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COMMON FIXED POINT THEOREMS FOR A CLASS OF WEAKLY COMPATIBLE MAPPINGS IN D-METRIC SPACES

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Abstract. In this paper, we give some new definitions of D-metric spaces and we prove a common fixed point theorem for a class of mappings under the condition of weakly compatible mappings in complete D-metric spaces. We get some improved versions of several fixed point theorems in complete D-metric spaces.

1. Introduction and preliminaries

In 1922, the Polish mathematician, Banach proved a theorem which ensures, under appropriate conditions, the existence and uniqueness of a fixed point. His result is called Banach's fixed point theorem or the Banach contraction principle. This theorem provides a technique for solving a variety of applied problems in mathematical science and engineering. Many authors have extended, generalized and improved Banach's fixed point theorem in different ways. In [11], Jungck introduced more generalized commuting mappings, called *compatible* mappings, which are more general than commuting and weakly commuting mappings. This concept has been useful for obtaining more comprehensive fixed point theorems(see, e.g.,([1, 2, 3, 4, 6, 8, 9, 12, 13, 14, 16]). One such generalization is generalized metric space or D-metric space initiated by Dhage [5] in 1992. He proved some results on fixed points for a self-map satisfying a contraction for complete and bounded D-metric spaces. Rhoades [11] generalized Dhage's contractive condition by increasing the number of factors and proved the existence of unique fixed point of a self-map in a D-metric space. Recently, motivated by the concept of compatibility for a metric space, Singh and Sharma [15] introduced the concept of D-compatibility of maps in a D-metric space and proved some fixed point theorems using a contractive condition.
In what follows $\mathbb{N}$ the set of all natural numbers, and $\mathbb{R}^+$ the set of all positive real numbers.

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Definition 1.1. Let X be a nonempty set. A generalized metric (or D-metric) on X is a function $D : X^3 \longrightarrow \mathbb{R}^+$ that satisfies the following conditions for each $x, y, z, a \in X$.

(1) $D(x, y, z) \geq 0$,
(2) $D(x, y, z) = 0$ if and only if $x = y = z$,
(3) $D(x, y, z) = D(p\{x, y, z\})$, (symmetry) where p is a permutation function,
(4) $D(x, y, z) \leq D(x, y, a) + D(a, z, z)$.

The pair (X, D) is called a generalized metric (or D-metric) space.

It is easy to show that the following functions D are D-metric.

(a) $D(x, y, z) = \max\{d(x, y), d(y, z), d(z, x)\}$,
(b) $D(x, y, z) = d(x, y) + d(y, z) + d(z, x)$,
where, d is the ordinary metric on X.
(c) If $X = \mathbb{R}^n$ then we define

$$D(x, y, z) = (||x - y||^p + ||y - z||^p + ||z - x||^p)^{\frac{1}{p}}$$

for every $p \in \mathbb{R}^+$.

(d) If $X = \mathbb{R}^+$ then we define

$$D(x, y, z) = \begin{cases} 0 & \text{if } x = y = z, \\ \max\{x, y, z\} & \text{otherwise.} \end{cases}$$

Remark 1.2. Let (X, D) be a D-metric space. Then we have $D(x, x, y) = D(x, y, y)$. Since

(i) $D(x, x, y) \leq D(x, x, x) + D(x, y, y) = D(x, y, y)$

and

(ii) $D(y, y, x) \leq D(y, y, y) + D(y, x, x) = D(y, x, x)$.

We get $D(x, x, y) = D(x, y, y)$.

Let (X, D) be a D-metric space. For $r > 0$ define

$$B_D(x, r) = \{y \in X : D(x, y, y) < r\}.$$

Example 1.3. Let $X = \mathbb{R}$ and $D(x, y, z) = |x - y| + |y - z| + |z - x|$ for all $x, y, z \in \mathbb{R}$. Then,

$$\begin{aligned} B_D(1, 2) = \{y \in \mathbb{R} : D(1, y, y) < 2\} &= \{y \in \mathbb{R} : |y - 1| + |y - 1| < 2\} \\ &= \{y \in \mathbb{R} : |y - 1| < 1\} = (0, 2). \end{aligned}$$

Definition 1.4. Let (X, D) be a D-metric space and $A \subset X$.

(1) A is said to be open if for every $x \in A$ there exist $r > 0$ such that $B_D(x, r) \subset A$.

(2) A is said to be D-bounded if there exists $r > 0$ such that $D(x, y, y) < r$ for all $x, y \in A$.

(3) A sequence $\{x_n\}$ in X converges to x if and only if

$$D(x_n, x_n, x) = D(x, x, x_n) \to 0$$

as $n \to \infty$. That is, for each $\epsilon > 0$ there exist $n_0 \in \mathbb{N}$ such that

$$(*) \qquad \forall n \geq n_0 \Longrightarrow D(x, x, x_n) < \epsilon.$$

This is equivalent with, for each $\epsilon > 0$ there exist $n_0 \in \mathbb{N}$ such that

$$(**) \qquad \forall n, m \geq n_0 \Longrightarrow D(x, x_n, x_m) < \epsilon.$$

Indeed, suppose that (*) hods. Then

$$D(x_n, x_m, x) = D(x_n, x, x_m) \leq D(x_n, x, x) + D(x, x_m, x_m) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \varepsilon.$$

Conversely, set $m = n$ in $(**)$ we have $D(x_n, x_n, x) < \epsilon$.

(4) $\{x_n\}$ in X is called a Cauchy sequence if for each $\epsilon > 0$, there exits $n_0 \in \mathbb{N}$ such that $D(x_n, x_n, x_m) < \epsilon$ for each $n, m \geq n_0$. The D-metric space (X, D) is said to be complete if every Cauchy sequence in X is convergent.

Let τ be the set of all open subset of X. Then τ is a topology on X induced by the D-metric D.

Lemma 1.5. *Let (X, D) be a D-metric space. If $r > 0$, then ball $B_D(x, r)$ with center $x \in X$ and radius r is open.*

Proof. Let $z \in B_D(x, r)$. Then $D(x, z, z) < r$. If set $D(x, z, z) = \delta$ and $r' = r - \delta$ then we prove that $B_D(z, r') \subseteq B_D(x, r)$. Let $y \in B_D(z, r')$. Then, by triangular inequality, we have $D(x, y, y) = D(y, y, x) \leq D(y, y, z) + D(z, x, x) < r' + \delta = r$. Hence $B_D(z, r') \subseteq B_D(x, r)$. This implies that $B_D(x, r)$ is open ball. □

Lemma 1.6. *Let (X, D) be a D-metric space. If sequence $\{x_n\}$ in X converges to x, then it is unique.*

Proof. Let $x_n \longrightarrow y$ and $y \neq x$. Since $\{x_n\}$ converges to x and y, for each $\epsilon > 0$ there exist $n_1, n_2 \in \mathbb{N}$ such that for every $n \geq n_1$, $D(x, x, x_n) < \frac{\epsilon}{2}$, and for every $n \geq n_2$, $D(y, y, x_n) < \frac{\epsilon}{2}$. If set $n_0 = \max\{n_1, n_2\}$, then for every $n \geq n_0$ we have

$$D(x, x, y) \leq D(x, x, x_n) + D(x_n, y, y) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \varepsilon.$$

Hence $D(x, x, y) = 0$, which is a contradiction. So, $x = y$. □

Lemma 1.7. *Let (X, D) be a D-metric space. If the sequence $\{x_n\}$ in X is convergent to x, then it is a Cauchy sequence.*

Proof. Since $x_n \longrightarrow x$, for each $\epsilon > 0$ there exists $n_1, n_2 \in \mathbb{N}$ such that for every $n \geq n_1$, $D(x_n, x_n, x) < \frac{\epsilon}{2}$, and for every $m \geq n_2$, $D(x, x_m, x_m) < \frac{\epsilon}{2}$. If set $n_0 = \max\{n_1, n_2\}$, then for every $n, m \geq n_0$ we have

$$D(x_n, x_n, x_m) \leq D(x_n, x_n, x) + D(x, x_m, x_m) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence sequence $\{x_n\}$ is a Cauchy sequence. □

Definition 1.8. Let (X, D) be a D-metric space. D is said to be a continuous function on X^3 if

$$\lim_{n\to\infty} D(x_n, y_n, z_n) = D(x, y, z),$$

where a sequence $\{(x_n, y_n, z_n)\}$ in X^3 converges to a point $(x, y, z) \in X^3$, i.e.,

$$\lim_{n\to\infty} x_n = x,\ \lim_{n\to\infty} y_n = y,\ \lim_{n\to\infty} z_n = z.$$

Lemma 1.9. *Let (X, D) be a D- metric space. Then D is continuous function on X^3.*

Proof. If the sequence $\{(x_n, y_n, z_n)\}$ in X^3 converges to a point $(x, y, z) \in X^3$, i.e.,

$$\lim_{n\to\infty} x_n = x,\ \lim_{n\to\infty} y_n = y,\ \lim_{n\to\infty} z_n = z,$$

then for each $\epsilon > 0$ there exist $n_1, n_2, n_3 \in \mathbb{N}$ such that for every $n \geq n_1$, $D(x, x, x_n) < \frac{\epsilon}{3}$, for every $n \geq n_2$, $D(y, y, y_n) < \frac{\epsilon}{3}$, and for every $n \geq n_3$, $D(z, z, z_n) < \frac{\epsilon}{3}$. If set $n_0 = \max\{n_1, n_2, n_3\}$, then for every $n \geq n_0$ we have

$$\begin{aligned}
D(x_n, y_n, z_n) &\leq D(x_n, y_n, z) + D(z, z_n, z_n) \\
&\leq D(x_n, z, y) + D(y, y_n, y_n) + D(z, z_n, z_n) \\
&\leq D(z, y, x) + D(x, x_n, x_n) + D(y, y_n, y_n) + D(z, z_n, z_n) \\
&< D(x, y, z) + \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\
&= D(x, y, z) + \epsilon.
\end{aligned}$$

Hence we have

$$D(x_n, y_n, z_n) - D(x, y, z) < \epsilon.$$

On the other hand,

$$\begin{aligned}
D(x, y, z) &\leq D(x, y, z_n) + D(z_n, z, z) \\
&\leq D(x, z_n, y_n) + D(y_n, y, y) + D(z_n, z, z) \\
&\leq D(z_n, y_n, x_n) + D(x_n, x, x) + D(y_n, y, y) + D(z_n, z, z) \\
&< D(x_n, y_n, z_n) + \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\
&= D(x_n, y_n, z_n) + \epsilon.
\end{aligned}$$

That is,

$$D(x, y, z) - D(x_n, y_n, z_n) < \epsilon.$$

Therefore we have $|D(x_n, y_n, z_n) - D(x, y, z)| < \epsilon$, that is

$$\lim_{n\to\infty} D(x_n, y_n, z_n) = D(x, y, z).$$

Hence D is a continuous function. □

Definition 1.10. Let (X, D) is a D-metric space. Then D is called of *first type* if for every $x, y \in X$ we have

$$D(x, x, y) \leq D(x, y, z)$$

for every $z \in X$.

In 1998, Jungck and Rhoades [8] introduced the following concept of weak compatibility.

Definition 1.11. Let A and S be mappings from a D-metric space (X, D) into itself. Then the pair (A, S) is said to be weak compatible if they commute at their coincidence point, that is, $Ax = Sx$ implies that $ASx = SAx$.

2. The main results

Our main result, for a complete D-metric space X, reads follows:

Theorem 2.1. *Let A, B, C, S, T and R be self-mappings of a complete D-metric space (X, D) where D is first type with :*

(i)$A(X) \subseteq T(X)$, $B(X) \subseteq S(X)$, $C(X) \subseteq R(X)$ and $A(X)$ or $B(X)$ or $C(X)$ is a closed subset of X,

(ii) $D(Ax, By, Cz)$

$$\leq \alpha D(Rx, Ty, Sz) + \beta \max\{D(Rx, Ax, By), D(Ty, By, Cz), D(Sz, Cz, Ax)\}$$
$$+ \gamma(D(Rx, By, Ty) + D(Ty, Cz, Sz) + D(Sz, Ax, Rx))$$

where $\alpha, \beta, \gamma \geq 0$ and $\alpha + \beta + 3\gamma < 1$, for every $x, y, z \in X$,

(iii) the pairs (A, R) , (B, T) and (S, C) are weak compatible.

Then A, B, C, S, T and R have a unique common fixed point in X.

Proof. Let $x_0 \in X$ be an arbitrary point. By (i), there exists $x_1, x_2, x_3 \in X$ such that

$$Ax_0 = Tx_1 = y_0, \; Bx_1 = Sx_2 = y_1 \text{ and } Cx_2 = Rx_3 = y_2.$$

Inductively, construct sequence $\{y_n\}$ in X such that

$$y_{3n} = Ax_{3n} = Tx_{3n+1}, \quad y_{3n+1} = Bx_{3n+1} = Sx_{3n+2} \quad \text{and}$$
$$y_{3n+2} = Cx_{3n+2} = Rx_{3n+3},$$

for $n = 0, 1, 2, \cdots$.

Now, we prove $\{y_n\}$ is a Cauchy sequence. Let $d_m = D(y_m, y_{m+1}, y_{m+2})$. Then, we have

$$\begin{aligned} d_{3n} &= D(y_{3n}, y_{3n+1}, y_{3n+2}) \\ &= D(Ax_{3n}, Bx_{3n+1}, Cx_{3n+2}) \\ &\leq \alpha D(Rx_{3n}, Tx_{3n+1}, Sx_{3n+2}) \\ &\quad + \beta \max\{D(Rx_{3n}, Ax_{3n}, Bx_{3n+1}), D(Tx_{3n+1}, Bx_{3n+1}, Cx_{3n+2}), \\ &\qquad D(Sx_{3n+2}, Cx_{3n+2}, Ax_{3n})\} \end{aligned}$$

$$
\begin{aligned}
&+\gamma(D(Rx_{3n},Bx_{3n+1},Tx_{3n+1})+D(Tx_{3n+1},Cx_{3n+2},Sx_{3n+2})\\
&\qquad\qquad\qquad\qquad+D(Sx_{3n+2},Ax_{3n},Rx_{3n}))\\
&=\alpha D(y_{3n-1},y_{3n},y_{3n+1})\\
&\quad+\beta 8\max\{D(y_{3n-1},y_{3n},y_{3n+1}),D(y_{3n},y_{3n+1},y_{3n+2}),\\
&\qquad\qquad\qquad\qquad D(y_{3n+1},y_{3n+2},y_{3n})\}\\
&\quad+\gamma(D(y_{3n-1},y_{3n+1},y_{3n})+D(y_{3n},y_{3n+2},y_{3n+1})+D(y_{3n+1},y_{3n},y_{3n-1}))\\
&=\alpha d_{3n-1}+\beta\max\{d_{3n-1},d_{3n},d_{3n}\}+\gamma(d_{3n-1}+d_{3n}+d_{3n-1}).
\end{aligned}
$$

We prove that $d_{3n}\le d_{3n-1}$, for every $n\in\mathbb{N}$. If $d_{3n}>d_{3n-1}$ for some $n\in\mathbb{N}$, by above inequality we have

$$d_{3n}\le \alpha d_{3n}+\beta d_{3n}+3\gamma d_{3n}=(\alpha+\beta+3\gamma)d_{3n}<d_{3n},$$

which is a contradiction. Now, if $m=3n+1$, then

$$
\begin{aligned}
d_{3n+1}&=D(y_{3n+1},y_{3n+2},y_{3n+3})\\
&=D(y_{3n+3},y_{3n+1},y_{3n+2})\\
&=D(Ax_{3n+3},Bx_{3n+1},Cx_{3n+2})\\
&\le\alpha D(Rx_{3n+3},Tx_{3n+1},Sx_{3n+2})+\beta\max\{D(Rx_{3n+3},Ax_{3n+3},Bx_{3n+1}),\\
&\qquad D(Tx_{3n+1},Bx_{3n+1},Cx_{3n+2}),D(Sx_{3n+2},Cx_{3n+2},Ax_{3n+3})\}\\
&\quad+\gamma(D(Rx_{3n+3},Bx_{3n+1},Tx_{3n+1})+D(Tx_{3n+1},Cx_{3n+2},Sx_{3n+2})\\
&\qquad\qquad\qquad\qquad+D(Sx_{3n+2},Ax_{3n+3},Rx_{3n+3}))\\
&=\alpha D(y_{3n+2},y_{3n},y_{3n+1})\\
&\quad+\beta\max\{D(y_{3n+2},y_{3n+3},y_{3n+1}),D(y_{3n},y_{3n+1},y_{3n+2}),\\
&\qquad\qquad\qquad\qquad D(y_{3n+1},y_{3n+2},y_{3n+3})\}\\
&\quad+\gamma(D(y_{3n+2},y_{3n+1},y_{3n})+D(y_{3n},y_{3n+2},y_{3n+1})\\
&\qquad\qquad\qquad+D(y_{3n+1},y_{3n+3},y_{3n+2}))\\
&=\alpha d_{3n}+\beta\max\{d_{3n+1},d_{3n},d_{3n+1}\}+\gamma(d_{3n}+d_{3n}+d_{3n+1}).
\end{aligned}
$$

Similarly, if $d_{3n+1}>d_{3n}$ for some $n\in\mathbb{N}$ we have $d_{3n+1}<d_{3n+1}$ which is a contradiction. If $m=3n+2$, then

$$
\begin{aligned}
d_{3n+2}&=D(y_{3n+2},y_{3n+3},y_{3n+4})=D(y_{3n+3},y_{3n+4},y_{3n+2})\\
&=D(Ax_{3n+3},Bx_{3n+4},Cx_{3n+2})\\
&\le\alpha D(Rx_{3n+3},Tx_{3n+4},Sx_{3n+2})\\
&\quad+\beta\max\{D(Rx_{3n+3},Ax_{3n+3},Bx_{3n+4}),D(Tx_{3n+4},Bx_{3n+4},Cx_{3n+2}),\\
&\qquad\qquad\qquad\qquad D(Sx_{3n+2},Cx_{3n+2},Ax_{3n+3})\}\\
&\quad+\gamma(D(Rx_{3n+3},Bx_{3n+4},Tx_{3n+4})+D(Tx_{3n+4},Cx_{3n+2},Sx_{3n+2})\\
&\qquad\qquad\qquad\qquad+D(Sx_{3n+2},Ax_{3n+3},Rx_{3n+3}))
\end{aligned}
$$

$$
\begin{aligned}
&= \alpha D(y_{3n+2}, y_{3n+3}, y_{3n+1}) \\
&\quad + \beta \max\{D(y_{3n+2}, y_{3n+3}, y_{3n+4}), D(y_{3n+3}, y_{3n+4}, y_{3n+2}), \\
&\qquad\qquad D(y_{3n+1}, y_{3n+2}, y_{3n+3})\} \\
&\quad + \gamma(D(y_{3n+2}, y_{3n+4}, y_{3n+3}) + D(y_{3n+3}, y_{3n+2}, y_{3n+1}) \\
&\qquad\qquad + D(y_{3n+1}, y_{3n+3}, y_{3n+2})) \\
&= \alpha d_{3n+1} + \beta \max\{d_{3n+2}, d_{3n+2}, d_{3n+1}\} + \gamma(d_{3n+2} + d_{3n+1} + d_{3n+1}).
\end{aligned}
$$

Similarly, if $d_{3n+2} > d_{3n+1}$ for some $n \in \mathbb{N}$ we have $d_{3n+2} < d_{3n+2}$ which is a contradiction. Hence for every $n \in \mathbb{N}$ we have $d_n \leq d_{n-1}$. Thus by above inequalities we have $d_n \leq q d_{n-1}$, where $q = \alpha + \beta + 3\gamma < 1$. That is

$$
d_n = D(y_n, y_{n+1}, y_{n+2}) \leq q D(y_{n-1}, y_n, y_{n+1}) \leq \cdots \leq q^n D(y_0, y_1, y_2).
$$

Since D is of first type, we have

$$
D(y_n, y_n, y_{n+1}) \leq q^n D(y_0, y_1, y_2).
$$

Therefore

$$
\begin{aligned}
D(y_n, y_n, y_m) \leq\ & D(y_n, y_n, y_{n+1}) + D(y_{n+1}, y_{n+1}, y_{n+2}) + \cdots \\
& + D(y_{m-1}, y_{m-1}, y_m).
\end{aligned}
$$

Hence

$$
\begin{aligned}
D(y_n, y_n, y_m) &\leq q^n D(y_0, y_1, y_2) + q^{n+1} D(y_0, y_1, y_2) + \cdots + q^{m-1} D(y_0, y_1, y_2) \\
&= \frac{q^n - q^m}{1 - q} D(y_0, y_1, y_2) \\
&\leq \frac{q^n}{1 - q} D(y_0, y_1, y_2) \longrightarrow 0.
\end{aligned}
$$

So, sequence $\{y_n\}$ is Cauchy in X and $\{y_n\}$ converges to y in X. That is, $\lim_{n\to\infty} y_n = y$,

$$
\begin{aligned}
\lim_{n\to\infty} y_n &= \lim_{n\to\infty} Ax_{3n} = \lim_{n\to\infty} Bx_{3n+1} = \lim_{n\to\infty} Cx_{3n+2} \\
&= \lim_{n\to\infty} Tx_{3n+1} = \lim_{n\to\infty} Rx_{3n+3} = \lim_{n\to\infty} Sx_{3n+2} = y.
\end{aligned}
$$

Let $C(X)$ be a closed subset of X. Then there exist $u \in X$ such that $Ru = y$. We prove that $Au = y$. For

$$
\begin{aligned}
&D(Au, Bx_{3n+1}, Cx_{3n+2}) \\
&\leq \alpha D(Ru, Tx_{3n+1}, Sx_{3n+2}) \\
&\quad + \beta \max\{D(Ru, Au, Bx_{3n+1}), D(Tx_{3n+1}, Bx_{3n+1}, Cx_{3n+2}), \\
&\qquad\qquad D(Sx_{3n+2}, Cx_{3n+2}, Au)\} \\
&\quad + \gamma(D(Ru, Bx_{3n+1}, Tx_{3n+1}) + D(Tx_{3n+1}, Cx_{3n+2}, Sx_{3n+2}) \\
&\qquad\qquad + D(Sx_{3n+2}, Au, Ru)).
\end{aligned}
$$

Letting $n \longrightarrow \infty$, we get

$$D(Au,y,y) \leq \alpha D(Ru,y,y) + \beta \max\{D(Ru,Au,y), D(y,y,y), D(y,y,Au)\} \\ + \gamma(D(Ru,y,y) + D(y,y,y) + D(y,Au,Ru)).$$

If $D(y,y,Au) > 0$, then we have $D(Au,y,y) < D(y,y,Au)$, which is a contradiction. Thus $Au = y$. By weak compatibility of the pair (R, A), we have $ARu = RAu$, hence $Ay = Ry$. We prove that $Ay = y$, if $Ay \neq y$, then

$$\begin{aligned} &D(Ay, Bx_{3n+1}, Cx_{3n+2}) \\ &\leq \alpha D(Ry, Tx_{3n+1}, Sx_{3n+2}) \\ &\quad + \beta \max\{D(Ry, Ay, Bx_{3n+1}), D(Tx_{3n+1}, Bx_{3n+1}, Cx_{3n+2}), \\ &\qquad D(Sx_{3n+2}, Cx_{3n+2}, Ax)\} \\ &\quad + \gamma(D(Ry, Bx_{3n+1}, Tx_{3n+1}) + D(Tx_{3n+1}, Cx_{3n+2}, Sx_{3n+2}) \\ &\qquad + D(Sx_{3n+2}, Ay, Ry)). \end{aligned}$$

Letting $n \longrightarrow \infty$, we have

$$D(Ay,y,y) \leq \alpha D(Ry,y,y) + \beta \max\{D(Ry,Ay,y), D(y,y,y), D(y,y,Ay)\} \\ + \gamma(D(Ry,y,y) + D(y,y,y) + D(y,Ay,Ry)).$$

This is a contradiction. Therefore, $Ry = Ay = y$, that is, y is a common fixed of R and A. Since $y = Ay \in A(X) \subseteq R(X)$, there exist $v \in X$ such that $Tv = y$. We prove that $Bv = y$. For

$$\begin{aligned} D(y, Bv, Cx_{3n+2}) &= D(Ay, Bv, Cx_{3n+2}) \\ &\leq \alpha D(Ry, Tv, Sx_{3n+2}) \\ &\quad + \beta \max\{D(Ry, Ay, Bv), D(Tv, Bv, Cx_{3n+2}), \\ &\qquad D(Sx_{3n+2}, Cx_{3n+2}, Ay)\} \\ &\quad + \gamma(D(Ry, Bv, Tv) + D(Tv, Cx_{3n+2}, Sx_{3n+2}) \\ &\qquad + D(Sx_{3n+2}, Ay, Ry)) \end{aligned}$$

Letting $n \longrightarrow \infty$ we get

$$D(y,Bv,y) \leq \alpha D(y,Tv,y) + \beta \max\{D(y,y,Bv), D(Tv,Bv,y), D(y,y,y)\} \\ + \gamma(D(y,Bv,Tv) + D(Tv,y,y) + D(y,y,y)).$$

Thus $Bv = y$. By weak compatibility of the pair (B, T) we have $TBv = BTv$, hence $By = Ty$. We prove that $By = y$, if $By \neq y$, then

$$\begin{aligned} D(Ay, By, Cx_{3n+2}) &\leq \alpha D(Ry, Ty, Sx_{3n+2}) \\ &\quad + \beta \max\{D(Ry, Ay, By), D(Ty, By, Cx_{3n+2}), \\ &\qquad D(Sx_{3n+2}, Cx_{3n+2}, Ay)\} \\ &\quad + \gamma(D(Ry, By, Ty) + D(Ty, Cx_{3n+2}, Sx_{3n+2}) \\ &\qquad + D(Sx_{3n+2}, Ay, Ry)). \end{aligned}$$

Letting $n \longrightarrow \infty$ we have

$$\begin{aligned} D(y,By,y) \leq \alpha D(y,Ty,y) + \beta \max\{D(y,y,By), D(Ty,By,y), D(y,y,y)\} \\ + \gamma(D(y,By,Ty) + D(Ty,y,y) + D(y,y,y)). \end{aligned}$$

This is a contradiction. Therefore, $By = Ty = y$, that is, y is a common fixed of B and T . Similarly, since $y = By \in B(X) \subseteq S(X)$, there exist $w \in X$ such that $Sw = y$. We prove that $Cw = y$. For

$$\begin{aligned} D(y,y,Cw) &= D(Ay,By,Cw) \\ &\leq \alpha D(Ry,Ty,Sw) \\ &\quad + \beta \max\{D(Ry,Ay,By), D(Ty,By,Cw), D(Sw,Cw,Ay)\} \\ &\quad + \gamma(D(Ry,By,Ty) + D(Ty,Cw,Sw) + D(Sw,Ay,Ry)). \end{aligned}$$

Thus $Cw = y$. By weak compatibility the pair (C,S) we have $CSw = SCw$, hence $Cy = Sy$. We prove that $Cy = y$, if $Cy \neq y$, then

$$\begin{aligned} D(y,y,Cy) &= D(Ay,By,Cy) \\ &\leq \alpha D(Ry,Ty,Sy) \\ &\quad + \beta \max\{D(Ry,Ay,By), D(Ty,By,Cy), D(Sy,Cy,Ay)\} \\ &\quad + \gamma(D(Ry,By,Ty) + D(Ty,Cy,Sy) + D(Sy,Ay,Ry)). \end{aligned}$$

This is a contradiction. Therefore, $Cy = Sy = y$, that is, y is a common fixed of C and S. Thus

$$Ay = Sy = Ty = By = Cy = Ry = y.$$

Now, we have to prove the uniqueness. Let v be another common fixed point of T, A, B, C, R, S.
If $D(y,y,v) > 0$, then

$$\begin{aligned} D(y,y,v) &= D(Ay,By,Cv)) \\ &\leq \alpha D(Ry,Ty,Sv) \\ &\quad + \beta \max\{D(Ry,Ay,By), D(Ty,By,Cv), D(Sv,Cv,Ay)\} \\ &\quad + \gamma(D(Ry,By,Ty) + D(Ty,Cv,Sv) + D(Sv,Ay,Ry)). \end{aligned}$$

this is a contradiction. Therefore, $y = v$.

□

Corollary 2.2. *Let S, T, R and $\{A_\alpha\}_{\alpha \in I}$, $\{B_\beta\}_{\beta \in J}$ and $\{C_\gamma\}_{\gamma \in K}$ be the set of all self-mappings of a complete D-metric space (X,D), where D is of first type satisfying:*

(i) *there exists $\alpha_0 \in I$, $\beta_0 \in J$ and $\gamma_0 \in K$ such that $A_{\alpha_0}(X) \subseteq T(X)$, $B_{\beta_0}(X) \subseteq S(X)$ and $C_{\gamma_0}(X) \subseteq R(X)$,*

(ii) *$A_{\alpha_0}(X)$, $B_{\beta_0}(X)$ or $C_{\gamma_0}(X)$ is a closed subset of X,*

(iii) $D(A_\alpha x, B_\beta y, C_\gamma z)$

$$\begin{aligned} &\le a_1 D(Rx, Ty, Sz) \\ &\quad + b_1 \max\{D(Rx, A_\alpha x, B_\beta y), D(Ty, B_\beta y, C_\gamma z), D(Sz, C_\gamma z, A_\alpha x)\} \\ &\quad + c_1(D(Rx, B_\beta y, Ty) + D(Ty, C_\gamma z, Sz) + D(Sz, A_\alpha x, Rx)) \end{aligned}$$

where $a_1, b_1, c_1 \ge 0$ *and* $a_1 + b_1 + 3c_1 < 1$, *for every* $x, y, z \in X$ *and for every* $\alpha \in I, \beta \in J, \gamma \in K$,

(iv) *the pairs* (A_{α_0}, R), (B_{β_0}, T) *or* (C_{γ_0}, S) *are weak compatible.*
Then A, B, C, S, T *and* R *have a unique common fixed point in* X.

Proof. By Theorem 2.1 R, S, T and A_{α_0}, B_{β_0} and C_{γ_0} for some $\alpha_0 \in I$, $\beta_0 \in J$, $\gamma_0 \in K$ have a unique common fixed point in X. That is, there exist a unique $a \in X$ such that $R(a) = S(a) = T(a) = A_{\alpha_0}(a) = B_{\beta_0}(a) = C_{\gamma_0}(a) = a$. Let there exist $\lambda \in J$ such that $\lambda \ne \beta_0$ and $D(a, B_\lambda a, a) > 0$. Then we have

$$\begin{aligned} &D(a, B_\lambda a, a) \\ &= D(A_{\alpha_0} a, B_\lambda a, C_{\gamma_0} a) \\ &\le a_1 D(Ra, Ta, Sa) \\ &\quad + b_1 \max\{D(Ra, A_{\alpha_0} a, B_\beta a), D(Ta, B_\beta a, C_{\gamma_0} a), D(Sa, C_{\gamma_0} a, A_{\alpha_0} a)\} \\ &\quad + c_1(D(Ra, B_\beta a, Ta) + D(Ta, C_{\gamma_0} a, Sa) + D(Sa, A_{\alpha_0} a, Ra)), \end{aligned}$$

which is a contradiction. Hence for every $\lambda \in J$ we have $B_\lambda(a) = a$. Similarly for every $\delta \in I$ and $\eta \in K$ we get $A_\delta(a) = C_\eta(a) = a$. Therefore for every $\delta \in I, \lambda \in J$ and $\eta \in K$ we have $A_\delta(a) = B_\lambda(a) = C_\eta(a) = R(a) = S(a) = T(a) = a$. □

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