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PATH-CONNECTED AND NON PATH-CONNECTED ORTHOMODULAR LATTICES

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Abstract. A block of an orthomodular lattice L is a maximal Boolean subalgebra of L. A site is a subalgebra of an orthomodular lattice L of the form $S = A \cap B$, where A and B are distinct blocks of L. An orthomodular lattice L is called with finite sites if $|A \cap B| < \infty$ for all distinct blocks A, B of L. We prove that there exists a weakly path-connected orthomodular lattice with finite sites which is not path-connected and if L is an orthomodular lattice such that the height of the join-semilattice $[Com\, L]_\vee$ generated by the commutators of L is finite, then L is path-connected.

1. Introduction

A path of an orthomodular lattice was defined by Bruns [1] and has been studied by several authors. We have the following classes of path-connected orthomodular lattices: every block-finite orthomodular lattice is path-connected [1], and every commutator-finite orthomodular lattice is path-connected [2], and every vertex-finite orthomodular lattice is path-connected [7]. We study some conditions such that an irreducible orthomodular lattice is to be simple, and some properties of paths of an orthomodular lattice has been used to prove that every block-finite irreducible orthomodular lattice is simple [9], and every vertex-finite irreducible orthomodular lattice is simple [7]. In this paper, we extend these results, and find some path-connected orthomodular lattices and some properties of path-connected orthomodular lattices.

An orthomodular poset (abbreviated by OMP) is a partially ordered set P which satisfies *the orthomodular law*: if $x \leq y$, then $y = x \vee (x' \wedge y)$ [6]. *An orthomodular lattice* (abbreviated by OML) is an ortholattice L which satisfies the orthomodular law [6]. *A Boolean algebra B* is an ortholattice satisfying *the distributive law*: $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)\ \forall x, y, z \in B$.

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A subalgebra of an OML L is a nonempty subset M of L which is closed under the operations $\vee$, $\wedge$ and $'$. We write $M \leq L$ if M is a subalgebra of L. If $M \leq L$ and $a, b \in M$ with $a \leq b$, then *the relative interval sublattice* $M[a,b] = \{x \in M \mid a \leq x \leq b\}$ is an OML with *the relative orthocomplementation* $\sharp$ on $M[a,b]$ given by $c^\sharp = (a \vee c') \wedge b = a \vee (c' \wedge b) \quad \forall c \in M[a,b]$. In particular, $L[a,b]$ will be denoted by $[a,b]$ if there is no ambiguity.

The commutator of a and b of an OML L is denoted by $a{*}b$, and is defined by $a{*}b = (a \vee b) \wedge (a \vee b') \wedge (a' \vee b) \wedge (a' \vee b')$. The set of all commutators of L is denoted by $ComL$ and L is said to be *commutator-finite* if $|ComL|$ is finite. For elements a, b of an OML, we say *a commutes with* b, in symbols $a \,\mathbf{C}\, b$, if $a * b = 0$. If M is a subset of an OML L, the set $\mathbf{C}(M) = \{x \in L \,|\, x \,\mathbf{C}\, m \quad \forall m \in M\}$ is called *the commutant* of M in L and the set $\mathbf{C}en(M) = \mathbf{C}(M) \cap M$ is called *the center* of M. The set $\mathbf{C}(L)$ is called the center of L and then $\mathbf{C}(L) = \bigcap \{\mathbf{C}(a) \,|\, a \in L\}$. An OML L is called *irreducible* if $\mathbf{C}(L) = \{0, 1\}$, and L is called *reducible* if it is not irreducible.

A block of an OML L is a maximal Boolean subalgebra of L. The set of all blocks of L is denoted by $\mathfrak{A}_L$. Note that $\bigcup \mathfrak{A}_L = L$ and $\bigcap \mathfrak{A}_L = \mathbf{C}(L)$. An OML L is said to be *block-finite* if $|\mathfrak{A}_L|$ is finite.

For any e in an OML L, the subalgebra $S_e = [0, e'] \cup [e, 1]$ is called *the* (*principal*) *section generated by e*. Note that for $A, B \in \mathfrak{A}_L$, if $e \in A \cap B$ and $A \cap B = S_e \cap (A \cup B)$, then $A \cap B = S_e \cap A = S_e \cap B$ [1].

Definition 1.1. For blocks A, B of an OML L define $A \overset{wk}{\sim} B$ if and only if $A \cap B = S_e \cap (A \cup B)$ for some $e \in A \cap B$; $A \sim B$ if and only if $A \neq B$ and $A \cup B \leq L$; $A \approx B$ if and only if $A \sim B$ and $A \cap B \neq \mathbf{C}(L)$.

A path in L is a finite sequence $B_0, B_1, \ldots, B_n$ $(n \geq 0)$ in $\mathfrak{A}_L$ satisfying $B_i \sim B_{i+1}$ whenever $0 \leq i < n$. The path is said to *join* the blocks B_0 and B_n. The number n is said to be *the length* of the path. A path is said to be *proper* if and only if $n = 1$ or $B_i \approx B_{i+1}$ holds whenever $0 \leq i < n$. A path is called to be *strictly proper* if and only if $B_i \approx B_{i+1}$ holds whenever $0 \leq i < n$ [1].

Let A, B be two blocks of an OML L. If $A \sim B$ holds, then there exists a unique element $e \in A \cap B$ satisfying $A \cap B = (A \cup B) \cap S_e$ [1]. Using this element e, we say that A and B are *linked at e* (*strongly linked at e*) if $A \sim B$ $(A \approx B)$, and use the notation $A \sim_e B$ $(A \approx_e B)$. The element e is called *a vertex* of L and it is the commutator of any $x \in A \setminus B$ and $y \in B \setminus A$ [1]. The set of all vertices of L is denoted by V_L and L is said to be *vertex-finite* if $|V_L|$ is finite.

Note that $A \approx B$ implies $A \sim B$, and $A \sim B$ implies $A \overset{wk}{\sim} B$. Some authors, for example Greechie, use the phrase "*A and B meet in the section S_e*" to describe $A \overset{wk}{\sim} B$ [4].

Definition 1.2. Let L be an OML, and $A, B \in \mathfrak{A}_L$. We will say that A and B are *path-connected in L*, *strictly path-connected in L* if A and B are joined by a proper path, a strictly proper path, respectively. An OML S is not path-connected if there exist two blocks in L which are not path-connected. An OML L is *path-connected in L, strictly path-connected in* L if any two blocks in L are joined by a proper path, a strictly proper path, respectively. An OML L is called *relatively path-connected* if and only if each $[0,x]$ is path-connected for all $x \in L$.

Let L be an OML, and $A, B, C \in \mathfrak{A}_L$. If A and B are joined with a strictly proper path $A = B_0 \approx B_1 \approx \cdots \approx B_{m-1} \approx B_m = B$ and if B and C are joined with a strictly proper path $B = C_0 \approx C_1 \approx \cdots \approx C_{n-1} \approx C_n = C$ then A and C are strictly path-connected by *the concatenated path* $A = B_0 \approx B_1 \approx \cdots \approx B_{m-1} \approx B \approx C_1 \approx \cdots \approx C_{n-1} \approx C_n = C$.

The following lemma is well known [1].

Lemma 1.3. *If* L_1, L_2 *are OMLs,* $L = L_1 \times L_2$, $A, B \in \mathfrak{A}_{L_1}$ *and* $C, D \in \mathfrak{A}_{L_2}$, *then* $A \times C \sim B \times D$ *holds in* L *if and only if either* $A = B$ *and* $C \sim D$ *or* $A \sim B$ *and* $C = D$. *If* A *and* B *are linked at* a *then* $A \times C$ *and* $B \times C$ *are linked at* $(a, 0)$. *If* C *and* D *are linked at* c *then* $A \times C$ *and* $A \times D$ *are linked at* $(0, c)$.

The following four theorems are well known [7].

Theorem 1.4. *Let* L *be an OML, and* $x \in L$. *Then* $\mathbf{C}(x)$ *is path-connected if and only if* $[0,x]$ *and* $[0,x']$ *are path-connected.*

Proof. We know that $\mathbf{C}(x) = [0,x] \oplus [0,x']$. First, if $[0,x]$ and $[0,x']$ are path-connected, then $\mathbf{C}(x)$ is path-connected by Lemma 1.3. Conversely, assume that $\mathbf{C}(x)$ is path-connected and let us prove that $[0,x]$ and $[0,x']$ are path-connected. It is sufficient to show that $[0,x]$ is path-connected by symmetry. Let A, B be distinct blocks in $[0,x]$ and let $D \in \mathfrak{A}_{[0,x']}$. We may assume that $A \cup B \not\leq [0,x]$, otherwise A and B are path-connected in $[0,x]$. Then $A \oplus D$ and $B \oplus D$ are blocks in $\mathbf{C}(x)$ and hence path-connected in $\mathbf{C}(x)$. Let $A \oplus D = C_0 \oplus E_0 \sim C_1 \oplus E_1 \sim \cdots \sim C_n \oplus E_n = B \oplus D$ $(n \geq 2)$ be a path joining $A \oplus D$ and $B \oplus D$ in $\mathbf{C}(x)$ where $C_i \in \mathfrak{A}_{[0,x]}$ and $E_i \in \mathfrak{A}_{[0,x']}$ $\forall (0 \leq i \leq n)$. Then the sequence $C_0, C_1, \ldots, C_n$ satisfies $C_i \sim C_{i+1}$ in $[0,x]$ or $C_i = C_{i+1}$ by Lemma 1.3. Let $M = \{i \,|\, C_i \sim C_{i+1},\ 1 \leq i \leq n-1\}$. Then $A = C_0 \sim C_{i_1} \sim \cdots \sim C_{i_k} \sim C_n = B$ where $i_j \in M$ such that $0 = i_0 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq n-1$. Thus A and B are path-connected in $[0,x]$, and hence $[0,x]$ is path-connected. This completes the proof. ☐

Theorem 1.5. *Every finite direct product of path-connected orthomodular lattices is path-connected.*

Theorem 1.6. *Every infinite direct product of path-connected OMLs containing infinitely many non-Boolean factors is not path-connected.*

Theorem 1.7. *Let* L *be an OML. Then the following are equivalent*:

(1) L *is relatively path-connected*;

(2) $\mathbf{C}(x)$ *is path-connected* $\forall x \in L$;
(3) S_x *is path-connected* $\forall x \in L$.

2. Path-connected orthomodular lattices

A site is a subalgebra of an OML L of the form $S = A \cap B$, where A and B are distinct blocks of L. An OML L is called *with uniformly finite sites* if there exists a natural number n such that for all distinct blocks A, B of L, $|A \cap B| < n$. An OML L is called *with finite sites* if for all distinct blocks A, B of L, $|A \cap B| < \infty$.

Using the pasting suggested by Greechie [4] and the inductive limit introduced by Dacey [3], we will present an OML with finite sites which is not path-connected.

A sublattice M of an OML L is said to be *a suborthomodular lattice* of L in case the restriction of the orthocomplementation on L makes M an OML. A suborthomodular lattice M of an OML L is called *subcomplete* in case $N \subset M$ and $\bigvee N$ exists as computed in L implies $\bigvee N$ is in M.

In what follows we assume that $(L_1, \leq_1, ^\sharp)$ and $(L_2, \leq_2, ^+)$ are two disjoint OMLs, that S^i is a proper suborthomodular lattice of L_i $(i = 1, 2)$, and that there exists an orthoisomorphism $\theta : S^1 \to S^2$.

Definition 2.1. (1) Let $L_0 = L_1 \cup L_2$.
(2) Let $P_1 = \{(x, y) \in L_0 \times L_0 \ : \ y = x\theta\}$.
(3) Let $\Delta = \{(x, x) \ : \ x \in L_0\}$.
(4) Let P be the equivalence relation defined by $P = \Delta \cup P_1 \cup {P_1}^{-1}$, where ${P_1}^{-1} = \{(y, x) \ : \ (x, y) \in P_1\}$.
(5) Let $L = L_0/P$.
(6) For $i = 1, 2$, let $R = \{([x], [y]) \in L \times L : \quad$ there exist $x_i \in [x]$ and $y_i \in [y]$ such that $x_i <_i y_i\}$;
(7) Let $\leq$ be the relation $(R_1 \cup R_2)^2$.
(8) Define $[0]$ to be $[0_1]$ and $[1]$ to be $[1_1]$, where 0_1 and 1_1 are the zero and unit elements of L_1.
(9) Define $' : L \to L$ by the following prescription: for $[x] \in L$,

$$[x]' = \begin{cases} [{x_1}^\sharp], & \text{if there exists } x_1 \in L_1 \text{ such that } x_1 \in [x], \\ [{x_2}^+], & \text{if there exists } x_2 \in L_2 \text{ such that } x_2 \in [x]. \end{cases}$$

(10) Two sections S^1 and S^2 are said to be *corresponding sections* of L_1 and L_2 in case there exists $M_i \subset S^i \subset L_i$ $(i = 1, 2)$ such that $M_1\theta = M_2$ and $S^1 = \bigcup\{S_{m^\sharp} \ : \ m \in M_1\}$ and $S^2 = \bigcup\{S_{m^+} \ : \ m \in M_2\}$.

The following theorem is well known [4].

Theorem 2.2 ([4]). *Let S^1 and S^2 be corresponding sections of L_1 and L_2. Let L_i be complete and let S^i be subcomplete $(i = 1, 2)$. Then L is a complete OML.*

Definition 2.3. An OML L is said to be obtained by *pasting two OMLs* L_1 *and* L_2 *along the sections* S^1 *and* S^2 if and only if all the conditions of Definition 2.1 are satisfied, and we write

$$L = P(L_1, L_2; S^1, S^2; \theta).$$

A poset D is called *directed* if any two elements subset of D has an upper bound in D.

Definition 2.4. *An inductive system* $(\mathcal{A}_\alpha, \phi^\alpha_\beta)_D$ of sets is defined to be a triplet of the following objects:

(1) a directed partially ordered set $(D, \leq)$;
(2) sets $\mathcal{A}_\alpha$ for each $\alpha \in D$;
(3) mappings ϕ^α_β for all $\alpha \leq \beta$, where ϕ^α_β maps $\mathcal{A}_\alpha$ into $\mathcal{A}_\beta$ such that $\phi^\beta_\gamma \phi^\alpha_\beta = \phi^\alpha_\gamma$ for $\alpha \leq \beta \leq \gamma$ and ϕ^α_α is the identity mapping for all $\alpha \in D$.

A limit $(\mathcal{A}, \phi^\alpha)$ of an inductive system (or *an inductive limit*) is a set $\mathcal{A}$ together with mappings $\phi^\alpha : \mathcal{A}_\alpha \to \mathcal{A}$, subject to the following conditions: $\phi^\beta \phi^\alpha_\beta = \phi^\alpha$ for $\alpha \leq \beta$ and, if mapping $\psi_\alpha : \mathcal{A}_\alpha \to \mathcal{B}$ are given with $\psi_\beta \phi^\alpha_\beta = \psi_\alpha$ for $\alpha \leq \beta$, then there exists a unique $\psi : \mathcal{A} \to \mathcal{B}$ such that $\psi_\alpha = \psi \phi^\alpha$ for $\alpha \in D$.

Let $(D, \leq)$ be a directed set. Assume that for each $\alpha \in D$, $\mathcal{A}_\alpha$ is an OMP and for $\alpha \leq \beta$, there is *an ortho-embedding* $\phi^\alpha_\beta : \mathcal{A}_\alpha \to \mathcal{A}_\beta$ such that the family (ϕ^α_β) satisfies $\phi^\alpha_\alpha = Id_\alpha, \phi^\beta_\gamma \circ \phi^\alpha_\beta = \phi^\alpha_\gamma$ for $\alpha \leq \beta \leq \gamma$. Then $(\mathcal{A}_\alpha, \phi^\alpha_\beta)_D$ is an inductive system in the category of OMPs and ortho-embeddings.

Let $(\mathcal{A}_\alpha, \phi^\alpha_\beta)_D$ be a fixed inductive system in *the category* of OMPs and ortho-embeddings. Let $X = \bigcup_{\alpha \in D} \mathcal{A}_\alpha$. Define a relation $\simeq$ on X by: $x \simeq y$, $x \in \mathcal{A}_\alpha$, $y \in \mathcal{A}_\beta$ if there exists $\gamma \in D$ such that $\alpha \leq \gamma$, $\beta \leq \gamma$ and $\phi^\alpha_\gamma(x) = \phi^\beta_\gamma(y)$. Then $\simeq$ is an equivalence relation on X [3]. Let $\overline{x} = \{y \in X | y \simeq x\}$ and let $\mathcal{O} = \{\overline{x} | x \in X\}$. Define an ordering $\leq$ on $\mathcal{O}$ by: $\overline{x} \leq \overline{y}$ if there exist $\alpha \in D$, $x_\alpha \in \overline{x} \cap \mathcal{A}_\alpha$ and $y_\alpha \in \overline{y} \cap \mathcal{A}_\alpha$ such that $x_\alpha \leq y_\alpha$ in $\mathcal{A}_\alpha$. For $\alpha \in D$, let $1 = \overline{1_\alpha}$, $0 = \overline{0_\alpha}$ and define $\overline{x_\alpha}' = \overline{x_\alpha{}'}$. Then $'$ is an orthocomplementation on $\mathcal{O}$ [3].

The following Theorem 2.5 and Corollary 2.6 are well known [3].

Theorem 2.5 ([3])**.** *If* $(\mathcal{A}_\alpha, \phi^\alpha_\beta)_D$ *is an inductive system in the category of OMPs and ortho-embeddings, then its inductive limit exists in the same category and equals* $(\mathcal{O}, \phi^\alpha)$*, where* $\mathcal{O}$ *is in the above, and* $\phi^\alpha : \mathcal{A}_\alpha \to \mathcal{O}$ *is defined by* $\phi^\alpha(x_\alpha) = \overline{x_\alpha}$.

Corollary 2.6. *The inductive limit* $\mathcal{O}$ *of an inductive system of OMPs is OML if and only if* $\mathcal{A}_\alpha$ *is an OML.*

Let $X = \{a_1, a_2, a_3, \ldots\}$, and let $\wp(X)$ be the power set of X. Then the Boolean algebra B consists of all finite and cofinite elements of the power set $\wp(X)$ of X is denoted by

$$B = \langle a_1, a_2, a_3, \ldots \rangle.$$

The pasting of two disjoint OMLs L_1 and L_2 along the principal sections $S_{c_1} \leq L_1$ and $S_{c_2} \leq L_2$ generated by c_1, c_2 respectively is denoted by

$$L = P(L_1, L_2; S_{c_1}, S_{c_2}; \theta)$$

(see Definition 2.3). We may omit the isomorphism θ if there is no difficulty.

Let $X = \{b_{ij}^k | 1 \leq i, k < \omega, 1 \leq j \leq 4\} \cup \{c_3, c_4\}$ be such that $b_{ij}^k \neq b_{mn}^l$ unless $(i, j, k) = (m, n, l)$, and $c_3 \neq c_4$ and $c_3, c_4 \neq b_{ij}^k$ $\forall i, j, k$. In the following construction each $[b_{ij}^i]$ represents equivalent class containing b_{ij}^i $\forall i, j$ which have been defined. Now we are ready to present an OML $\mathfrak{L}$ with finite sites which is not path-connected (Figure 1).

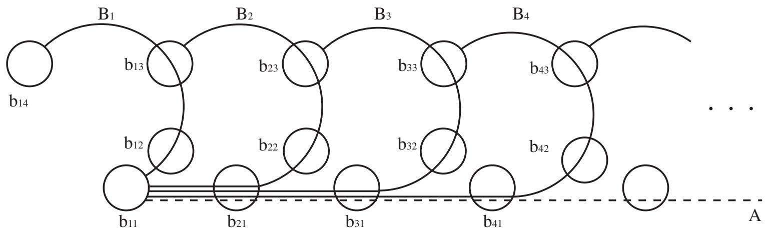

FIGURE 1. Greechie Diagram of the OML $\mathfrak{L}$ in Theorem 2.9

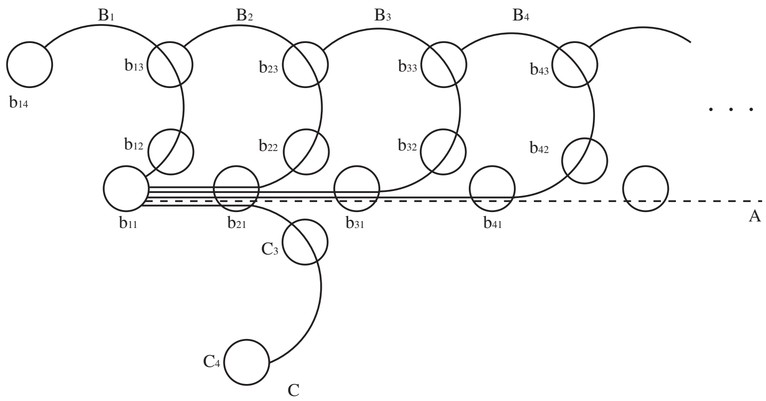

FIGURE 2. Greechie Diagram of the OML $\mathfrak{L}_0$ in Theorem 2.11

In the above two figures, the bars labeling each element and each block are omitted in order to make the diagrams simple; thus b_{ij} (B_i) represents $\overline{b_{ij}}$ $(\overline{B_i})$.

Let $B_1 = \langle b_{11}^1, b_{12}^1, b_{13}^1, b_{14}^1 \rangle$, $B_2 = \langle b_{11}^2, b_{21}^2, b_{22}^2, b_{23}^2, b_{13}^2 \rangle$, $B_3 = \langle b_{11}^3, b_{21}^3, b_{31}^3, b_{32}^3, b_{33}^3, b_{23}^3 \rangle, \ldots$, and $B_n = \langle b_{11}^n, b_{21}^n, \ldots, b_{n1}^n, b_{n2}^n, b_{n3}^n, b_{(n-1)3}^n \rangle$.

We construct L_n ($1 \leq n < \omega$) by induction (Figure 1). Let $L_1 = B_1$. Then L_1 is an OML. Let $L_2 = P(L_1, B_2; S_{e_1}, S_{e_1^2}; \theta_1)$, where $e_1 = (b^1_{11} \vee b^1_{13})'$, $e^2_1 = (b^2_{11} \vee b^2_{13})'$ and θ_1 is induced by the mapping sending b^1_{ij} in L_1 to b^2_{ij} in B_2 for $(i,j) \in \{(1,1),(1,3)\}$. Then L_2 is an OML by Theorem 2.2. Let $[b^2_{ij}]$ be the equivalent class in L_2 containing b^k_{ij} $\forall i,j,k$ such that $1 \leq i,k \leq 2$ and $1 \leq j \leq 4$. Note that b^2_{ij} is equivalent to b^1_{ij}. Let $L_3 = P(L_2, B_3; S_{e_2}, S_{e_2^3}; \theta_2)$, where $e_2 = ([b^2_{11}] \vee [b^2_{21}] \vee [b^2_{23}])'$, $e^3_2 = (b^3_{11} \vee b^3_{21} \vee b^3_{23})'$ and θ_2 is induced by the mapping sending $[b^2_{ij}]$ in L_2 to b^3_{ij} in B_3 for $(i,j) \in \{(1,1),(2,1),(2,3)\}$. Then L_3 is an OML by Theorem 2.2. Let $[b^3_{ij}]$ be the equivalent class in L_3 containing b^k_{ij} $\forall i,j,k$ such that $1 \leq i,k \leq 3$ and $1 \leq j \leq 4$. Note that b^3_{ij} is equivalent to b^k_{ij} $\forall k$ such that $k < 3$. $L_4 = P(L_3, B_4; S_{e_3}, S_{e_3^4}; \theta_3)$, where $e_3 = ([b^3_{11}] \vee [b^3_{21}] \vee [b^3_{31}] \vee [b^3_{33}])'$, $e^4_3 = (b^4_{11} \vee b^4_{21} \vee b^4_{31} \vee b^4_{33})'$ and θ_3 is induced by the mapping sending $[b^3_{ij}]$ in L_3 to b^4_{ij} in B_4 for $(i,j) \in \{(1,1),(2,1),(3,1),(3,3)\}$. Then L_4 is an OML by Theorem 2.2. Let $[b^4_{ij}]$ be the equivalent class in L_4 containing b^k_{ij} $\forall i,j,k$ such that $1 \leq i,k \leq 4$ and $1 \leq j \leq 4$. Note that b^4_{ij} is equivalent to b^k_{ij} $\forall k$ such that $k < 4$. Assume that L_{n-1} has been constructed. Let $[b^{n-1}_{ij}]$ be the equivalent class in L_{n-1} containing b^k_{ij} $\forall i,j,k$ such that $1 \leq i,k \leq n-1$ and $1 \leq j \leq 4$. Note that b^{n-1}_{ij} is equivalent to b^k_{ij} $\forall k$ such that $k < n-1$. Let $L_n = P(L_{n-1}, B_n; S_{e_{(n-1)}}, S_{e^n_{(n-1)}}; \theta_{n-1})$, where $e_{(n-1)} = ((\bigvee_{i=1}^{n-1}[b^{(n-1)}_{i1}]) \vee [b^{(n-1)}_{(n-1)\,3}])'$, $e^n_{(n-1)} = ((\bigvee_{i=1}^{n-1} b^n_{i1}) \vee b^n_{(n-1)\,3})'$ and θ_{n-1} is induced by the mapping sending $[b^{n-1}_{ij}]$ in L_{n-1} to b^n_{ij} in B_n for $(i,j) \in \{(1,1),(2,1),\ldots,((n-1),1),((n-1),3)\}$. Then L_n is an OML by Theorem 2.2. Let $[b^n_{ij}]$ be an equivalent class in L_n containing b^k_{ij} $\forall i,j,k$ such that $1 \leq i,k \leq n$ and $1 \leq j \leq 4$. Note that b^n_{ij} is equivalent to b^k_{ij} $\forall k$ such that $k < n$.

Let ϕ^i_j be an ortho-embedding from L_i into L_j $\forall i,j$ ($1 \leq i \leq j < \omega$). Then $(L_i, \phi^i_j)_{(1 \leq i \leq j < \omega)}$ is an inductive system in the category of orthomodular posets and ortho-embeddings.

Let $X = \bigcup_{(1 \leq i < \omega)} L_i$. Define a relation $\simeq$ on X by: $x \simeq y$, $x \in L_i$, $y \in L_j$ if and only if there exists $1 \leq k < \omega$ such that $1 \leq i \leq k < \omega$, $1 \leq j \leq k < \omega$ and $\phi^i_k(x) = \phi^j_k(y)$. Then $\simeq$ is an equivalence relation on X [3]. Let $\overline{x} = \{y \in X | y \simeq x\}$ and let $\mathfrak{L} = \{\overline{x} \mid x \in X\}$ (Figure 1). Define an ordering $\leq$ on $\mathfrak{L}$ by: $\overline{x} \leq \overline{y}$ if and only if there exist i, x_i and y_i such that $1 \leq i < \omega$, $x_i \in \overline{x} \cap L_i$, $y_i \in \overline{y} \cap L_i$ and $x_i \leq y_i$ in L_i. For $1 \leq i < \omega$, let $1 = \overline{1_i}$, $0 = \overline{0_i}$, and define $\overline{x_i}' = \overline{x_i'}$. Then $'$ in an orthocomplementation on $\mathfrak{L}$ [3]. Thus $(\mathfrak{L}, \phi^i)$ is the inductive limit of the inductive system $(L_i, \phi^i_j)_{1 \leq i \leq j < \omega}$, where $\phi^i : L_i \to \mathfrak{L}$ is defined by $\phi^i(x_i) = \overline{x_i}$ by Theorem 2.5. Moreover, $\mathfrak{L}$ is an OML by Corollary 2.6 since L_i is an OML $\forall i$ ($1 \leq i < \omega$).

Lemma 2.7. $\mathfrak{A}_{\mathfrak{L}} = \{\overline{B_i} \mid 1 \leq i < \omega\} \cup \{A\}$.

Proof. We know that $\{\overline{B_i} \mid 1 \le i < \omega\} \subset \mathfrak{A}_{\mathfrak{L}}$ by our inductive construction. Let $[B^i], [C^i] \in \mathfrak{A}_{L_i}$. Then $\phi^i_j([B^i]) = [B^j]$ for some $[B^j] \in \mathfrak{A}_{L_j}$ $\forall i, j (1 \le i \le j < \omega)$ and $\phi^i_j([B^i]) \ne \phi^i_j([C^i])$ if $[B^i] \ne [C^i]$.

Let $[B^i], [C^i] \in \mathfrak{A}_{L_i}$. Then $\phi^i([B^i]) = \overline{B}$ for some $\overline{B} \in \mathfrak{A}_{\mathfrak{L}}$ $\forall i (1 \le i < \omega)$ and $\phi^i([B^i]) \ne \phi^i([C^i])$ if $[B^i] \ne [C^i]$.

Therefore $\forall i$ $(1 \le i < \omega)$ $\overline{B_i} \in \mathfrak{A}_{\mathfrak{L}}$, where $\overline{B_i}$ is the equivalent class containing $[B^j_i]$ $\forall j (1 \le j < \omega)$. Moreover, $\mathfrak{L} = \bigcup\{\overline{B} \mid B \in \mathfrak{A}_{L_i}\}$.

Let $A = \langle \overline{b_{11}}, \overline{b_{21}}, \overline{b_{31}}, \ldots, \overline{b_{n1}}, \ldots \rangle$, and let $\overline{x}, \overline{y} \in A$. Then there exists $\overline{B_i} \in \mathfrak{A}_{\mathfrak{L}}$ such that $\overline{x}, \overline{y} \in \overline{B_i}$ and hence $\overline{x}\mathbf{C}\overline{y}$. Therefore A is a commuting set. If $\overline{x} \notin A$, then by our construction there exists $\overline{B_k} \in \mathfrak{A}_{\mathfrak{L}}$ such that $\overline{x} \in \overline{B_k}$ and hence $\overline{b_{(k+2),1}} \in \overline{B_{k+2}} \cap A$ with $\overline{x} \not\mathbf{C} \overline{b_{(k+2),1}}$. Hence A is a maximal commuting set. We claim that $\mathfrak{A}_{\mathfrak{L}} = \{\overline{B_i} \mid 1 \le i < \omega\} \cup \{A\}$. Let $C \in \mathfrak{A}_{\mathfrak{L}}$. We may assume that $C \ne A$. Thus there exists an atom $\overline{z} \in C \setminus A$ and hence there exists $\overline{B_h} \in \mathfrak{A}_{\mathfrak{L}}$ such that $\overline{z} \in \overline{B_h}$. We have the following three cases: (1) there exists a unique h such that $\overline{z} \in \overline{B_h}$ and hence $C = \overline{B_h}$; (2) $\overline{z} \in \overline{B_h} \cap \overline{B_{h+1}}$ and hence $C = \overline{B_h}$ or $C = \overline{B_{h+1}}$; (3) $\overline{z} \in \overline{B_{h-1}} \cap \overline{B_h}$ and hence $C = \overline{B_{h-1}}$ or $C = \overline{B_h}$. This completes the proof. □

An OML L is called *the horizontal sum* of a family $(L_i)_{i \in I}$ (denoted by $\circ(L_i)_{i\in I}$) of at least two subalgebras, if $\bigcup L_i = L$, and $L_i \cap L_j = \{0, 1\}$ whenever $i \ne j$, and one of the following equivalent conditions is satisfied:

(1) if $x \in L_i \setminus L_j$ and $y \in L_j \setminus L_i$, then $x \vee y = 1$;
(2) every block of L belongs to some L_i;
(3) if S_i is a subalgebra of L_i, then $\bigcup S_i$ is a subalgebra of L [2].

An OML L is said to be *the weak horizontal sum* of a family $(L_i)_{i\in I}$ of subalgebras if and only if there exists an isomorphism f of L onto a product of $L_0 \times L'$ of a Boolean algebra L_0 and an OML L' such that the subalgebras L_i of L correspond via f to subalgebras of the form $L_0 \times L'_i$ and L' is the horizontal sum of the family $(L'_i)_{i\in I}$ [1].

In the following Lemma 2.8, Theorem 2.9, Lemma 2.10 and Theorem 2.11, the bars labeling each element and each block are omitted in order to make the notation simple; thus b_{ij} and B_i represent $\overline{b_{ij}}$ and $\overline{B_i}$, respectively.

Lemma 2.8. *Two blocks of the type B_i and B_j $(1 \le i \le j < \omega)$ in $\mathfrak{L}$ have the following properties: $B_i \cup B_j \le \mathfrak{L}$ if $j = i + 1$ or $j \ge i + 4$, and $B_i \cup B_j \not\le \mathfrak{L}$ if $j = i + 2, i + 3$.*

Proof. We know that $B_i \cup B_j \le \mathfrak{L}$ if $j = i + 1$ or $j \ge i + 4$ since $B_i \cup B_j$ is a weak horizontal sum of B_i and B_j. Let us prove that $B_i \cup B_j \not\le \mathfrak{L}$ if $j = i + 2$ or $i + 3$.

Let $x = b_{11} \vee b_{21} \vee \cdots \vee b_{(i+1)\,1} \vee b_{(i+1)\,3} \in B_{i+2}$. Then $x \vee b_{i3} = b'_{(i+1)\,2} \in B_{i+1}$ and $x \vee b_{i3} = b'_{(i+1)\,2} \notin \mathfrak{L} \setminus B_{i+1}$. Thus $B_i \cup B_{i+2} \not\le \mathfrak{L}$.

Let $y = b_{11} \vee b_{21} \vee \cdots \vee b_{(i+2)\,1} \vee b_{(i+2)\,3} \in B_{i+3}$. Then $y \vee b_{i3} = b'_{(i+1)\,3} \in B_{i+1} \cap B_{i+2}$ and $y \vee b_{i3} = b'_{(i+1)\,3} \notin \mathfrak{L} \setminus (B_{i+1} \cup B_{i+2})$. Thus $B_i \cup B_{i+3} \not\lesssim \mathfrak{L}$. $\square$

We know that every OML with uniformly finite sites is path-connected [8]. The following theorem shows that there exists an OML with finite sites which is not path-connected.

Theorem 2.9. *There exists a weakly path-connected OML with finite sites which is not path-connected.*

Proof. Let $A = \langle b_{11}, b_{21}, b_{31}, \ldots, b_{n1}, \ldots \rangle$. Then $\mathfrak{A}_{\mathfrak{L}} = \{A\} \cup \{B_i | 1 \leq i < \omega\}$ by Lemma 2.7.

First, let us prove that $\mathfrak{L}$ is with finite sites. Since $|A \cap B_i| = |S_{f_i}| = 2^{i+1}$, where $f_i = (\bigvee_{k=1}^{i} b_{k1})'$ and $|B_i \cap B_j| \leq 2^{i+1}\ \forall\ (1 \leq i \leq j < \omega)$, $\mathfrak{L}$ is an OML with finite sites.

Second, let us prove that A is not path-connected with any $B_i \in \mathfrak{A}_{\mathfrak{L}}\ \forall i\ (1 \leq i < \omega)$. Fix such i and let $x = \bigvee_{1 \leq k \leq i+2} b_{k1} \in A$, and let $y = b_{i3} \in B_i$. Then $x \vee y = b'_{(i+1)\,3} \notin A \cup B_i$. Thus $A \cup B_i \not\lesssim \mathfrak{L}\ \ \forall i\ (1 \leq i < \omega)$ since $\mathfrak{A}_{\mathfrak{L}} = \{A\} \cup \{B_i | 1 \leq i < \omega\}$. Hence $\mathfrak{L}$ is not path-connected since A is not path-connected with any other block of L except itself.

Finally, let us prove that $\mathfrak{L}$ is weakly path-connected. B_i and B_j are path-connected for all $1 \leq i < j < \omega$ by a path $B_i \sim B_{i+1} \sim \cdots \sim B_j$ by Lemma 2.8 and hence weakly path-connected. A is weakly path-connected with B_i since $A \cap B_i = S_{f_i} \cap (A \cup B_i)\quad (1 \leq i < \omega)$, where $f_i = (\bigvee_{k=1}^{i} b_{k1})'$. Therefore $\mathfrak{L}$ is weakly path-connected. $\square$

Let $\{\widetilde{b}_{11}, \widetilde{b}_{21}\} \cap \{\overline{b_{ij}} \mid \overline{b_{ij}} \in \mathfrak{L}\} = \emptyset$ and $C = \langle \widetilde{b}_{11}, \widetilde{b}_{21}, c_3, c_4 \rangle$. Let $\mathfrak{L}_0 = P(\mathfrak{L}, C; S_{(\overline{b_{11}} \vee \overline{b_{21}})'}, S_{(\widetilde{b}_{11} \vee \widetilde{b}_{21})'}; \theta)$, where θ is induced by the map sending $\overline{b_{ij}}$ to $\widetilde{b}_{ij}$ for $(i,j) \in \{(1,1),(2,1)\}$ (Figure 2). Then $\mathfrak{L}_0$ is an OML by Theorem 2.2 and $\mathfrak{A}_{\mathfrak{L}_0} \cong \mathfrak{A}_{\mathfrak{L}} \cup C$.

Lemma 2.10. *Every $\mathbf{C}(b_{i1})$ $(1 < i < \omega)$ in $\mathfrak{L}$ is isomorphic to $2^{i-1} \times \mathfrak{L}$, and every $\mathbf{C}(b_{i1})$ $(2 < i < \omega)$ in $\mathfrak{L}_0$ is $\mathfrak{L}$ is isomorphic to $2^{i-1} \times \mathfrak{L}$.*

Proof. We know that $\mathbf{C}(b_{i1}) = \mathbf{C}(b_{i1})[0, \bigvee_{k=1}^{i-1} b_{k1}] \oplus \mathbf{C}(b_{i1})[0, (\bigvee_{k=1}^{i-1} b_{k1})']$.

Moreover, $\mathbf{C}(b_{i1})[0, \bigvee_{k=1}^{i-1} b_{k1}] \cong 2^{i-1}$, and $\mathbf{C}(b_{i1})[0, (\bigvee_{k=1}^{i-1} b_{k1})'] \cong \mathfrak{L}$ with the isomorphism $\phi : \mathbf{C}(b_{i1})[0, (\bigvee_{k=1}^{i-1} b_{k1})'] \to \mathfrak{L}$ induced by $\phi(b_{kj}) = b_{(k-i+1)j}$ $\forall k$ such that $i \leq k$, i.e., $\phi(L_j[0, (\bigvee_{k=1}^{i-1} b_{k1})']) = L_{j-i+1}\ \forall j \geq i$. Similarly, every $\mathbf{C}(b_{i1})$ in $\mathfrak{L}_0$ is isomorphic to $2^{i-1} \times \mathfrak{L}$. This completes the proof. $\square$

Let L be an OML. A subalgebra S of L is said to be a *full subalgebra* if every blocks of S is a block of L. Note that $\mathbf{C}(x)$ is a full subalgebra of L for all $x \in L$ since $\mathfrak{A}_{\mathbf{C}(x)} = \{B \in \mathfrak{A}_L | x \in B\}$.

Theorem 2.11. *There exists a path-connected OML such that $\mathbf{C}(x)$ is not path-connected for some $x \in L$.*

Proof. First, let us show that $\mathfrak{L}_0$ is path-connected. $A \cup C \leq \mathfrak{L}_0$ and $A \cap C = S_{(b_{11} \vee b_{21})'} \neq \mathbf{C}(\mathfrak{L}_0) = \{0, b_{11}, b'_{11}, 1\}$ and hence $A \approx C$. Then A is strictly path-connected with each B_i since $A \approx C \approx B_i$ $(1 < i < \omega)$, and $B_i \sim C$ for all $1 \leq i \leq \omega$. Thus $\mathfrak{L}_0$ is path-connected.

Finally, $\mathbf{C}(b_{31})$ is a full subalgebra of $\mathfrak{L}_0$ which is not path-connected since $\mathbf{C}(b_{31}) \cong 2^2 \times \mathfrak{L}$ by Lemma 2.10, and $\mathfrak{A}_{\mathbf{C}(b_{31})} = \{B \in \mathfrak{A}_{\mathfrak{L}_0} | b_{31} \in B\} = \{A\} \cup \{B_i \in \mathfrak{L} \mid B_i, 3 \leq i < \omega\}$. Thus $A \in \mathfrak{A}_{\mathbf{C}(b_{31})}$ is not path-connected in $\mathbf{C}(b_{31})$ with each B_i $\forall$ $(3 \leq i < \omega)$, by the proof in Theorem 2.9. This completes the proof. □

We need the following Theorem 2.12 [5] to get a class of path-connected OMLs.

Theorem 2.12 ([5]). *Let L be an OML. Then the set $\mathbf{CA}(L)$ of all central Abelian elements of L is the set of orthocomplements of the upper bounds for the set $Com\, L$, and $\bigvee \mathbf{CA}(L)$ exists if and only if $\bigvee Com\, L$ exists. If $h = \bigvee Com\, L$ exists, then $\mathbf{CA}(L) = [0, h']$ and $[0, h]$ contains no nonzero elements which are central Abelian elements of $[0, h]$ (and, therefore, of L).*

We denote *the join-semilattice* generated by $M \subset L$ of a lattice L by $[M]_\vee$. $[M]_\vee$ consists of all $\bigvee M_0$ with M_0 a finite subset of M. Then we have the following structure theorem.

Theorem 2.13. *If L is a non-Boolean OML such that the height of the join-semilattice $[Com\, L]_\vee$ generated by the commutators of L is finite, then L has a unique orthogonal decomposition $L = [0, e_0] \oplus [0, e_1] \oplus \cdots \oplus [0, e_n]$, where e_0 is the largest central Abelian element of L, and each $[0, e_i]\,(1 \leq k \leq n)$ is an irreducible non-Boolean OML such that the height of the join-semilattice $[Com\,[0, e_i]]_\vee$ generated by the commutators of $[0, e_i]$ is finite.*

Proof. Let L be a non-Boolean OML such that the height of the join-semilattice $[Com\, L]_\vee$ generated by the commutators of L is finite. Then $\bigvee Com\, L$ exists. Let $e'_0 = \bigvee Com\, L$. Since e_0 is central, $L = [0, e_0] \oplus [0, e'_0]$. Thus $Com\, L = Com\,[0, e'_0]$ by Theorem 2.12. If $[0, e'_0] = \bigoplus_{i \in I} [0, e_i]$ with each $e_i > 0$, then each summand has at least two commutators since each $[0, e_i]$ is a non-Boolean OML and hence the height $h([Com\,[0, e_i]]_\vee) \geq 1$. We may assume that I has the maximal cardinality among all such decompositions of $[0, e'_0]$. Then $|I| < \infty$ and each interval $[0, e_i]$ is irreducible. Moreover, each e_i $(i \geq 1)$ is an atom of $\mathbf{C}(L)$. Since any such decomposition of $[0, e'_0]$ is determined by the atoms of $\mathbf{C}en([0, e'_0])$, the decomposition is unique. □

We need the following Lemma 2.14 to prove Theorem 2.15.

Lemma 2.14. *Let L be an OML, and $A, B \in \mathfrak{A}_L$. If $A \cap B = \mathbf{C}(L)$ and $A \cup B \nleq L$, then there exist $C, D \in \mathfrak{A}_L$ such that $A \cap C \neq \mathbf{C}(L)$, $C \cap D \neq \mathbf{C}(L)$ and $D \cap B \neq \mathbf{C}(L)$.*

Proof. There exist c, d such that $c, d \in A \cup B$ and $c \vee d \notin A \cup B$ since $A \cup B \not\leq L$. Hence $c \vee d \notin \mathbf{C}(L) = \bigcap \mathfrak{A}_L$. We may assume that $c \in A \setminus B$ and $d \in B \setminus A$. Therefore there exist $C, D \in \mathfrak{A}_L$ such that $c, c \vee d \in C$ and $d, c \vee d \in D$. Then $c, d, c \vee d \notin \mathbf{C}(L)$ with $c \in A \cap C$, $c \vee d \in C \cap D$ and $d \in D \cap B$. □

We can find the following class of path-connected OMLs which contains all commutator-finite OMLs [5] and all block-finite OMLs [1]. This containment is proper as may be proved simply by considering any orthocomplemented projective plane.

Theorem 2.15. *If L is an OML such that the height of the join-semilattice $[Com\, L]_\vee$ generated by the commutators of L is finite, then L is path-connected.*

Proof. Let L be an OML such that the height of the join-semilattice $[Com\, L]_\vee$ generated by the commutators of L is finite. Then we may assume that L is irreducible by Theorem 2.13 and Theorem 1.5. Let us prove that L is path-connected by induction on the height k of $[Com\, L]_\vee$ (with the ordering inherited from L). If $k = 0$, then L is path-connected since L is a Boolean algebra. Assume that the conclusion of the theorem is true for each OML such that the height of the join-semilattice generated by the commutators of that OML is less than or equal to $n - 1$. If $k = n \geq 1$, then L is not a Boolean algebra. Thus there exist two distinct blocks A, B of L.

Assume first that $A \cap B \neq \{0, 1\} = \mathbf{C}(L)$. Let $m \in A \cap B \setminus \{0, 1\}$. If the height of the join-semilattice $[Com\, \mathbf{C}(m)]_\vee$ generated by $Com\, \mathbf{C}(m)$ is less than the height of the join-semilattice $[Com\, L]_\vee$, then A and B are path-connected in $\mathbf{C}(m)$ and hence in L by the inductive hypothesis. Thus we may assume that $h([Com\, \mathbf{C}(m)]_\vee) = h([Com\, L]_\vee)$. Suppose $\bigvee Com\, \mathbf{C}(m) < \bigvee Com\, L$. Then $h([Com\, \mathbf{C}(m)]_\vee) < h([Com\, L]_\vee)$ contradicting $h([Com\, \mathbf{C}(m)]_\vee) = h([Com\, L]_\vee)$. Hence $\bigvee Com\, \mathbf{C}(m) = \bigvee Com\, L = 1$ since L is irreducible. Thus $\mathbf{C}(m)$ has no nontrivial Boolean factors by Theorem 2.12. Therefore $\mathbf{C}(m)[0, m](= [0, m]\ )$ and $\mathbf{C}(m)[0, m'](= [0, m'])$ are non-Boolean. Then $[0, m]$, $[0, m']$ are path-connected since $h([Com\, [0, m]]_\vee) < h([Com\, L]_\vee)$ and $h([Com\, [0, m']]_\vee) < h([Com\, L]_\vee)$ by the inductive hypothesis. Thus $\mathbf{C}(m) = [0, m] \oplus [0, m']$ is path-connected by Theorem 1.4. Therefore A and B are path-connected in the full subalgebra $\mathbf{C}(m)$ of L. Thus A, B are path-connected in L.

Assume finally that $A \cap B = \{0, 1\}$. If $A \cup B \leq L$, then A and B are path-connected. If $A \cup B \not\leq L$, then there exist $C, D \in \mathfrak{A}_L$ such that $A \cap C \neq \{0, 1\}$, $C \cap D \neq \{0, 1\}$ and $D \cap B \neq \{0, 1\}$ by Lemma 2.14. Thus A and B are path-connected with a concatenated path by the first case. □

As a special case of Theorem 2.15, if L is an OML such that $\alpha \vee \beta = 1$ for any distinct commutators $\alpha, \beta \notin \{0, 1\}$, then L is path-connected. The fact that every commutator-finite OML is path-connected [2], is also a corollary of this theorem.

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