STRONG CONVERGENCE OF MODIFIED ISHIKAWA ITERATION FOR TWO RELATIVELY NONEXPANSIVE MAPPINGS IN A BANACH SPACE

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ABSTRACT. In this paper, we prove a strong convergence theorem for a common fixed point of two relatively nonexpansive mappings in a Banach space by using the modified Ishikawa iteration method. Our results improved and extend the corresponding results announced by many others.

1. Introduction

Let E be a Banach space, E^* be the dual space of E. $\langle \cdot, \cdot \rangle$ denotes the duality pairing of E and E^*. The function $\phi : E \times E \to R$ is defined by

$$\phi(y, x) = \|y\|^2 - 2\langle y, Jx \rangle + \|x\|^2,$$

for all $x, y \in E$, where J is the normalized duality mapping from E to E^*. Let C be a closed convex subset of E, and let T be a mapping from C into itself. We denote by $F(T)$ the set of fixed points of T. A point p in C is said to be an asymptotic fixed point of T[1] if C contains a sequence $\{x_n\}$ which converges weakly to p such that the strong $\lim_{n \to \infty}(x_n - Tx_n) = 0$. The set of asymptotic fixed points of T will be denoted by $\hat{F}(T)$. A mapping T from C into itself is called nonexpansive if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$. and relatively nonexpansive [1] if $\hat{F}(T) = F(T)$ and $\phi(p, Tx) \le \phi(p, x)$ for all $x \in C$ and $p \in F(T)$. The asymptotic behavior of a relatively nonexpansive mapping was studied in [1-4].

Two classical iteration processes are often used to approximate a fixed point of a nonexpansive mapping. The first one is introduced in 1953 by Mann [5] which well-known as Mann's iteration process and is defined as follows:

$$(1.1) \qquad \begin{cases} x_0 & \in & C \qquad \text{chosen arbitrarily,} \\ x_{n+1} & = & \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad n \ge 0, \end{cases}$$

where the sequence $\{\alpha_n\}$ is chosen in $[0, 1]$. Twenty-one years later, Ishikawa [6] enlarged and improved Mann's iteration (1.1) to the new iteration method,

it is often cited as Ishikawas iteration process which is defined recursively by

$$(1.2) \qquad \begin{cases} x_0 & \in & C \qquad \text{chosen arbitrarily} \\ y_n & = & \beta_n x_n + (1 - \beta_n) T x_n, \\ x_{n+1} & = & \alpha_n x_n + (1 - \alpha_n) T y_n, \quad n \geq 0, \end{cases}$$

where α_n and β_n are sequences in the interval $[0, 1]$.

Both iterations processes (1.1) and (1.2) have only weak convergence, in general Banach space (see [7] for more details). As a matter of fact, process (1.1) may fail to converge while process (1.2) can still converge for a Lipschitz pseudo-contractive mapping in a Hilbert space [8].

Some attempts to modify the Mann iteration method so that strong convergence is guaranteed have recently been made. Nakajo and Takahashi [9] proposed the following modification of the Mann iteration method for a single nonexpansive mapping T in a Hilbert space H:

$$(1.3) \qquad \begin{cases} x_0 & = & x \in C, \\ y_n & = & \alpha_n x_n + (1 - \alpha_n) T x_n, \\ C_n & = & \{ z \in C : \| z - y_n \| \leq \| z - x_n \| \}, \\ Q_n & = & \{ z \in C : \langle x_n - z, x - x_n \rangle \geq 0 \}, \\ x_{n+1} & = & P_{C_n \bigcap Q_n} x, \quad n = 0, 1, 2..., \end{cases}$$

where P_K denotes the metric projection from H onto a closed convex subset K of H. They proved that if the sequence $\{ \alpha_n \}$ is bounded above from one, then $\{ x_n \}$ defined by (1.3) converges strongly to $P_{F(T)} x$.

Recently, Martinez-Yanes and Xu [10] has adapted Nakajo and Takahashi's [9] idea to modify the process (1.2) for a single nonexpansive mapping T in a Hilbert space H :

$$(1.4) \qquad \begin{cases} x_0 & \in & C, \\ z_n & = & \beta_n x_n + (1 - \beta_n) T x_n, \\ y_n & = & \alpha_n x_n + (1 - \alpha_n) T z_n, \\ C_n & = & \{ v \in C : \| y_n - v \|^2 \leq \| x_n - v \|^2 \\ & & \qquad + (1 - \alpha_n)(\| z_n \|^2 - \| x_n \|^2 + 2 \langle x_n - z_n, v \rangle) \}, \\ Q_n & = & \{ v \in C : \langle x_n - v, x_0 - x_n \rangle \geq 0 \}, \\ x_{n+1} & = & P_{C_n \bigcap Q_n} x_0, \end{cases}$$

where P_K denotes the metric projection from H onto a closed convex subset K of H. They proved that if $\{ \alpha_n \}$ and $\{ \beta_n \}$ are sequences in $[0, 1]$ such that $\alpha_n \leq 1 - \delta$ for some $\delta \in (0, 1]$ and $\beta_n \to 1$, then the sequence $\{ x_n \}$ generated by (1.4) converges strongly to $P_{F(T)} x_0$.

The ideas to generalize the processes (1.3),(1.4) from Hilbert space to Banach space have recently been made. By using available properties on uniformly convex and uniformly smooth Banach space, Matsushita and Takahashi [1] presented their ideas as the following method for a single relatively nonexpansive

mapping T in a Banach space E :

$$(1.5) \quad \begin{cases} x_0 & = & x \in C, \\ y_n & = & J^{-1}(\alpha_n J x_n + (1 - \alpha_n) J T x_n), \\ H_n & = & \{z \in C : \phi(z, y_n) \leq \phi(z, x_n)\}, \\ W_n & = & \{z \in C : \langle x_n - z, Jx - Jx_n \rangle \geq 0\}, \\ x_{n+1} & = & \Pi_{H_n \cap W_n} x, n = 0, 1, 2, ..., \end{cases}$$

where $\alpha_n \subset [0, 1), \limsup_{n \to \infty} \alpha_n < 1$, and $\Pi_{H_n \cap W_n}$ is the generalized projection from C into $H_n \cap W_n$. They proved $\{x_n\}$ converges strongly $\Pi_{F(T)} x_0$.

Qin and Su[11]proposed the following modified Ishikawa iteration process for a single relatively nonexpansive mapping T in a Banach space E :

$$(1.6) \quad \begin{cases} x_0 & \in & C, \\ z_n & = & J^{-1}(\beta_n J x_n + (1 - \beta_n) J T x_n), \\ y_n & = & J^{-1}(\alpha_n J x_n + (1 - \alpha_n) J T z_n), \\ C_n & = & \{v \in C : \phi(v, y_n) \leq \alpha_n \phi(v, x_n) + (1 - \alpha_n) \phi(v, z_n)\}, \\ Q_n & = & \{v \in C : \langle x_n - v, Jx_0 - Jx_n \rangle \geq 0\}, \\ x_{n+1} & = & \Pi_{C_n \cap Q_n} x_0, \end{cases}$$

where $\alpha_n \subset [0, 1), \limsup_{n \to \infty} \alpha_n < 1$, $\beta_n \to 1$, and $\Pi_{C_n \cap Q_n}$ is the generalized projection from C into $C_n \cap Q_n$. They proved if T is uniformly continuous, then $\{x_n\}$ converges strongly to $\Pi_{F(T)} x_0$.

Inspired and motivated by these facts, our purpose in this paper is to develop the modified Ishikawa iteration process (1.6) to two relatively nonexpansive mappings.

2. Preliminaries

We denote by $J : E \to 2^{E^*}$ the normalized duality mapping from E to 2^{E^*}, defined by

$$J(x) := \{v \in E^* : \langle v, x \rangle = \|v\|^2 = \|x\|^2\}, \quad \forall x \in E.$$

The duality mapping J has the following properties:

(i) if E is smooth, then J is single-valued;
(ii) if E is strictly convex, then J is one-to-one;
(iii) if E is reflexive, then J is surjective.
(iv) if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E.

Let E be a reflexive, strictly convex, smooth Banach space and J the duality mapping from E into E^*. Then J^{-1} is also single-valued, one-to-one, surjective, and it is the duality mapping from E^* into E.

When $\{x_n\}$ is a sequence in E, we denote strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and weak convergence by $x_n \rightharpoonup x$.

Let $U = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. A Banach space E is said to be strictly convex if $\|\frac{x+y}{2}\| < 1$ for all $x, y \in U$ and $x \neq y$. It is also said to be uniformly convex if $\lim_{n \to \infty} \|x_n - y_n\| = 0$ for any two sequences $\{x_n\}, \{y_n\}$ in U and $\lim_{n \to \infty} \|\frac{x_n + y_n}{2}\| = 1$. A Banach space is said to be smooth provided

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in U$. It is also said to be uniformly smooth if the limit is attained uniformly for $x, y \in U$. A Banach space E is said to have the K-K property if a sequence $\{x_n\}$ of E satisfying that $x_n \rightharpoonup x \in E$ and $\|x_n\| \to \|x\|$, then $x_n \to x$. It is known that if E is uniformly convex, then E has the K-K property. Let E be a smooth Banach space. The function $\phi : E \times E \to R$ is defined by

$$\phi(y, x) = \|y\|^2 - 2\langle y, Jx \rangle + \|x\|^2$$

for all $x, y \in E$. It is obvious from the definition of the function ϕ that

(A1) $(\|x\| - \|y\|)^2 \leq \phi(y, x) \leq (\|x\| + \|y\|)^2$.
(A2) $\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy \rangle$.
(A3) $\phi(x, y) = \langle x, Jx - Jy \rangle + \langle y - x, Jy \rangle \leq \|x\|\|Jx - Jy\| + \|y - x\|\|y\|$.

Remark 2.1. From the Remark 2.1 of reference [1], we can know that if E is a strictly convex and smooth Banach space, then for $x, y \in E, \phi(y, x) = 0$ if and only if $x = y$.

Lemma 2.1 (see [1]). *Let E be a uniformly convex and smooth Banach space and let $\{y_n\}, \{z_n\}$ be two sequences of E. If $\phi(y_n, z_n) \to 0$, and either $\{y_n\}$ or $\{z_n\}$ is bounded, then $y_n - z_n \to 0$.*

Let C be a nonempty closed convex subset of E. Suppose that E is reflexive, strictly convex and smooth. Then, for any $x \in E$, there exists a unique point $x_0 \in C$ such that

$$\phi(x_0, x) = \min_{y \in C} \phi(y, x).$$

The mapping $\Pi_C : E \to C$ defined by $\Pi_C x = x_0$ is called the generalized projection [1, 12]. In a Hilbert space, $\Pi_C = P_C$ (metric projection). The following are well-known results.

Lemma 2.2 (see [11-12]). *Let C be a nonempty closed convex subset of a smooth Banach space E and $x \in E$. Then, $x_0 = \Pi_C x$ if and only if*

$$\langle x_0 - y, Jx - Jx_0 \rangle \geq 0$$

for all $y \in C$.

Lemma 2.3 (see [12]). *Let E be a reflexive, strictly convex and smooth Banach space, let C be a nonempty closed convex subset of E and let $x \in E$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x)$$

for all $y \in C$.

Lemma 2.4 (see [1]). *Let E be a strictly convex and smooth Banach space, let C be a closed convex subset of E, and let T be a relatively nonexpansive mapping from C into itself. Then $F(T)$ is closed and convex.*

Lemma 2.5 (see [13]). *Let E be a uniformly convex Banach space and let $r > 0$. Then there exists a continuous strictly increasing convex function $g : [0, 2r] \to R$ such that $g(0) = 0$ and*

$$\|tx + (1 - t)y\|^2 \leq t\|x\|^2 + (1 - t)\|y\|^2 - t(1 - t)g(\|x - y\|),$$

for all $x, y \in B_r$ and $t \in [0, 1]$, where $B_r = \{z \in E : \|z\| \leq r\}$.

3. Main results

For any $x_0 \in C$, we define the iteration process $\{x_n\}$ as follows:

$$(3.1) \quad \begin{cases} x_0 & \in & C \quad \text{chosen arbitrarily,} \\ z_n & = & \Pi_C J^{-1}(\beta_n J x_n + (1 - \beta_n) J S x_n), \\ y_n & = & J^{-1}(\alpha_n J x_n + (1 - \alpha_n) J T z_n), \\ C_n & = & \{v \in C : \phi(v, y_n) \leq \phi(v, x_n)\}, \\ Q_n & = & \{v \in C : \langle x_n - v, J x_0 - J x_n \rangle \geq 0\}, \\ x_{n+1} & = & \Pi_{C_n \bigcap Q_n} x_0, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}$ satisfy: $0 \leq \alpha_n < 1$, for all $n \in N \bigcup\{0\}$ and $\limsup\limits_{n \to \infty} \alpha_n < 1$, $\quad 0 < \beta_n < 1$, and $\liminf\limits_{n \to \infty} \beta_n(1 - \beta_n) > 0$.

Theorem 3.1. *Let E be a uniformly convex and uniformly smooth Banach space. Let C be a nonempty, closed convex subset of E. Assume that T, S are two relatively nonexpansive mappings from C into itself such that $F = F(T) \bigcap F(S) \neq \emptyset$, then the sequence $\{x_n\}$ defined by (3.1) converges strongly to $\Pi_F x_0$, where Π_F is the generalized projection from E onto F.*

Proof. We first show that C_n and Q_n are closed and convex for each $n \in N \bigcup\{0\}$. From the definition of C_n and Q_n, it is obvious that C_n is closed and Q_n is closed and convex for each $n \in N \bigcup\{0\}$. We show that C_n is convex. Since $\phi(v, y_n) \leq \phi(v, x_n)$ is equivalent to

$$2\langle v, J x_n - J y_n \rangle + \|y_n\|^2 - \|x_n\|^2 \leq 0,$$

it follows that C_n is convex. Next, we show that $F \subset C_n \bigcap Q_n$ for all $n \in N \bigcup\{0\}$. Put $\omega_n = J^{-1}(\beta_n J x_n + (1 - \beta_n) J S x_n)$, we have $z_n = \Pi_C \omega_n$. Let $p \in F$, then, by Lemma 2.3 and the convexity of $\|\cdot\|^2$, we have

$$(3.2) \quad \begin{aligned} \phi(p, z_n) & \leq & \phi(p, \omega_n) = \|p\|^2 - 2\langle p, \beta_n J x_n + (1 - \beta_n) J S x_n \rangle \\ & & + \|\beta_n J x_n + (1 - \beta_n) J S x_n\|^2 \\ & \leq & \|p\|^2 - 2\beta_n \langle p, J x_n \rangle - 2(1 - \beta_n)\langle p, J S x_n \rangle \\ & & + \beta_n \|x_n\|^2 + (1 - \beta_n)\|S x_n\|^2 \\ & = & \beta_n \phi(p, x_n) + (1 - \beta_n)\phi(p, S x_n) \\ & \leq & \phi(p, x_n), \end{aligned}$$

and then

$$
\begin{aligned}
\phi(p, y_n) &= \|p\|^2 - 2\langle p, \alpha_n Jx_n + (1-\alpha_n)JTz_n\rangle \\
&\quad + \|\alpha_n Jx_n + (1-\alpha_n)JTz_n\|^2 \\
&\leq \|p\|^2 - 2\alpha_n\langle p, Jx_n\rangle - 2(1-\alpha_n)\langle p, JTz_n\rangle \\
&\quad + \alpha_n\|x_n\|^2 + (1-\alpha_n)\|Tz_n\|^2 \\
&= \alpha_n\phi(p, x_n) + (1-\alpha_n)\phi(p, Tz_n) \\
&\leq \alpha_n\phi(p, x_n) + (1-\alpha_n)\phi(p, z_n) \\
&\leq \phi(p, x_n).
\end{aligned}
$$

Thus, we have $p \in C_n$. Therefore we obtain $F \subset C_n$ for each $n \in N \bigcup\{0\}$. Using the same argument presented in the proof of [1, Theorem 3.1;pp.261-262] we have $F \subset C_n \bigcap Q_n$ for each $n \in N \bigcup\{0\}$. This implies that $\{x_n\}$ is well defined. It follows from the definition of Q_n and lemma 2.2 that $x_n = \Pi_{Q_n} x_0$. Using $x_n = \Pi_{Q_n} x_0$ and lemma 2.3, we have

$$
\phi(x_n, x_0) \leq \phi(p, x_0) - \phi(p, x_n) \leq \phi(p, x_0)
$$

for each $p \in F \subset Q_n$ for each $n \in N \bigcup\{0\}$. Therefore, $\{\phi(x_n, x_0)\}$ is bounded. Moreover, from (A1), we have that $\{x_n\}$ is bounded.

Since $x_{n+1} = \Pi_{C_n \bigcap Q_n} x_0 \in Q_n$ and lemma 2.3, we have $\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0)$ for each $n \in N \bigcup\{0\}$. Therefore, $\{\phi(x_n, x_0)\}$ is nondecreasing. So there exists the limit of $\phi(x_n, x_0)$. From the lemma 2.3, we have

$$
\phi(x_{n+1}, x_n) \leq \phi(x_{n+1}, x_0) - \phi(x_n, x_0)
$$

for each $n \in N \bigcup\{0\}$. This implies that $\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0$. Since $x_{n+1} = \Pi_{C_n \bigcap Q_n} x_0 \in C_n$, from the definition of C_n, we also have

$$
\phi(x_{n+1}, y_n) \leq \phi(x_{n+1}, x_n)
$$

for each $n \in N \bigcup\{0\}$. Tending $n \to \infty$, we have $\lim_{n\to\infty} \phi(x_{n+1}, y_n) = 0$. Using the lemma 2.1, we obtain

$$
\lim_{n\to\infty} \|x_{n+1} - y_n\| = \lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.
$$

From $\|x_n - y_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\|$, we have

$$
(3.3) \qquad \|x_n - y_n\| \to 0, \quad (n \to \infty).
$$

Since J is uniformly norm-to-norm continuous on bounded sets, we have

$$
(3.4) \quad \lim_{n\to\infty} \|Jx_{n+1} - Jy_n\| = \lim_{n\to\infty} \|Jx_{n+1} - Jx_n\| = \lim_{n\to\infty} \|Jx_n - Jy_n\| = 0.
$$

On the other hand, we have, for each $n \in N \bigcup\{0\}$,

$$
\begin{aligned}
\|Jx_{n+1} - Jy_n\| &= \|\alpha_n(Jx_{n+1} - Jx_n) + (1-\alpha_n)(Jx_{n+1} - JTz_n)\| \\
&\geq (1-\alpha_n)\|Jx_{n+1} - JTz_n\| - \alpha_n\|Jx_n - Jx_{n+1}\|.
\end{aligned}
$$

and hence

$$
\|Jx_{n+1} - JTz_n\| \leq \frac{1}{1-\alpha_n}(\|Jx_{n+1} - Jy_n\| + \|Jx_n - Jx_{n+1}\|).
$$

From (3.4) and $\limsup\limits_{n\to\infty} \alpha_n < 1$, we obtain

$$\|Jx_{n+1} - JTz_n\| \to 0, \quad (n \to \infty).$$

Since J^{-1} is also uniformly norm-to-norm continuous on bounded sets, we have $\lim\limits_{n\to\infty} \|x_{n+1} - Tz_n\| = 0$. From $\|x_n - Tz_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - Tz_n\|$, we have

$$(3.5) \qquad \lim_{n\to\infty} \|x_n - Tz_n\| = 0.$$

Next, we show that $\|x_n - Sx_n\| \to 0$ and $\|z_n - Tz_n\| \to 0$. Since $\{x_n\}$ is bounded, $\phi(p, Sx_n) \le \phi(p, x_n)$, where $p \in F$, we also obtain $\{Jx_n\}, \{JSx_n\}$ are bounded, then there exists $r > 0$ such that $\{Jx_n\}, \{JSx_n\} \subset B_r$. Therefore lemma 2.5 is applicable and we observe that

$$(3.6) \qquad \begin{aligned} \phi(p, z_n,) &\le \phi(p, \omega_n) \\ &\le \|p\|^2 - 2\langle p, \beta_n Jx_n + (1-\beta_n)JSx_n \rangle + \beta_n \|x_n\|^2 \\ &\quad (1-\beta_n)\|Sx_n\|^2 - \beta_n(1-\beta_n)g(\|Jx_n - JSx_n\|) \\ &\le \phi(p, x_n) - \beta_n(1-\beta_n)g(\|Jx_n - JSx_n\|), \end{aligned}$$

and hence

$$(3.7) \qquad \phi(p, y_n) \le \phi(p, x_n) - (1-\alpha_n)\beta_n(1-\beta_n)g(\|Jx_n - JSx_n\|).$$

That is

$$(3.8) \qquad (1-\alpha_n)\beta_n(1-\beta_n)g(\|Jx_n - JSx_n\|) \le \phi(p, x_n) - \phi(p, y_n).$$

From (3.3),(3.4) and

$$\begin{aligned} \phi(p, x_n) - \phi(p, y_n) &= 2\langle p, Jy_n - Jx_n \rangle + \|x_n\|^2 - \|y_n\|^2 \\ &= 2\langle p, Jy_n - Jx_n \rangle + (\|x_n\| + \|y_n\|)(\|x_n\| - \|y_n\|). \end{aligned}$$

we have

$$\phi(p, x_n) - \phi(p, y_n) \to 0, \quad (n \to \infty).$$

By $\limsup\limits_{n\to\infty} \alpha_n < 1, \liminf\limits_{n\to\infty} \beta_n(1-\beta_n) > 0$, and (3.8), we have

$$g(\|Jx_n - JSx_n\|) \to 0.$$

From the properties of the function g, we obtain $\lim\limits_{n\to\infty} \|Jx_n - JSx_n\| = 0$. Since J^{-1} is also uniformly norm-to-norm continuous on bounded sets, we obtain

$$(3.9) \qquad \lim_{n\to\infty} \|x_n - Sx_n\| = 0.$$

Since $\|J\omega_n - Jx_n\| = (1-\beta_n)\|JSx_n - Jx_n\| \to 0$ as $n \to \infty$, therefore, $\|\omega_n - x_n\| \to 0$ as $n \to \infty$. By (A3), we have $\phi(x_n, \omega_n) \to 0$, as $n \to \infty$. Since $\phi(x_n, z_n) = \phi(x_n, \Pi_C \omega_n) \le \phi(x_n, \omega_n)$, then, we get $\lim\limits_{n\to\infty} \phi(x_n, z_n) = 0$. By Lemma 2.1, we have

$$(3.10) \qquad \lim_{n\to\infty} \|x_n - z_n\| = 0.$$

Since $\|z_n - Tz_n\| \leq \|z_n - x_n\| + \|x_n - Tz_n\|$, from (3.10) and (3.5), we have

$$(3.11) \qquad\qquad \lim_{n\to\infty} \|z_n - Tz_n\| = 0.$$

Therefore, from (3.10), if $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$ such that $x_{n_k} \rightharpoonup \hat{x} \in C$, then $z_{n_k} \rightharpoonup \hat{x}$. From (3.9),(3.11) and the definition of relatively nonexpansive mapping, we have $\hat{x} \in F$.

Finally, we show that $x_n \to \Pi_F x_0$. Let $w = \Pi_F x_0$. For any $n \in N$, from $x_{n+1} = \Pi_{C_n \bigcap Q_n} x_0$ and $w \in F \subset C_n \bigcap Q_n$, we have

$$\phi(x_{n+1}, x_0) \leq \phi(w, x_0).$$

On the other hand, from weakly lower semicontinuity of the norm, we have

$$\begin{aligned}
\phi(\hat{x}, x_0) &= \|\hat{x}\|^2 - 2\langle \hat{x}, Jx_0\rangle + \|x_0\|^2 \\
&\leq \liminf_{k\to\infty}(\|x_{n_k}\|^2 - 2\langle x_{n_k}, Jx_0\rangle + \|x_0\|^2) \\
&= \liminf_{k\to\infty} \phi(x_{n_k}, x_0) \\
&\leq \limsup_{k\to\infty} \phi(x_{n_k}, x_0) \\
&\leq \phi(w, x_0).
\end{aligned}$$

From the definition of $\Pi_F x_0$, we obtain $\hat{x} = w$ and hence, $\lim_{k\to\infty} \phi(x_{n_k}, x_0) = \phi(\hat{x}, x_0)$. So, we have $\lim_{k\to\infty} \|x_{n_k}\| = \|\hat{x}\|$. Using the K-K property of E, we obtain $x_{n_k} \to \Pi_F x_0$. Since x_{n_k} is an arbitrary convergent subsequence of $\{x_n\}$, we can conclude that $\{x_n\}$ converges strongly to $\Pi_F x_0$. $\qquad\square$

Note that $\beta_n = \frac{1}{2} - \frac{1}{n}$, an example of the sequence $\{\beta_n\}$.

If $S = T$, then we obtain the following modified Ishikawa iteration for a single relatively nonexpansive mapping.

Corollary 3.2. *Let E be a uniformly convex and uniformly smooth Banach space. Let C be a nonempty, closed convex subset of E. Assume that T is a relatively nonexpansive mapping from C into itself such that $F = F(T) \neq \emptyset$, then the sequence $\{x_n\}$ generated by*

$$\begin{cases}
x_0 &\in\ C \qquad chosen\ arbitrarily, \\
z_n &=\ \Pi_C J^{-1}(\beta_n Jx_n + (1-\beta_n)JTx_n) \\
y_n &=\ J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTz_n), \\
C_n &=\ \{v \in C : \phi(v, y_n) \leq \phi(v, x_n)\}, \\
Q_n &=\ \{v \in C : \langle x_n - v, Jx_0 - Jx_n\rangle \geq 0\}, \\
x_{n+1} &=\ \Pi_{C_n \bigcap Q_n} x_0.
\end{cases}$$

where $\{\alpha_n\},\{\beta_n\}$ satisfy: $0 \leq \alpha_n < 1$, for all $n \in N \bigcup\{0\}$ and $\limsup_{n\to\infty}\alpha_n < 1$, $0 < \beta_n < 1$, and $\liminf_{n\to\infty}\beta_n(1-\beta_n) > 0$, converges strongly to $\Pi_F x_0$.

Remark 3.1. Corollary 3.2 in this paper removes the uniformly continuity of the relatively nonexpansive mapping T in the reference [11].

If $S = I$, then we obtain the following result:

Corollary 3.3. *(*Matsushita and Takahashi [1, Theorem 3.1]*)Let E be a uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of E, let T be a relatively nonexpansive mapping from C into itself, and let $\{\alpha_n\}$ be sequence of real numbers such that $0 \leq \alpha_n < 1$ and $\limsup\limits_{n \to \infty} \alpha_n < 1$. If $F(T)$ is nonempty, then the sequence $\{x_n\}$ generated by (1.5) converges strongly to $P_{F(T)}x$. where $P_{F(T)} = \Pi_{F(T)}$ is the generalized projection from C onto $F(T)$.*

References

[1] S.Y. Matsushita, W. Takahashi, *A strong convergence theorem for relatively nonexpansive mappings in a Banach space*, Journal of Approximation Theory. **134** (2005), 257–266.

[2] D. Butnariu, S. Reich, and A.J. Zaslavski,*Asymptotic behavior of relatively nonexpansive operators in Banach spaces*, J. Appl. Anal.**7** (2001), 151–174.

[3] D. Butnariu, S. Reich, and A.J. Zaslavski,*Weak convergence of orbits of nonlinear operators in reflexive Banach spaces*, Numer. Funct. Anal. Optim.**24** (2003), 489–508.

[4] Y. Censor, S. Reich,*Iterations of paracontractions and firmly nonexpansive operators with applications to feasibility and optimization*, Optimization. **37** (1996), 323–339.

[5] W.R. Mann,*Mean value methods in iteration*, Proc. Amer. Math. Soc. **4** (1953), 506–510.

[6] S. Ishikawa,*Fixed points by a new iteration method*, Proc. Amer. Math. Soc. **44** (1974), 147–150.

[7] A. Genel, J. Lindenstrass,*An example concerning fixed points*, Israel J. Math. **22** (1975), 81–86.

[8] C.E. Chidume, S.A. Mutangadura,*An example on the Mann iteration method for Lipschitz pseudocontractions*, Proc. Amer. Math. Soc. **129** (2001), 2359–2363.

[9] K. Nakajo, W. Takahashi,*Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups*, J.Math.Anal. Appl. **279** (2003), 372–379.

[10] M.Y. Carlos, H.K. Xu,*Strong convergence of the CQ method for fixed point iteration processes*, Nonlinear Anal. **64** (2006), 2400–2411.

[11] X.L.Qin, Y.F.Su,*Strong convergence theorems for relatively nonexpansive mappings in a Banach space*, Nonlinear Anal. **67** (2007), 1958–1965.

[12] Ya.I. Alber, S. Reich,*An iterative method for solving a class of nonlinear operator equations in Banach spaces*, Panamer. Math.J. **4** (1994), 39–54.

[13] H.K.Xu,*Inequalities in Banach spaces with applications*, Nonlinear Anal. **16** (1991), 1127–1138.

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