uniquely determined by the above four properties. If $\mathcal{R}$ is a (semi-)prime ring, then $\mathcal{U}$ is also a (semi-)prime ring. The center of $\mathcal{U}$ is called the *extended centroid* of $\mathcal{R}$ and is denoted by $\mathcal{C}$. It is well known that $\mathcal{C}$ is a von Neumann regular ring. It turns out that $\mathcal{C}$ is a field if and only if $\mathcal{R}$ is a prime ring. The set of all idempotents of $\mathcal{C}$ is denoted by $\mathcal{E}$. The element of $\mathcal{E}$ are called *central idempotents*.

Another related object we have to mention is the generalized differential identities on (semi-)prime rings. A generalized differential polynomial over $\mathcal{U}$ means a generalized polynomial with coefficients in $\mathcal{U}$ and with noncommutative variables involving generalized derivations. A generalized differential identity for some subset of $\mathcal{U}$ is a generalized differential polynomial satisfied by the given subset. Obviously, the definition of a generalized differential polynomial (or identity) is a common generalization of the definition of a differential polynomial (or identity). We are ready to state the first main result of this paper.

**Theorem 3.1.** *Let $n$ be a fixed positive integer, $\mathcal{R}$ be a $2n!$-torsion free prime ring and $\mu, \nu$ be a pair of generalized derivations on $\mathcal{R}$. If $\langle \mu^2(x)+\nu(x), x^n\rangle = 0$ for all $x \in \mathcal{R}$, then $\mu$ and $\nu$ are either left multipliers or right multipliers.*

*Proof.* By assumption we have

$$\langle \mu^2(x) + \nu(x), x^n\rangle = 0 \tag{1}$$

for all $x \in \mathcal{R}$. Substituting $x + \lambda y$ for $x$ in (1) yields that

$$\lambda P_1(x, y) + \lambda^2 P_2(x, y) + \cdots + \lambda^n P_n(x, y) = 0,$$

where $\lambda \in \mathbb{Z}$, $x, y \in \mathcal{R}$, $P_i(x, y)$ denotes the sum of terms involving $i$ factors of $y$ in the expansion of $\langle \mu^2(x + \lambda y) + \nu(x + \lambda y), (x + \lambda y)^n\rangle = 0$. It follows from [5, Lemma 1] that

$$\begin{aligned} P_1(x, y) = {} & \langle \mu^2(y) + \nu(y), x^n\rangle \\ & + \langle \mu^2(x) + \nu(x), x^{n-1}y + x^{n-2}yx + x^{n-3}yx^2 + \cdots + yx^{n-1}\rangle = 0 \end{aligned}$$

for all $x, y \in \mathcal{R}$. It is well known that $\mathcal{R}$ and $\mathcal{U}$ satisfy the same differential identities [11, Theorem 2] and hence satisfy the same generalized differential identities. Thus

$$\langle \mu^2(y)+\nu(y), x^n\rangle + \langle \mu^2(x)+\nu(x), x^{n-1}y+x^{n-2}yx+x^{n-3}yx^2+\cdots+yx^{n-1}\rangle = 0 \tag{2}$$

for all $x, y \in \mathcal{U}$. Note that $\mathcal{U}$ has the identity element $e$. Taking $x = e$ in (1), we obtain

$$\mu^2(e) + \nu(e) = 0,$$

since $\mathcal{U}$ is also $2n!$-torsion free. Taking $y = e$ in (2) and using the relation $\mu^2(e) + \nu(e) = 0$, we have

$$n\langle \mu^2(x) + \nu(x), x^{n-1}\rangle = 0$$

for all $x \in \mathcal{U}$. Since $\mathcal{U}$ is also $2n!$-torsion free,

$$\langle \mu^2(x) + \nu(x), x^{n-1}\rangle = 0$$

for all $x \in \mathcal{U}$. Continuing this process, we assert that

$$\langle \mu^2(x) + \nu(x), x\rangle = 0$$

for all $x \in \mathcal{U}$. Applying [3, Theorem 1] yields that

$$\mu^2(x) + \nu(x) = 0 \tag{3}$$

for all $x \in \mathcal{U}$. The relation (3) implies that $\mu^2$ is a generalized derivation on $\mathcal{U}$ and hence

$$\mu^2(xy) = \mu^2(x)y + xd_1(y) \tag{4}$$

for all $x, y \in \mathcal{U}$, where $d_1$ is the associated derivation of $\mu^2$. On the other hand

$$\mu^2(xy) = \mu(\mu(x)y + xd_2(y)) = \mu^2(x)y + 2\mu(x)d_2(y) + xd_2^2(y) \tag{5}$$

for all $x, y \in \mathcal{U}$, where $d_2$ is the associated derivation of $\mu$. It follows from (4) and (5) that

$$xd_1(y) = 2\mu(x)d_2(y) + xd_2^2(y) \tag{6}$$

for all $x, y \in \mathcal{U}$. Taking $x = e$ in (6), we get

$$d_1(y) = 2\mu(e)d_2(y) + d_2^2(y) \tag{7}$$

for all $y \in \mathcal{U}$. Substituting $yx$ for $y$ in (7) produces

$$d_1(y)x + yd_1(x) = 2\mu(e)d_2(y)x + 2\mu(e)yd_2(x) + d_2^2(y)x + 2d_2(y)d_2(x) + yd_2^2(x)$$

for all $x, y \in \mathcal{U}$. Right multiplication of (7) by $x$ leads to

$$d_1(y)x = 2\mu(e)d_2(y)x + d_2^2(y)x \tag{8}$$

for all $x, y \in \mathcal{U}$. Subtracting (8) from (7) we have

$$yd_1(x) = 2\mu(e)yd_2(x) + 2d_2(y)d_2(x) + yd_2^2(x) \tag{9}$$

for all $x, y \in \mathcal{U}$. Combining (9) with (7) it is easy to see that

$$\mu(e)xd_2(y) + d_2(x)d_2(y) - x\mu(e)d_2(y) = 0$$

for all $x, y \in \mathcal{U}$. By [14, Lemma 1], we obtain

$$d_2(y) = 0 \text{ or } d_2(x) = [x, \mu(e)]$$

for all $x, y \in \mathcal{U}$. If $d_2(y) = 0$, then $\mu, \mu^2$ and $\nu$ are both left multipliers by the relation (5) and (3). If $d_2(x) = [x, \mu(e)]$, then

$$\mu(x) = \mu(e)x + d_2(x) = x\mu(e)$$

for all $x \in \mathcal{U}$. It is easy to check that $\mu, \mu^2$ and $\nu$ are both right multipliers. This theorem is completed. □

As consequences of Theorem 3.1, we immediately get.

**Corollary 3.2.** *Let $n$ be a fixed positive integer, $\mathcal{R}$ be a $2n!$-torsion free prime ring and $\mu$ be a generalized derivation on $\mathcal{R}$. If $\mu(x)x^n + x^n\mu(x) = 0$ for all $x \in \mathcal{R}$, then $\mu = 0$.*

**Corollary 3.3.** *Let $n$ be a fixed positive integer, $\mathcal{R}$ be a $2n!$-torsion free prime ring and $d, g$ be a pair of derivations on $\mathcal{R}$. If $\langle d^2(x) + g(x), x^n\rangle = 0$ for all $x \in \mathcal{R}$, then $d = g = 0$.*

Furthermore, Corollary 3.3 can be also extended to the following more general form.

**Theorem 3.4.** *Let $n$ be a fixed positive integer, $\mathcal{R}$ be a noncommutative $2n!$-torsion free prime ring and $d, g$ be a pair of derivations on $\mathcal{R}$. If $\langle d^2(x) + g(x), x^n\rangle \in \mathcal{C}_{\mathcal{R}}$ for all $x \in \mathcal{R}$, then $d = g = 0$.*

*Proof.* By assumption we have

$$[\langle d^2(x) + g(x), x^n\rangle, z] = 0 \tag{10}$$

for all $x, z \in \mathcal{R}$. Substituting $x + \lambda y$ for $x$ in (10) yields that

$$\lambda P_1(x, y, z) + \lambda^2 P_2(x, y, z) + \cdots + \lambda^n P_n(x, y, z) = 0,$$

where $\lambda \in \mathbb{Z}$, $x, y, z \in \mathcal{R}$, $P_i(x, y, z)$ denotes the sum of terms involving $i$ factors of $y$ in the expansion of $[\langle d^2(x + \lambda y) + g(x + \lambda y), (x + \lambda y)^n\rangle, z] = 0$. It follows from [5, Lemma 1] that

$$\begin{aligned} &P_1(x, y, z) \\ &= [\langle d^2(y) + g(y), x^n\rangle + \langle d^2(x) + g(x), x^{n-1}y + x^{n-2}yx + x^{n-3}yx^2 + \cdots + yx^{n-1}\rangle, z] = 0 \end{aligned}$$

for all $x, y, z \in \mathcal{R}$. This shows that

$$\langle d^2(y)+g(y), x^n\rangle+\langle d^2(x)+g(x), x^{n-1}y+x^{n-2}yx+x^{n-3}yx^2+\cdots+yx^{n-1}\rangle \in \mathcal{C}_{\mathcal{R}}$$

for all $x, y \in \mathcal{R}$. It is well known that $\mathcal{R}$ and $\mathcal{U}$ satisfy the same differential identities [11, Theorem 2]. Therefore

$$\langle d^2(y)+g(y), x^n\rangle+\langle d^2(x)+g(x), x^{n-1}y+x^{n-2}yx+x^{n-3}yx^2+\cdots+yx^{n-1}\rangle \in \mathcal{C}_{\mathcal{R}} \tag{11}$$

for all $x, y \in \mathcal{U}$. Note that $\mathcal{U}$ has the identity element $e$. Taking $y = e$ in (11) and considering the fact that $d(e) = g(e) = 0$ immediately get

$$n\langle d^2(x) + g(x), x^{n-1}\rangle \in \mathcal{C}_{\mathcal{R}}$$

for all $x \in \mathcal{U}$. Thus

$$n\langle d^2(x) + g(x), x^{n-1}\rangle \in \mathcal{C}_{\mathcal{R}}$$

for all $x \in \mathcal{R}$. Since $\mathcal{U}$ is also $2n!$-torsion free,

$$\langle d^2(x) + g(x), x^{n-1}\rangle \in \mathcal{C}_{\mathcal{R}}$$

for all $x \in \mathcal{R}$. Continuing this process, we ultimately get that

$$2(d^2(x) + g(x)) \in \mathcal{C}_{\mathcal{R}}$$

for all $x \in \mathcal{R}$. This implies that

$$[d^2(x) + g(x), x] = 0$$

for all $x \in \mathcal{R}$. Applying [20, Theorem 1] yields that $d = g = 0$. □

We next use the orthogonal completeness method to extend Theorem 3.4 to the case of semiprime rings.

**Theorem 3.5.** *Let $n$ be a fixed positive integer, $\mathcal{R}$ be a noncommutative $2n!$-torsion free semiprime ring and $d, g$ be a pair of derivations on $\mathcal{R}$. If $\langle d^2(x) + g(x), x^n\rangle \in \mathcal{C}_{\mathcal{R}}$ for all $x \in \mathcal{R}$, then $d$ and $g$ both map $\mathcal{R}$ into $\mathcal{C}_{\mathcal{R}}$.*

*Proof.* Let $\mathcal{B}$ be the complete Boolean algebra of $\mathcal{E}$. We choose a maximal ideal $\mathcal{M}$ of $\mathcal{B}$. According to [2], $\mathcal{M}\mathcal{U}$ is a prime ideal of $\mathcal{U}$, which is invariant under any derivation of $\mathcal{U}$. It was well known that the pair of derivations $d, g$ on $\mathcal{R}$ can be uniquely extended to be a pair of derivations on $\mathcal{U}$. Let $\overline{d}, \overline{g}$ be the canonical pair of derivations on $\overline{\mathcal{U}} = \mathcal{U}/\mathcal{M}\mathcal{U}$ induced by $d, g$, respectively. The assumption implies that

$$[\langle d^2(x) + g(x), x^n\rangle, z] = 0$$

for all $x, z \in \mathcal{R}$. It follows from [11, Theorem 2] that $\mathcal{R}$ and $\mathcal{U}$ satisfy the same differential identities. Thus

$$[\langle d^2(x) + g(x), x^n\rangle, z] = 0$$

for all $x, z \in \mathcal{U}$. Furthermore,

$$[\langle \overline{d}^2(\overline{x}) + \overline{g}(\overline{x}), \overline{x}^n\rangle, \overline{z}] = 0$$

for all $\overline{x}, \overline{z} \in \overline{\mathcal{U}}$. By Theorem 3.4, we know that either $\overline{d}(\overline{x}) = 0$ and $\overline{g}(\overline{x}) = 0$ or $[\overline{\mathcal{U}}, \overline{\mathcal{U}}] = 0$. In any case we both have

$$d(\mathcal{U})[\mathcal{U}, \mathcal{U}] \in \mathcal{M}\mathcal{U}$$

and

$$g(\mathcal{U})[\mathcal{U}, \mathcal{U}] \in \mathcal{M}\mathcal{U}$$

for all $\mathcal{M}$. Note that $\bigcap\{\mathcal{M}\mathcal{U} \mid \mathcal{M} \text{ is any maximal ideal of } \mathcal{B}\} = 0$. Hence $d(\mathcal{U})[\mathcal{U}, \mathcal{U}] = 0$ and $g(\mathcal{U})[\mathcal{U}, \mathcal{U}] = 0$. In particular, $d(\mathcal{R})[\mathcal{R}, \mathcal{R}] = 0$ and $g(\mathcal{R})[\mathcal{R}, \mathcal{R}] = 0$. These imply that

$$0 = d(\mathcal{R})[\mathcal{R}^2, \mathcal{R}] = d(\mathcal{R})\mathcal{R}[\mathcal{R}, \mathcal{R}] + d(\mathcal{R})[\mathcal{R}, \mathcal{R}]\mathcal{R} = d(\mathcal{R})\mathcal{R}[\mathcal{R}, \mathcal{R}]$$

and

$$0 = g(\mathcal{R})[\mathcal{R}^2, \mathcal{R}] = g(\mathcal{R})\mathcal{R}[\mathcal{R}, \mathcal{R}] + g(\mathcal{R})[\mathcal{R}, \mathcal{R}]\mathcal{R} = g(\mathcal{R})\mathcal{R}[\mathcal{R}, \mathcal{R}].$$

Therefore $[\mathcal{R}, d(\mathcal{R})]\mathcal{R}[\mathcal{R}, d(\mathcal{R})] = 0$ and $[\mathcal{R}, g(\mathcal{R})]\mathcal{R}[\mathcal{R}, g(\mathcal{R})] = 0$. By semiprimeness of $\mathcal{R}$ we obtain that $[\mathcal{R}, d(\mathcal{R})] = 0$ and $[\mathcal{R}, g(\mathcal{R})] = 0$. These show that $d(\mathcal{R}) \in \mathcal{C}_{\mathcal{R}}$ and $g(\mathcal{R}) \in \mathcal{C}_{\mathcal{R}}$. □

## 4. Pair of (generalized-)derivations on Banach algebras

In this section we will study the images of pair of (generalized-, Jordan-) derivations on Banach algebras and discuss some open problems with related to the well known noncommutative Singer-Wermer conjecture from the point of view of ring theory.

**Theorem 4.1.** *Let $n$ be a fixed positive integer, $\mathcal{A}$ be a unital Banach algebra and $\mu$ be a continuous generalized derivations on $\mathcal{A}$. If $\mu(x)x^n + x^n\mu(x) \in rad(\mathcal{A})$ for all $x \in \mathcal{A}$, then $\mu(\mathcal{A}) \subseteq rad(\mathcal{A})$.*

*Proof.* Let $\mathcal{P}$ be any primitive ideal of $\mathcal{A}$. Since $\mu$ is continuous, $\mu(\mathcal{P}) \subseteq \mathcal{P}$ by the similar argument of [15, Lemma 3.2]. Thus $\mu$ can be induced to a generalized derivation of quotient Banach algebra $\mathcal{A}/\mathcal{P}$ as follows

$$\tilde{\mu}(\tilde{x}) = \mu(x) + \mathcal{P}$$

for all $\tilde{x} \in \mathcal{A}/\mathcal{P}$ and $x \in \mathcal{A}$. Since $\mathcal{P}$ is a primitive ideal, the quotient Banach algebra $\mathcal{A}/\mathcal{P}$ is prime and semisimple. The assumption of the theorem implies that

$$\tilde{\mu}(\tilde{x})\tilde{x}^n + \tilde{x}^n\tilde{\mu}(\tilde{x}) = \tilde{0}$$

for all $\tilde{x} \in \mathcal{A}/\mathcal{P}$ and $x \in \mathcal{A}$. Note that Corollary 3.2 holds for both the case of commutative and the case of noncommutative. In any case $\tilde{\mu} = 0$ and hence $\mu(\mathcal{A}) \subseteq \mathcal{P}$. Since $\mathcal{P}$ is arbitrary, $\mu(\mathcal{A}) \subseteq rad(\mathcal{A})$. □

**Corollary 4.2.** *Let $n$ be a fixed positive integer, $\mathcal{A}$ be a semisimple Banach algebra and $\mu$ be a generalized derivation on $\mathcal{A}$. If $\mu(x)x^n + x^n\mu(x) \in rad(\mathcal{A})$ for all $x \in \mathcal{A}$, then $\mu = 0$.*

**Lemma 4.3** ([19, Lemma 1.2])**.** *Let $d$ be a derivation on Banach algebra $\mathcal{A}$ and $\mathcal{J}$ be a primitive ideal of $\mathcal{A}$. If there exists a real constant $k > 0$ such that $\|Q_{\mathcal{J}}d^n\| \leq k^n$ for all $n \in \mathbb{N}$, then $d(\mathcal{J}) \subseteq \mathcal{J}$.*

Now we give the main result of this section.

**Theorem 4.4.** *Let $n$ be a fixed positive integer, $\mathcal{A}$ be a Banach algebra and $d, g$ be a pair of derivations on $\mathcal{A}$. If $\langle d^2(x) + g(x), x^n\rangle \in \mathcal{C}_{\mathcal{A}}$ for all $x \in \mathcal{A}$, then $d(\mathcal{A}) \subseteq rad(\mathcal{A})$ and $g(\mathcal{A}) \subseteq rad(\mathcal{A})$.*

*Proof.* Let $\mathcal{J}$ be any primitive ideal of $\mathcal{A}$. By Zorn's lemma, there exists a minimal prime ideal $\mathcal{P}$ of $\mathcal{A}$ contained in $\mathcal{J}$ such that $d(\mathcal{P}) \subseteq \mathcal{P}$ and $g(\mathcal{P}) \subseteq \mathcal{P}$

by [12, Lemma 1]. If $\mathcal{P}$ is closed, then the pair of derivations $d$ and $g$ can be induced to a pair of derivations on the Banach algebra $\mathcal{A}/\mathcal{P}$ as follows

$$\tilde{d}(\tilde{x}) = d(x) + \mathcal{P}, \quad \tilde{g}(\tilde{x}) = g(x) + \mathcal{P}$$

for all $\tilde{x} \in \mathcal{A}/\mathcal{P}$ and $x \in \mathcal{A}$. If $\mathcal{A}/\mathcal{P}$ is commutative, both $\tilde{d}(\mathcal{A}/\mathcal{P})$ and $\tilde{g}(\mathcal{A}/\mathcal{P})$ are contained in the Jacobson radical of $\mathcal{A}/\mathcal{P}$ by [18, Theorem 4.4]. If $\mathcal{A}/\mathcal{P}$ is noncommutative, by the assumption we have

$$[\langle \tilde{d}^2(\tilde{x}) + \tilde{g}(\tilde{x}), \tilde{x}^n \rangle, \tilde{z}] = \tilde{0}$$

for all $\tilde{x}, \tilde{z} \in \mathcal{A}/\mathcal{P}$ and $x, z \in \mathcal{A}$. By the primeness of $\mathcal{A}/\mathcal{P}$ and Theorem 3.4, it follows that $\tilde{d} = \tilde{0}$ and $\tilde{g} = \tilde{0}$ on $\mathcal{A}/\mathcal{P}$. In any case, we get both $d(\mathcal{A}) \subseteq \mathcal{J}$ and $g(\mathcal{A}) \subseteq \mathcal{J}$. If $\mathcal{P}$ is not closed, then $\mathcal{S}(d) \subseteq \mathcal{P}$ by [6, Lemma 2.3], where $\mathcal{S}(d)$ is the separating space of linear operator $d$. By [16, Lemma 1.3], we have $\mathcal{S}(Q_{\hat{\mathcal{P}}} d) = \widehat{Q_{\hat{\mathcal{P}}}(\mathcal{S}(d))} = 0$ whence $Q_{\hat{\mathcal{P}}} d$ is continuous on $\mathcal{A}$. This implies that $Q_{\hat{\mathcal{P}}} d(\hat{\mathcal{P}}) = 0$ on $\mathcal{A}/\mathcal{P}$ and hence $d(\hat{\mathcal{P}}) \subseteq \hat{\mathcal{P}}$. Thus $d$ can be induced to a derivation on the Banach algebra $\mathcal{A}/\hat{\mathcal{P}}$ as follows

$$\tilde{d}(\tilde{x}) = d(x) + \hat{\mathcal{P}}$$

for all $\tilde{x} \in \mathcal{A}/\hat{\mathcal{P}}$ and $x \in \mathcal{A}$. Let us define the following mapping

$$\xi \tilde{d}^n Q_{\hat{\mathcal{P}}} : \mathcal{A} \longrightarrow \mathcal{A}/\hat{\mathcal{P}} \longrightarrow \mathcal{A}/\hat{\mathcal{P}} \longrightarrow \mathcal{A}/\mathcal{J}$$

through $\xi \tilde{d}^n Q_{\hat{\mathcal{P}}}(x) = Q_{\mathcal{J}} d^n(x)$ for all $x \in \mathcal{A}$ and $n \in \mathbb{N}$, where $\xi$ is the canonical inclusion mapping from $\mathcal{A}/\hat{\mathcal{P}}$ onto $\mathcal{A}/\mathcal{J}$ and $\xi$ indeed exists since $\hat{\mathcal{P}} \subseteq \mathcal{J}$. By [16, Lemma 1.4], we assert that $\tilde{d}$ is continuous on $\mathcal{A}/\hat{\mathcal{P}}$ and hence that $\|Q_{\mathcal{J}} d^n\| \leq \|\tilde{d}\|^n$ for all $n \in \mathbb{N}$. Applying Lemma 4.3 yields that $d(\mathcal{J}) \subseteq \mathcal{J}$. Using the same argument with $g$, we also get that $g(\mathcal{J}) \subseteq \mathcal{J}$. Then the pair of derivations $d$ and $g$ can be induced to a pair of derivations on the Banach algebra $\mathcal{A}/\mathcal{J}$ as follows

$$\tilde{d}(\tilde{x}) = d(x) + \mathcal{J}, \quad \tilde{g}(\tilde{x}) = g(x) + \mathcal{J}$$

for all $\tilde{x} \in \mathcal{A}/\mathcal{J}$ and $x \in \mathcal{A}$. The remainder follows the similar argument to the case of when $\mathcal{P}$ is closed since the primitive algebra $\mathcal{A}/\mathcal{J}$ is prime. Therefore we show that $d(\mathcal{A}) \subseteq \mathcal{J}$ and $g(\mathcal{A}) \subseteq \mathcal{J}$. So $d(\mathcal{A}) \subseteq \mathcal{J}$ and $g(\mathcal{A}) \subseteq \mathcal{J}$ for every primitive ideal $\mathcal{J}$. These imply that $d(\mathcal{A}) \subseteq rad(\mathcal{A})$ and $g(\mathcal{A}) \subseteq rad(\mathcal{A})$. □

As a consequence of Theorem 4.4, we immediately get.

**Corollary 4.5.** *Let $n$ be a fixed positive integer, $\mathcal{A}$ be a semisimple Banach algebra and $d, g$ be a pair of derivations on $A$. If $\langle d^2(x) + g(x), x^n \rangle \in \mathcal{C}_{\mathcal{A}}$ for all $x \in \mathcal{A}$, then $d = 0$ and $g = 0$.*

Let us see the pair of Jordan derivations on a Banach algebra.

**Theorem 4.6.** *Let $n$ be a fixed positive integer, $\mathcal{A}$ be a Banach algebra and $d, g$ be a pair of continuous Jordan derivations on $\mathcal{A}$. If $\langle d^2(x) + g(x), x^n \rangle \in rad(\mathcal{A})$ for all $x \in \mathcal{A}$, then $d(\mathcal{A}) \subseteq rad(\mathcal{A})$ and $g(\mathcal{A}) \subseteq rad(\mathcal{A})$.*

*Proof.* Let $\mathcal{P}$ be any primitive ideal of $\mathcal{A}$. Since $d$ and $g$ are both continuous, $d(\mathcal{P}) \subseteq \mathcal{P}$ and $g(\mathcal{P}) \subseteq \mathcal{P}$ by [15, Lemma 3.2]. Then $d$ and $g$ can be induced to a pair of Jordan derivations on the Banach algebra $\mathcal{A}/\mathcal{P}$ as follows

$$\tilde{d}(\tilde{x}) = d(x) + \mathcal{P}, \quad \tilde{g}(\tilde{x}) = g(x) + \mathcal{P}$$

for all $\tilde{x} \in \mathcal{A}/\mathcal{P}$ and $x \in \mathcal{A}$. Since $\mathcal{P}$ is a primitive ideal of $\mathcal{A}$, the quotient algebra $\mathcal{A}/\mathcal{P}$ is prime and semisimple. On the other hand, we should remark that the pair of Jordan derivations $\tilde{d}$ and $\tilde{g}$ on $\mathcal{A}/\mathcal{P}$ are also a pair of derivations on $\mathcal{A}/\mathcal{P}$ by Brešar's theorem. It is well known that every derivation on a semisimple Banach algebra is continuous. Combing this result with the well known Singer-Wermer theorem, we know that there are no nonzero derivations on a commutative semisimple Banach algebra. Hence we have $\tilde{d} = 0$ and $\tilde{g} = 0$ when $\mathcal{A}/\mathcal{P}$ is commutative. It remains to show that $\tilde{d} = 0$ and $\tilde{g} = 0$ in the case of when $\mathcal{A}/\mathcal{P}$ is noncommutative. The assumption of the theorem leads to

$$\langle \tilde{d}^2(\tilde{x}) + \tilde{g}(\tilde{x}), \tilde{x}^n \rangle = \tilde{0}$$

for all $\tilde{x} \in \mathcal{A}/\mathcal{P}$ and $x \in \mathcal{A}$. It follows from Theorem 3.4 that $\tilde{d} = 0$ and $\tilde{g} = 0$. In any case both $\tilde{d} = 0$ and $\tilde{g} = 0$. These imply that $d(\mathcal{A}) \subseteq \mathcal{P}$ and $g(\mathcal{A}) \subseteq \mathcal{P}$ for arbitrary primitive ideal $\mathcal{P}$ of $\mathcal{A}$ and hence $d(\mathcal{A}) \subseteq rad(\mathcal{A})$ and $g(\mathcal{A}) \subseteq rad(\mathcal{A})$. ☐

**Corollary 4.7.** *Let $n$ be a fixed positive integer, $\mathcal{A}$ be a semisimple Banach algebra and $d, g$ be a pair of Jordan derivations on $\mathcal{A}$. If $\langle d^2(x) + g(x), x^n \rangle \in rad(\mathcal{A})$ for all $x \in \mathcal{A}$, then $d = 0$ and $g = 0$.*

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