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RIDGELET TRANSFORM ON SQUARE INTEGRABLE BOEHMIANS

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Abstract. The ridgelet transform is extended to the space of square integrable Boehmians. It is proved that the extended ridgelet transform $\mathscr{R}$ is consistent with the classical ridgelet transform R, linear, one-to-one, onto and both $\mathscr{R}, \mathscr{R}^{-1}$ are continuous with respect to δ-convergence as well as Δ-convergence.

1. Introduction

Motivated by the concept of Boehme's regular operators [1], the theory of Boehmians is developed in the literature. Roughly speaking, Boehmians are convolution quotients f_n/δ_n of sequences, with (δ_n) converges to the Dirac distribution. After the work [6] of P. Mikusiński, various Boehmian spaces have been defined and also various integral transforms have been extended on them. See [4, 5, 7, 9, 11, 12, 13, 16].

Now we recall the ridgelet transform which is introduced in [2, 15]. Let $\mathscr{L}^2(\mathbb{R})$ denote the space of all complex valued Lebesgue measurable functions ψ on the set $\mathbb{R}$ of all real numbers with $\int_{-\infty}^{\infty} |\psi(x)|^2\, dx < \infty$, $\mathscr{L}^2(\mathbb{R}^2)$ denote the space of all complex valued Lebesgue measurable functions f on the set $\mathbb{R}^2$ with

$$\|f\| = \left(\int_{\mathbb{R}^2} |f(\mathbf{x})|^2\, d\mathbf{x} \right)^{\frac{1}{2}} < \infty$$

and $\mathscr{L}^2(\mathbb{Y})$ denote the space of all complex valued Lebesgue measurable functions F on $\mathbb{Y} = \mathbb{R}^+ \times \mathbb{R} \times [0, 2\pi]$ with

$$|||F||| = \left(\int_{\mathbb{Y}} |F(a, b, \theta)|^2\, d\mu \right)^{\frac{1}{2}} < \infty,$$

where $d\mu = d\mu(a, b, \theta) = \frac{da}{a^3} db \frac{d\theta}{4\pi}$.

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For a wavelet $\psi \in \mathscr{L}^2(\mathbb{R})$ and $(a,b,\theta) \in \mathbb{Y}$, we define

$$\psi_{a,b,\theta}(\mathbf{x}) = a^{-\frac{1}{2}}\psi((\mathbf{x}\cdot e^{\mathrm{i}\theta} - b)/a),\ \forall \mathbf{x} \in \mathbb{R}^2.$$

Definition 1. The ridgelet transform of $f \in \mathscr{L}^2(\mathbb{R}^2)$ is defined by

$$(Rf)(a,b,\theta) = \int_{\mathbb{R}^2} f(\mathbf{x})\psi_{a,b,\theta}(\mathbf{x})\,d\mathbf{x},\ \forall (a,b,\theta) \in \mathbb{Y}.$$

It is known that $R : \mathscr{L}^2(\mathbb{R}^2) \to \mathscr{L}^2(\mathbb{Y})$ with the inversion formula

$$f(\mathbf{x}) = \int_{\mathbb{Y}} (Rf)(a,b,\theta)\psi_{a,b,\theta}(\mathbf{x})\,d\mu,\ \forall \mathbf{x} \in \mathbb{R}^2 \tag{1}$$

and it satisfies the Parsevel's identity $\|f\| = |||Rf|||$ holds (see [15]).

For $f \in \mathscr{L}^2(\mathbb{R}^2)$ and $\phi \in \mathscr{D}(\mathbb{R}^2)$, we define the usual convolution $f * \phi$ by

$$(f*\phi)(\mathbf{x}) = \int_{\mathbb{R}^2} f(\mathbf{x}-\mathbf{y})\phi(\mathbf{y})\,d\mathbf{y},\ \forall \mathbf{x} \in \mathbb{R}^2.$$

2. Boehmian spaces

Boehmians are introduced as quotients of sequences to generalize the distributions. In this section, we briefly recall the construction of an abstract Boehmian space and we construct the Boehmian space

$$\mathscr{B}_2(\mathscr{L}^2(\mathbb{Y}), (\mathscr{D}(\mathbb{R}^2), *), \star, \Delta)$$

which is required to extend the ridgelet transform.

Using the notations as in [8], we let Γ be a topological vector space, $(S, *)$ be a commutative semi group, and $\star : \Gamma \times S \to \Gamma$ is satisfying the following properties:

(1) $(f+g)\star\phi = (f\star\phi) + (g\star\phi)$, $\forall f, g \in \Gamma$ and $\forall \phi \in S$,
(2) $(\alpha f)\star\phi = \alpha(f\star\phi)$, $\forall \alpha \in \mathbb{C}$, $\forall f \in \Gamma$ and $\forall \phi \in S$,
(3) $f\star(\phi*\psi) = (f\star\phi)\star\psi$, $\forall f \in \Gamma$ and $\forall \phi, \psi \in S$,
(4) If $f_n \to f$ as $n \to \infty$ in Γ and $\phi \in S$, then $f_n \star \phi \to f \star \phi$ as $n \to \infty$ in Γ,

and a collection Δ of all sequences (δ_n) from S, satisfying the following conditions:

(Δ_1): If $(\delta_n), (\epsilon_n) \in \Delta$, then $(\delta_n * \epsilon_n) \in \Delta$;
(Δ_2): If $f_n \to f$ as $n \to \infty$ in Γ and $(\delta_n) \in \Delta$, then $f_n \star \delta_n \to f$ as $n \to \infty$ in Γ.

Every member of $\mathscr{B} = \mathscr{B}(\Gamma, (S,*), \star, \Delta)$ is of the form $[\frac{f_n}{\delta_n}]$, where $f_n \in \Gamma$, $\forall n \in \mathbb{N}$, $(\delta_n) \in \Delta$ and

$$f_n \star \delta_m = f_m \star \delta_n,\ \forall m, n \in \mathbb{N}.$$

Two Boehmians $[\frac{f_n}{\delta_n}]$, $[\frac{g_n}{\phi_n}]$ in $\mathscr{B}$ are said to be equal if

$$f_n \star \phi_m = g_m \star \delta_n,\ \forall m, n \in \mathbb{N}.$$

The space Γ is identified as a subset of $\mathscr{B}$ by the identification $f \mapsto [\frac{f\star\delta_n}{\delta_n}]$, where $(\delta_n) \in \Delta$ is arbitrary.

Addition, scalar multiplication and the operation $\star$ in the context of $\mathscr{B}$ are defined as follows:

$$\left[\frac{f_n}{\delta_n}\right] + \left[\frac{g_n}{\phi_n}\right] = \left[\frac{f_n \star \phi_n + g_n \star \delta_n}{\delta_n * \phi_n}\right],$$

$$\alpha\left[\frac{f_n}{\delta_n}\right] = \left[\frac{\alpha f_n}{\delta_n}\right],$$

$$\left[\frac{f_n}{\delta_n}\right] \star \phi = \left[\frac{f_n \star \phi}{\delta_n}\right].$$

There are two notions of convergences on Boehmian spaces defined as follows:

Definition 2 (δ-convergence)**.** (x_n) is said to δ-converge to x, denoted by $x_n \xrightarrow{\delta} x$, if there exists $(\delta_k) \in \Delta$ such that $x_n \star \delta_k \in \Gamma$, $\forall n, k \in \mathbb{N}$, $x \star \delta_k \in \Gamma$, $\forall k \in \mathbb{N}$ and for each $k \in \mathbb{N}$, $x_n \star \delta_k \to x \star \delta_k$ as $n \to \infty$ in Γ.

Definition 3 (Δ-convergence)**.** (x_n) is said to Δ-converge to x, denoted by $x_n \xrightarrow{\Delta} x$, if there exists $(\delta_n) \in \Delta$ such that $(x_n - x) \star \delta_n \in \Gamma$, $\forall n \in \mathbb{N}$ and $(x_n - x) \star \delta_n \to 0$ as $n \to \infty$ in Γ.

The following lemma gives a necessary and sufficient condition for δ-convergence, which is stated and proved in [6].

Lemma 1. *$(x_n) \xrightarrow{\delta} x$ as $n \to \infty$ in $\mathscr{B}$ if and only if there exist $f_{n,k}$, $f_k \in \Gamma$ and $(\phi_k) \in \Delta$ such that $x_n = [\frac{f_{n,k}}{\phi_k}]$, $x = [\frac{f_k}{\phi_k}]$ and for each k, $f_{n,k} \to f_k$ as $n \to \infty$ in Γ.*

The $\mathscr{L}^p$-Boehmians are constructed in [3] for $p > 1$. So just we state the definition of the square integrable Boehmians. The space

$$\mathscr{B}_1 = \mathscr{B}(\mathscr{L}^2(\mathbb{R}^2), (\mathscr{D}(\mathbb{R}^2), *), *, \Delta)$$

of square integrable Boehmians is defined by taking the topological vector space Γ as $\mathscr{L}^2(\mathbb{R}^2)$, commutative semigroup $(S, *)$ as $(\mathscr{D}(\mathbb{R}^2), *)$, where $\mathscr{D}(\mathbb{R}^2)$ is the Schwartz testing function space consisting of smooth complex valued functions on $\mathbb{R}^2$ with compact supports, $*$ is the usual convolution of functions on $\mathbb{R}^2$ defined by

$$(f * \phi)(\mathbf{x}) = \int_{\mathbb{R}^2} f(\mathbf{x} - \mathbf{y})\phi(\mathbf{y})\, d\mathbf{y},$$

$\star$ as the same usual convolution and Δ as the collection of all sequences (δ_n) from $\mathscr{D}(\mathbb{R}^2)$ satisfying the following properties:

(1) $\int_{\mathbb{R}^2} \delta_n(\mathbf{x})\, d\mathbf{x} = 1$ for all n in the set $\mathbb{N}$ of all natural numbers;
(2) $\int_{\mathbb{R}^2} |\delta_n(\mathbf{x})|\, d\mathbf{x} \le M$, $\forall n \in \mathbb{N}$ for some $M > 0$;
(3) Given $\epsilon > 0$ there exists $m \in \mathbb{N}$ such that supp $\delta_n \subset B(\mathbf{0}; \epsilon)$, $\forall n \ge m$, where $B(\mathbf{0}; \epsilon) = \{\mathbf{x} \in \mathbb{R}^2 : |\mathbf{x}| < \epsilon\}$ and $|\mathbf{x}|$ is the Euclidean norm of $\mathbf{x}$ in $\mathbb{R}^2$.

To construct the Boehmian space $\mathscr{B}_2 = \mathscr{B}(\mathscr{L}^2(\mathbb{Y}), (\mathscr{D}(\mathbb{R}^2), *), \star, \Delta)$, we shall prove the following auxiliary results.

Definition 4. Let $F \in \mathscr{L}^2(\mathbb{Y})$ and $\phi \in \mathscr{D}(\mathbb{R}^2)$. For $(a, b, \theta) \in \mathbb{Y}$, we define

$$(F \star \phi)(a, b, \theta) = \int_{\mathbb{R}^2} F(a, b - \mathbf{x} \cdot e^{\mathrm{i}\theta}, \theta)\phi(\mathbf{x})\, d\mathbf{x}.$$

Lemma 2. *If $F \in \mathscr{L}^2(\mathbb{Y})$ and $\phi \in \mathscr{D}(\mathbb{R}^2)$, then $|||F \star \phi||| \leq C\, |||\, F|||$ for some $C > 0$ and hence $F \star \phi \in \mathscr{L}^2(\mathbb{Y})$.*

Proof. If ϕ is identically zero, then $F \star \phi = 0$ and hence, in this case, the lemma follows. Assume that $\phi \neq 0$. If $d\nu = |\phi(x)|dx$ and $C = \int_{\mathbb{R}^2} |\phi(x)|dx$, then $\frac{d\nu}{C}$ is a probability measure on $\mathbb{R}^2$. Therefore

$$\begin{aligned}
|||F \star \phi|||^2 &\leq \int_{\mathbb{Y}} \left(\int_{\mathbb{R}^2} |F(a, b - \mathbf{x} \cdot e^{\mathrm{i}\theta}, \theta)| \cdot |\phi(\mathbf{x})|\, d\mathbf{x} \right)^2 d\mu \\
&= \int_{\mathbb{Y}} C^2 \left(\int_{\mathbb{R}^2} |F(a, b - \mathbf{x} \cdot e^{\mathrm{i}\theta}, \theta)|\, \frac{d\nu}{C} \right)^2 d\mu \\
&\qquad \text{(by using Jensen's inequality [14, p. 62])} \\
&\leq \int_{\mathbb{Y}} C^2 \int_{\mathbb{R}^2} |F(a, b - \mathbf{x} \cdot e^{\mathrm{i}\theta}, \theta)|^2 \frac{d\nu}{C}\; d\mu \\
&= C \int_{\mathbb{Y}} \int_{\mathbb{R}^2} \left| F(a, b - \mathbf{x} \cdot e^{\mathrm{i}\theta}, \theta) \right|^2 \cdot |\phi(\mathbf{x})|\, d\mathbf{x} d\mu \\
&= C \int_{\mathbb{R}^2} \int_{\mathbb{Y}} \left| F(a, b - \mathbf{x} \cdot e^{\mathrm{i}\theta}, \theta) \right|^2 d\mu\, |\phi(\mathbf{x})|\, d\mathbf{x} \\
&\qquad \text{(by using Fubini's theorem [14, p. 164])} \\
&= C^2\, |||\, F\, |||^2 .
\end{aligned}$$

Hence the lemma follows. □

Lemma 3. *If $F_1, F_2 \in \mathscr{L}^2(\mathbb{Y})$, $\phi \in \mathscr{D}(\mathbb{R}^2)$ and $\alpha \in \mathbb{C}$, then*

(1) $(F_1 + F_2) \star \phi = F_1 \star \phi + F_2 \star \phi$.

(2) $(\alpha F) \star \phi = \alpha(F \star \phi)$.

The proof of the lemma follows from the linearity of the integral operator $\int_{\mathbb{R}^2}$.

Lemma 4. *If $F \in \mathscr{L}^2(\mathbb{Y})$ and $\phi_1, \phi_2 \in \mathscr{D}(\mathbb{R}^2)$, then $F \star (\phi_1 * \phi_2) = (F \star \phi_1) \star \phi_2$.*

Proof. For $(a, b, \theta) \in \mathbb{Y}$,

$$\begin{aligned}
(F \star (\phi_1 * \phi_2))(a, b, \theta) &= \int_{\mathbb{R}^2} F(a, b - \mathbf{x} \cdot e^{\mathrm{i}\theta}, \theta) \int_{\mathbb{R}^2} \phi_1(\mathbf{x} - \mathbf{y})\phi_2(\mathbf{y})\, d\mathbf{y}\, d\mathbf{x} \\
&= \int_{\mathbb{R}^2} \phi_2(\mathbf{y}) \int_{\mathbb{R}^2} F(a, b - \mathbf{x} \cdot e^{\mathrm{i}\theta}, \theta)\phi_1(\mathbf{x} - \mathbf{y})\, d\mathbf{x}\, d\mathbf{y} \\
&\qquad \text{(by using Fubini's theorem)}
\end{aligned}$$

$$= \int_{\mathbb{R}^2} \phi_2(\mathbf{y}) \int_{\mathbb{R}^2} F(a, b - (\mathbf{z}+\mathbf{y}) \cdot e^{\mathrm{i}\theta}, \theta)\phi_1(\mathbf{z})\, d\mathbf{z}\, d\mathbf{y}$$

$$\text{(by putting } \mathbf{z} = \mathbf{x} - \mathbf{y})$$

$$= \int_{\mathbb{R}^2} \phi_2(\mathbf{y})(F \star \phi_1)(a, b - \mathbf{y} \cdot e^{\mathrm{i}\theta}, \theta)\, d\mathbf{y}$$

$$= ((F \star \phi_1) \star \phi_2)(a, b, \theta).$$

□

Lemma 5. *If (F_n) converges to F in $\mathscr{L}^2(\mathbb{Y})$ and $\phi \in \mathscr{D}(\mathbb{R}^2)$, then $F_n \star \phi \to F \star \phi$ as $n \to \infty$ in $\mathscr{L}^2(\mathbb{Y})$.*

Proof. We have $|||F_n - F||| \to 0$ as $n \to \infty$. In view of Lemma 2, and by using Lemma 3(1), we get

$$|||F_n \star \phi - F \star \phi||| = |||(F_n - F) \star \phi||| \leq C\, |||\, F_n - F|||$$

for some $C > 0$. Hence the lemma follows. □

Lemma 6. *If $F \in \mathscr{L}^2(\mathbb{Y})$ and $(\delta_n) \in \Delta$, then $F \star \delta_n \to F$ as $n \to \infty$ in $\mathscr{L}^2(\mathbb{Y})$.*

Proof. Let $\epsilon > 0$ be given. Since the space $C_c(\mathbb{Y})$ of all complex valued continuous functions on $\mathbb{Y}$ with compact supports is dense in $\mathscr{L}^2(\mathbb{Y})$, there exits $G \in C_c(\mathbb{Y})$ such that

$$|||F - G||| < \min\left\{\frac{\epsilon}{3M}, \frac{\epsilon}{3}\right\}, \tag{2}$$

where $M > 0$ is as in property (2) for the delta sequence (δ_n) (see Section 2). We note that

$$|||F \star \delta_n - F||| \leq |||(F - G) \star \delta_n\, ||| + |||\, G \star \delta_n - G\, ||| + |||\, G - F\, |||\,. \tag{3}$$

In view of Lemma 2, we have

$$|||(F - G) \star \delta_n|||^2 < |||F - G\, |||^2 \left(\int_{\mathbb{R}^2} |\delta_n(\mathbf{x})|\, d\mathbf{x}\right)^2 \leq M^2\, |||\, F - G\, |||^2\,.$$

Therefore

$$|||(F - G) \star \delta_n||| \leq M\frac{\epsilon}{3M} \;=\; \frac{\epsilon}{3}. \tag{4}$$

Next, let K_1, K_2 and K_3 be compact subsets of $[0, 2\pi]$, $\mathbb{R}$ and $\mathbb{R}^+$ respectively, such that supp $G \subset K_1 \times K_2 \times K_3$.

Let $\mathbb{K} = K_1 \times (K_2 + [-1, 1]) \times K_3$ and C be such that $\int_{\mathbb{K}} d\mu < C < +\infty$. Since G is uniformly continuous on $\mathbb{Y}$, we choose $0 < r < 1$ such that if

$$(a_1, b_1, \theta_1), (a_2, b_2, \theta_2) \in \mathbb{Y} \text{ with } |(a_1, b_1, \theta_1) - (a_2, b_2, \theta_2)| < r,$$

then

$$|G(a_1, b_1, \theta_1) - G(a_2, b_2, \theta_2)| < \frac{\epsilon}{3M\sqrt{C}}, \tag{5}$$

where M is as in the property (2) of (δ_n) and $|(a_1, b_1, \theta_1) - (a_2, b_2, \theta_2)|$ is the Euclidean norm of $(a_1, b_1, \theta_1) - (a_2, b_2, \theta_2)$ in $\mathbb{R}^3$.

Let $m_1 \in \mathbb{N}$ such that supp $\delta_n \subset B(\mathbf{0}; r)$, $\forall n \geq m_1$. For $n \geq m_1$, we get

$$|||G \star \delta_n - G|||^2 \leq \int_{\mathbb{Y}} \left(\int_{\mathbb{R}^2} |(G(a, b - \mathbf{x} \cdot e^{\mathrm{i}\theta}, \theta) - G(a, b, \theta))\delta_n(\mathbf{x})| \, d\mathbf{x} \right)^2 d\mu.$$

If $d\nu_n = |\delta_n(\mathbf{x})| \, d\mathbf{x}$, then it is a positive finite measure on $\mathbb{R}^2$ and hence $\frac{d\nu_n}{\sigma_n}$ is a probability measure on $\mathbb{R}^2$, where $\sigma_n = \int_{\mathbb{R}^2} |\delta_n(\mathbf{x})| \, d\mathbf{x}$. Therefore the last integral is equal to

$$\int_{\mathbb{Y}} \sigma_n^2 \left(\int_{\mathbb{R}^2} |G(a, b - \mathbf{x} \cdot e^{\mathrm{i}\theta}, \theta) - G(a, b, \theta)| \frac{d\nu_n}{\sigma_n} \right)^2 d\mu$$

$$\leq \sigma_n^2 \int_{\mathbb{Y}} \int_{\mathbb{R}^2} |G(a, b - \mathbf{x} \cdot e^{\mathrm{i}\theta}, \theta) - G(a, b, \theta)|^2 \frac{d\nu_n}{\sigma_n} d\mu$$

(by using Jensen's inequality)

$$\leq M \int_{\mathbb{Y}} \int_{B(\mathbf{0}; r)} |G(a, b - \mathbf{x} \cdot e^{\mathrm{i}\theta}, \theta) - G(a, b, \theta)|^2 \cdot |\delta_n(\mathbf{x})| \, d\mathbf{x} d\mu.$$

(since supp $\delta_n \subseteq B(\mathbf{0}; r)$, $n \geq m_1$ and $\sigma_n \leq M$, $\forall n \in \mathbb{N}$)

Since supp $G(a, b, \theta) \subset K_1 \times K_2 \times K_3$ and $\mathbf{x} \in B(\mathbf{0}; r)$ (with $0 < r < 1$), supp $G(a, b - \mathbf{x} \cdot e^{\mathrm{i}\theta}, \theta) \subset \mathbb{K}$. Therefore the last integral is equal to $M \int_{\mathbb{K}} \int_{B(\mathbf{0}; r)} |G(a, b - \mathbf{x} \cdot e^{\mathrm{i}\theta}, \theta) - G(a, b, \theta)|^2 \cdot |\delta_n(\mathbf{x})| \, d\mathbf{x} d\mu$. Next we observe that $|(a, b - \mathbf{x} \cdot e^{\mathrm{i}\theta}, \theta) - (a, b, \theta)| < r$. Therefore, by using the inequality (5), the last integral is dominated by

$$M \int_{\mathbb{K}} \int_{B(\mathbf{0}; r)} \left(\frac{\epsilon}{3M\sqrt{C}} \right)^2 \cdot |\delta_n(\mathbf{x})| \, d\mathbf{x} d\mu \leq M^2 C \left(\frac{\epsilon}{3M\sqrt{C}} \right)^2 = \left(\frac{\epsilon}{3} \right)^2.$$

Using (2), (4), and the estimate of $|||G \star \delta_n - G|||$ in (3), it follows that $|||F \star \delta_n - F||| < \epsilon$ when $n \geq m_1$. Hence the lemma follows. ☐

Lemma 7. *If $F_n \to F$ as $n \to \infty$ in $\mathscr{L}^2(\mathbb{Y})$ and $(\delta_n) \in \Delta$, then $F_n \star \delta_n \to F$ as $n \to \infty$.*

Proof. We note that $|||F_n \star \delta_n - F||| \leq |||F_n \star \delta_n - F \star \delta_n ||| + ||| F \star \delta_n - F |||$. By Lemma 6, $|||F \star \delta_n - F||| \to 0$ as $n \to \infty$. Now by Lemma 2 and the property (2) of (δ_n), $|||(F_n - F) \star \delta_n||| \leq M \, ||| F_n - F||| \to 0$ as $n \to \infty$. ☐

3. Extended ridgelet transform

Lemma 8. *If $f \in \mathscr{L}^2(\mathbb{R}^2)$ and $\phi \in \mathscr{D}(\mathbb{R}^2)$, then $R(f * \phi) = (Rf) \star \phi$.*

Proof. For $(a, b, \theta) \in \mathbb{Y}$,

$$R(f * \phi)(a, b, \theta) = \int_{\mathbb{R}^2} a^{-\frac{1}{2}} \psi((\mathbf{x} \cdot e^{\mathrm{i}\theta} - b)/a)(f * \phi)(\mathbf{x}) \, d\mathbf{x}$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^2} a^{-\frac{1}{2}}\psi((\mathbf{x}\cdot e^{\mathrm{i}\theta}-b)/a)\int_{\mathbb{R}^2} f(\mathbf{x}-\mathbf{y})\phi(\mathbf{y})\,d\mathbf{y}\,d\mathbf{x}\\
&= \int_{\mathbb{R}^2} \phi(\mathbf{y})\int_{\mathbb{R}^2} a^{-\frac{1}{2}}\psi((\mathbf{x}\cdot e^{\mathrm{i}\theta}-b)/a)f(\mathbf{x}-\mathbf{y})\,d\mathbf{x}\,d\mathbf{y}\\
&\qquad\qquad\qquad\text{(by using Fubini's theorem)}\\
&= \int_{\mathbb{R}^2} \phi(\mathbf{y})\int_{\mathbb{R}^2} a^{-\frac{1}{2}}\psi(((\mathbf{z}+\mathbf{y})\cdot e^{\mathrm{i}\theta}-b)/a)f(\mathbf{z})\,d\mathbf{z}\,d\mathbf{y}\\
&= \int_{\mathbb{R}^2} \phi(\mathbf{y})\int_{\mathbb{R}^2} a^{-\frac{1}{2}}\psi((\mathbf{z}\cdot e^{\mathrm{i}\theta}-(b-\mathbf{y}\cdot e^{\mathrm{i}\theta}))/a)f(\mathbf{z})\,d\mathbf{z}\,d\mathbf{y}\\
&= \int_{\mathbb{R}^2} \phi(\mathbf{y})(Rf)(a,b-\mathbf{y}\cdot e^{\mathrm{i}\theta},\theta)\,d\mathbf{y}\\
&= ((Rf)\star\phi)(a,b,\theta).
\end{aligned}
$$

□

Definition 5. The extended ridgelet transform $\mathscr{R}X$ of a Boehmian $X = [\frac{f_n}{\delta_n}] \in \mathscr{B}_1$ is defined as $[\frac{Rf_n}{\delta_n}] \in \mathscr{B}_2$.

Lemma 9. *The extended ridgelet transform is well defined.*

Proof. Let $X = [\frac{f_n}{\delta_n}] \in \mathscr{B}_1$. Then we have $f_n * \delta_m = f_m * \delta_n,\ \forall m,n \in \mathbb{N}$. Applying the ridgelet transform on both sides and using the Lemma 8, we get $(Rf_n)\star\delta_m = (Rf_m)\star\delta_n,\ \forall m,n \in \mathbb{N}$. Therefore $\frac{Rf_n}{\delta_n}$ represents a Boehmian in $\mathscr{B}_2$. If $\frac{g_n}{\phi_n}$ is another representative of X, then we have $f_n * \phi_m = g_m * \delta_n,\ \forall m,n \in \mathbb{N}$. Again applying the ridgelet transform and using Lemma 8, we get that $(Rf_n)\star\phi_m = (Rg_m)\star\delta_n,\ \forall m,n \in \mathbb{N}$. This shows that $\frac{Rf_n}{\delta_n}$ and $\frac{Rg_n}{\phi_n}$ represent the same Boehmian in $\mathscr{B}_2$. Hence the lemma follows. □

Lemma 10. *The ridgelet transform on $\mathscr{B}_1$ is consistent with the classical ridgelet transform on $\mathscr{L}^2(\mathbb{R})$.*

Proof. Let $f \in \mathscr{L}^2(\mathbb{R})$. Then the Boehmian representing f in $\mathscr{B}_1$ is $[\frac{f*\delta_n}{\delta_n}]$. It is clear that $\mathscr{R}[\frac{f*\delta_n}{\delta_n}] = [\frac{R(f*\delta_n)}{\delta_n}] = [\frac{Rf\star\delta_n}{\delta_n}]$, which is the Boehmian representing the Rf in $\mathscr{B}_2$. Hence the lemma follows. □

Theorem 1. *The extended ridgelet transform $\mathscr{R} : \mathscr{B}_1 \to \mathscr{B}_2$ is a linear map.*

Proof. Let $X = [\frac{f_n}{\delta_n}]$, $Y = [\frac{g_n}{\phi_n}] \in \mathscr{B}_1$ and $\alpha \in \mathbb{C}$. By using the linearity of the ridgelet transform R on $\mathscr{L}^2(\mathbb{R}^2)$ and by Lemma 8, we get

$$
\begin{aligned}
\mathscr{R}(X+Y) &= \mathscr{R}\left[\frac{f_n*\phi_n + g_n*\delta_n}{\delta_n*\phi_n}\right] = \left[\frac{R(f_n*\phi_n + g_n*\delta_n)}{\delta_n*\phi_n}\right]\\
&= \left[\frac{R(f_n*\phi_n) + R(g_n*\delta_n)}{\delta_n*\phi_n}\right] = \left[\frac{(Rf_n)\star\phi_n + (Rg_n)\star\delta_n}{\delta_n*\phi_n}\right]\\
&= \left[\frac{Rf_n}{\delta_n}\right] + \left[\frac{Rg_n}{\phi_n}\right] = \mathscr{R}X + \mathscr{R}Y.
\end{aligned}
$$

□

Theorem 2. *The extended ridgelet transform $\mathscr{R} : \mathscr{B}_1 \to \mathscr{B}_2$ is an one-to-one map.*

Proof. Let $X = [\frac{f_n}{\delta_n}]$, $Y = [\frac{g_n}{\phi_n}] \in \mathscr{B}_1$. If $\mathscr{R}X = \mathscr{R}Y$, then we have $[\frac{Rf_n}{\delta_n}] = [\frac{Rg_n}{\phi_n}]$. This implies that $(Rf_n) \star \phi_m = (Rg_m) \star \delta_n$, $\forall m, n \in \mathbb{N}$. Using Lemma 8, we have $R(f_n * \phi_m) = R(g_m * \delta_n)$, $\forall m, n \in \mathbb{N}$. Since the classical ridgelet transform $R : \mathscr{L}^2(\mathbb{R}^2) \to \mathscr{L}^2(\mathbb{Y})$ is one-to-one, we have $f_n * \phi_m = g_m * \delta_n$, $\forall m, n \in \mathbb{N}$. Thus we have proved that $X = Y$. Hence the lemma follows. □

Theorem 3. *The extended ridgelet transform $\mathscr{R} : \mathscr{B}_1 \to \mathscr{B}_2$ is an onto map.*

Proof. Let $H = [\frac{F_n}{\delta_n}] \in \mathscr{B}_2$. Then $F_n \in \mathscr{L}^2(\mathbb{Y})$ for all $n \in \mathbb{N}$. Using the inversion formula (1) of the classical ridgelet transform, if $f_n = R^{-1}F_n$ then we have

$$f_n(\mathbf{x}) = \int_{\mathbb{Y}} F_n(a, b, \theta)\psi_{a,b,\theta}(\mathbf{x})\, d\mu, \ \forall \mathbf{x} \in \mathbb{R}^2.$$

Using Lemma 8, we have for all $m, n \in \mathbb{N}$, $f_n * \delta_m = (R^{-1}F_n) * \delta_m = R^{-1}(F_n \star \delta_m) = R^{-1}(F_m \star \delta_n) = (R^{-1}F_m) * \delta_n = f_m * \delta_n$. Therefore $[\frac{f_n}{\delta_n}] \in \mathscr{B}_1$. It is easy to verify that $\mathscr{R}[\frac{f_n}{\delta_n}] = H$. Hence the lemma follows. □

In view of proof of the above lemma, one can get the inversion formula for the extended ridgelet transform, and the inverse of the extended ridgelet transform is obtained as

$$\mathscr{R}^{-1}Y = \left[\frac{R^{-1}F_n}{\delta_n}\right] \in \mathscr{B}_1 \text{ for every } Y = \left[\frac{F_n}{\delta_n}\right] \in \mathscr{B}_2.$$

Theorem 4. *If $X \in \mathscr{B}_1$, $Y \in \mathscr{B}_2$ and $\phi \in \mathscr{D}(\mathbb{R}^2)$, then*
(1) *$\mathscr{R}(X * \phi) = \mathscr{R}X \star \phi$ and*
(2) *$\mathscr{R}^{-1}(Y \star \phi) = \mathscr{R}^{-1}Y * \phi$.*

Proof. Let $X = [\frac{f_n}{\delta_n}]$. Then by using Lemma 8 we get, $\mathscr{R}(X * \phi) = \mathscr{R}[\frac{f_n * \phi}{\delta_n}] = [\frac{R(f_n * \phi)}{\delta_n}] = [\frac{Rf_n \star \phi}{\delta_n}] = [\frac{Rf_n}{\delta_n}] \star \phi = \mathscr{R}X \star \phi$.

Let $Y = [\frac{F_n}{\phi_n}]$. By replacing Rf by F and by applying R^{-1} on both sides of the identity $R(f * \phi) = Rf \star \phi$ in Lemma 8, we obtain that $R^{-1}F * \phi = R^{-1}(F \star \phi)$, $\forall F \in \mathscr{L}^2(\mathbb{Y})$, $\forall \phi \in \mathscr{D}(\mathbb{R}^2)$. Therefore

$$\begin{aligned}\mathscr{R}^{-1}(Y \star \phi) &= \mathscr{R}^{-1}\left[\frac{F_n \star \phi}{\phi_n}\right] = \left[\frac{R^{-1}(F_n \star \phi)}{\phi_n}\right] = \left[\frac{R^{-1}F_n * \phi}{\phi_n}\right] \\ &= \left[\frac{R^{-1}F_n}{\phi_n}\right] * \phi = \mathscr{R}^{-1}Y * \phi.\end{aligned}$$

□

Theorem 5. *The extended ridgelet transform $\mathscr{R} : \mathscr{B}_1 \to \mathscr{B}_2$ and its inverse are continuous with respect to the δ-convergence.*

Proof. Let $X_n \xrightarrow{\delta} X$ as $n \to \infty$ in $\mathscr{B}_1$. Then by Lemma 2.4 of [6], there exists $f_{n,k}, f_k \in \mathscr{L}^2(\mathbb{R}^2)$ and $(\delta_k) \in \Delta$ such that $X_n = [\frac{f_{n,k}}{\delta_k}], X = [\frac{f_k}{\delta_k}]$ and for each $k \in \mathbb{N}$,

$$f_{n,k} \to f_k \text{ as } n \to \infty \text{ in } \mathscr{L}^2(\mathbb{R}^2).$$

Since the classical ridgelet transform is continuous we have

$$Rf_{n,k} \to Rf_k \text{ as } n \to \infty \text{ in } \mathscr{L}^2(\mathbb{Y}).$$

Since $\mathscr{R}X_n = [\frac{Rf_{n,k}}{\delta_k}], \mathscr{R}X = [\frac{Rf_k}{\delta_k}]$, we get $\mathscr{R}X_n \xrightarrow{\delta} \mathscr{R}X$ as $n \to \infty$ in $\mathscr{B}_2$.

Let $Y_n \xrightarrow{\delta} Y$ as $n \to \infty$ in $\mathscr{B}_2$. Then there exist $F_{n,k}, F_k \in \mathscr{L}^2(\mathbb{Y})$ and $(\delta_k) \in \Delta$ such that $Y_n = [\frac{F_{n,k}}{\delta_k}], Y = [\frac{F_k}{\delta_k}]$ and for each $k \in \mathbb{N}$,

$$F_{n,k} \to F_k \text{ as } n \to \infty \text{ in } \mathscr{L}^2(\mathbb{Y}).$$

Since the inverse ridgelet transform is continuous on $\mathscr{L}^2(\mathbb{Y})$ we have

$$R^{-1}F_{n,k} \to R^{-1}F_k \text{ as } n \to \infty \text{ in } \mathscr{L}^2(\mathbb{R}^2).$$

Since $\mathscr{R}^{-1}Y_n = [\frac{R^{-1}F_{n,k}}{\delta_k}], \mathscr{R}^{-1}Y = [\frac{R^{-1}F_k}{\delta_k}]$, we get $\mathscr{R}^{-1}Y_n \xrightarrow{\delta} \mathscr{R}^{-1}Y$ as $n \to \infty$ in $\mathscr{B}_1$. □

Theorem 6. *The extended ridgelet transform $\mathscr{R} : \mathscr{B}_1 \to \mathscr{B}_2$ and its inverse are continuous with respect to the Δ-convergence.*

Proof. Let $X_n \xrightarrow{\Delta} X$ as $n \to \infty$ in $\mathscr{B}_1$. Then by definition there exists $(\delta_n) \in \Delta$ such that $(X_n - X) * \delta_n \in \mathscr{L}^2(\mathbb{R}^2)$, $\forall n \in \mathbb{N}$ and $(X_n - X) * \delta_n \to 0$ as $n \to \infty$ in $\mathscr{L}^2(\mathbb{R}^2)$. This means that there exist $g_n \in \mathscr{L}^2(\mathbb{R}^2)$ such that $(X_n - X) * \delta_n = [\frac{g_n * \delta_k}{\delta_k}], \forall n \in \mathbb{N}$ and $g_n \to 0$ as $n \to 0$ in $\mathscr{L}^2(\mathbb{R}^2)$.

Since the ridgelet transform $R : \mathscr{L}^2(\mathbb{R}^2) \to \mathscr{L}^2(\mathbb{Y})$ is continuous, $Rg_n \to 0$ as $n \to 0$ in $\mathscr{L}^2(\mathbb{Y})$. Using Theorem 4, we get $(\mathscr{R}X_n - \mathscr{R}X) \star \delta_n = \mathscr{R}(X_n - X) \star \delta_n = \mathscr{R}((X_n - X) * \delta_n) = \mathscr{R}[\frac{g_n * \delta_k}{\delta_k}] = [\frac{R(g_n * \delta_k)}{\delta_k}] = [\frac{Rg_n \star \delta_k}{\delta_k}], \forall n \in \mathbb{N}$. Therefore it follows that $\mathscr{R}X_n \xrightarrow{\Delta} \mathscr{R}X$ as $n \to \infty$ in $\mathscr{B}_2$. Hence $\mathscr{R}$ is continuous with respect to the Δ-convergence.

If $Y_n \xrightarrow{\Delta} Y$ as $n \to \infty$ in $\mathscr{B}_2$. Then there exist $(\phi_n) \in \Delta$ and $G_n \in \mathscr{L}^2(\mathbb{Y})$ such that $(Y_n - Y) \star \phi_n = [\frac{G_n \star \phi_k}{\phi_k}], \forall n \in \mathbb{N}$ and $G_n \to 0$ as $n \to \infty$ in $\mathscr{L}^2(\mathbb{Y})$. Since $R^{-1} : \mathscr{L}^2(\mathbb{Y}) \to \mathscr{L}^2(\mathbb{R}^2)$ is continuous, $R^{-1}G_n \to 0$ as $n \to \infty$. Again by using Theorem 4, we get $(\mathscr{R}^{-1}Y_n - \mathscr{R}^{-1}Y) * \phi_n = \mathscr{R}^{-1}(Y_n - Y) * \phi_n = \mathscr{R}^{-1}((Y_n - Y) \star \phi_n) = \mathscr{R}^{-1}[\frac{G_n \star \phi_k}{\phi_k}] = [\frac{R^{-1}(G_n \star \phi_k)}{\phi_k}] = [\frac{R^{-1}G_n * \phi_k}{\phi_k}]$. Thus we have proved that $\mathscr{R}^{-1}$ is continuous with respect to the Δ-convergence. □

Finally we observe that the space $\mathscr{B}_1$ is properly larger than $\mathscr{L}^2(\mathbb{R}^2)$. In deed, the example of a Boehmian not representing any distribution given in [6] can be modified to get a member of $\mathscr{B}_1 \setminus \mathscr{L}^2(\mathbb{R}^2)$.

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