East Asian Mathematical Journal
Vol. 25 (2009), No. 1, pp. 89–96

COMMON FIXED POINTS FOR A RATIONAL INEQUALITY UNDER WEAK COMPATIBLE MAPS OF TYPE (A) (DEDICATED TO THE LATE P. V. LAKSHMAIAH)

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Abstract. In this paper, we present a unique common fixed point theorem under weak compatible mappings of type(A), which extends and generalizes a result of Achari [1].

1. Introduction

Jungck [7] established a common fixed point theorem with commutativity and continuity of mappings. Jungck fixed point theorem is a remarkable generalization of Banach fixed point theorem in a complete metric space. Since 1976 a number of generalization of this result appeared refer Murthy [15] for variety of non-commuting maps under fixed point considerations. Sessa [17] generalized the commutativity by introducing weakly commuting pair of maps in a metric spaces. It is remarkable that a pair of commuting maps implies weakly commuting maps but the converse is not true. This influenced many researchers and consequently a number of new results followed in these lines. Again Jungck [8] introduced another weak commutativity, weaker than weakly commuting mappings called compatible pair of maps. This also influenced many researchers to obtained more generalized results. The first author of this paper with Jungck and Cho [10] introduced a new concept of non-commutative pair of maps which is equivalent to compatible pair of maps in a metric space. When we refer this result it is remarkable that the continuity of maps plays very crucial role for inter-relationship between compatible maps and compatible maps of type(A).

In this note we establish another generalization of Banach contraction principle through rational inequality. Further our result also includes many other generalization of Banach contraction principle. Some authors have obtained their results in 2- metric space and 2- Banach space with commutativity or weak commutativity or compatibility. However, in 2- metric and 2- Banach

Received September 22, 2008; Accepted December 15, 2008.
2000 *Mathematics Subject Classification.* 47H10, 46A03.
Key words and phrases. Common fixed point, compatible maps of type(A), weak compatible maps of type(A).

versions are easily obtained by an obvious modification. Therefore for simplicity, we have confined this work to metric spaces.

2. Non-commuting maps

Jungck, Murthy and Cho [10] introduced the concept of *compatible mappings of type (A)* and given inter-relationship between compatible of type (A) and compatible mappings.

Definition 2.1. Let S and T be mappings of a metric space (X, d) into itself. Then S and T are compatible mappings if $\lim_{n\to\infty} d(STx_n, TSx_n) = 0$ whenever a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} S(x_n) = \lim_{n\to\infty} T(x_n) = t$ for some $t \in X$.

Definition 2.2. Let S and T be mappings of metric space (X, d) into itself, then the pair $\{S, T\}$ is called compatible of type(A) if

$$\lim_{n\to\infty} d(STx_n, TTx_n) = 0 \text{ and } \lim_{n\to\infty} d(TSx_n, SSx_n) = 0$$

whenever $\{x_n\} \subseteq X$ be a sequence such that

$$\lim_{n\to\infty} S(x_n) = \lim_{n\to\infty} T(x_n) = t \text{ for some } t \in X.$$

Definition 2.3. Let S and T be mappings of metric space (X, d) into itself, then the pair $\{S, T\}$ is called weak compatible of type(A) if

$$\lim_{n\to\infty} d(STx_n, TTx_n) = 0$$

whenever $\{x_n\} \subseteq X$ be a sequence such that

$$\lim_{n\to\infty} S(x_n) = \lim_{n\to\infty} T(x_n) = t \text{ for some } t \in X.$$

The following propositions are easy to prove so we are omitting proofs.

Proposition 2.1. *Let $S, T : (X, d) \to (X, d)$ be mappings. If S and T are weak compatible of type (A) and $S(t) = T(t)$ for some $t \in X$, then $ST(t) = TT(t)$.*

Proposition 2.2. *Let $S, T : (X, d) \to (X, d)$ be mappings. If S and T are weak compatible of type (A) and $S(x_n), T(x_n) \to t$ for some $t \in X$. Then we have $\lim_{n\to\infty} TS(x_n) = S(t)$, if S is continuous.*

Refer to [10] for examples and counter examples.

3. Common fixed points

Theorem 3.1. *Let S and T be mappings of a complete metric space (X, d) into itself with S or T continuous. Suppose there exist self mappings A and B of X satisfying:*

(i) $A, B : X \to S(X) \cap T(X)$
(ii) *$\{A, S\}$ and $\{B, T\}$ are the pairs of weak compatible mappings of type(A),*

(iii) *for each* $x, y \in X$ *with* $\max\{d(Sx, By), d(Ty, Ax)\} \neq 0$,

$$\begin{aligned} d(Ax, By) \leq k\ & \max[d(Sx, Ty).d(Sx, By), d(Sx, Ty).d(Ty, Ax), \\ & d(Sx, Ax).d(Sx, By), d(Ty, By).d(Ty, Ax)] \\ & \div \max[d(Sx, By), d(Ty, Ax)], \end{aligned}$$

where $0 < k < 1$ *and* $d(Ax, By) = 0$, *when* $\max\{d(Sx, By), d(Ty, Ax)\} = 0$.

Then A, B, S *and* T *have a unique common fixed point in* X.

Proof. Let x_o be any point in X and choose $\{x_n\}$ and $\{y_n\}$ as follows:

$$\begin{aligned} y_1 &= Sx_1 = Bx_o, \\ y_2 &= Tx_2 = Ax_1, \\ &\dots \\ y_{2n-1} &= Sx_{2n-1} = Bx_{2n-2}, \\ y_{2n} &= Tx_{2n} = Ax_{2n-1}, \\ &\dots \end{aligned}$$

Define $d_n = d(y_n, y_{n+1})$ and suppose first of all that $y_n \neq y_{n+2}$ for $n = 1, 2, 3, \dots$.
Then

$$\begin{aligned} d_{2n} = d(y_{2n}, y_{2n+1}) &= d(Ax_{2n-1}, Bx_{2n}) \\ &\leq k\ \max\{d(Sx_{2n-1}, Tx_{2n}).d(Sx_{2n-1}, Bx_{2n}), \\ &\qquad d(Sx_{2n-1}, Tx_{2n}).d(Tx_{2n}, Ax_{2n-1}), \\ &\qquad d(Sx_{2n-1}, Ax_{2n-1}).d(Sx_{2n-1}, Bx_{2n}), \\ &\qquad d(Tx_{2n}, Bx_{2n}).d(Tx_{2n}, Ax_{2n-1})\} \\ &\quad \div \max\{d(Sx_{2n-1}, B_{x2n}), d(Tx_{2n}, Ax_{2n-1})\} \\ &\leq k.d_{2n-1}. \end{aligned}$$

Similarly, $d_{2n-1} \leq kd_{2n-2}$. Hence $\{d_n\}$ is monotone decreasing sequence and converges to 0 as $0 < k < 1$. Therefore $\{y_n\}$ is a Cauchy sequence. By completeness of X, $\{y_n\}$ converges to a point $z \in X$. Thus $\lim_{n\to\infty} Ax_{2n-1} = z$, $\lim_{n\to\infty} Bx_{2n} = z$, $\lim_{n\to\infty} Tx_{2n} = z$ and $\lim_{n\to\infty} Sx_{2n-1} = z$.

Suppose S is continuous, then $SAx_{2n-1} \to Sz$. Since the pair $\{A, S\}$ is weak compatible of type(A). Then by Proposition 2.2 $\lim_{\to\infty} ASx_{2n-1} = Sz$.
Now suppose that $Sz \neq z$. Then

$$\begin{aligned} d(Sz, z) &= \lim_{n\to\infty} d(ASx_{2n-1}, Bx_{2n-2}) \\ &\leq \lim_{n\to\infty} k[\max\{d(SSx_{2n-1}, Tx_{2n-1}).d(SSx_{2n-1}, Bx_{2n-2}), \\ &\qquad d(SSx_{2n-1}, Tx_{2n-2}).d(Tx_{2n-2}, ASx_{2n-1}), \end{aligned}$$

$$
\begin{aligned}
&\quad d(SSx_{2n-1}, ASx_{2n-1}).d(SSx_{2n-1}, Bx_{2n-2}),\\
&\quad d(Tx_{2n-2}, Bx_{2n-2}).d(Tx_{2n-2}, ASx_{2n-1})]\\
&\div \max\{d(SSx_{2n-1}, Bx_{2n-2}), d(Tx_{2n-2}, ASx_{2n-1})\}\\
&= k\ \max\{d^2(Sz, z), d^2(Sz, z), 0, 0\} \div \max\{d(Sz, z), d(Sz, z)\}\\
&= k\ d(Sz, z)
\end{aligned}
$$

Thus $d(Sz, z) < d(Sz, z)$, a contradiction, hence $Sz = z$.
We claim that $Az = z$ for if $Az \neq z$, then

$$
\begin{aligned}
d(Az, z) &= \lim_{n\to\infty} d(Az, Bx_{2n-2})\\
&\le \lim_{n\to\infty} k\ \max\{d(Sz, Tx_{2n-2}).d(Sz, Bx_{2n-2}),\\
&\qquad d(Sz, Tx_{2n-2}).d(Tx_{2n-2}, Az), d(Sz, Az).d(Sz, Bx_{2n-2}),\\
&\qquad d(Tx_{2n-2}, Bx_{2n-2}).d(Tx_{2n-2}, Az)\}\\
&\quad \div \max\{d(Sz, Bx_{2n-2}), d(Tx_{2n-2}, Az)\}\\
&= k\ \max\{0, 0, 0, 0\} \div \max\{0, d(z, Az)\}\\
&= 0.
\end{aligned}
$$

Hence $Az = z$.

From 3.1 (i) $A(X) \subset S(X) \cap T(X)$ implies $A(X) \subset T(X)$, then there exists $w \in X$ such that $z = Az = Tw$. Suppose $Bw = z$ if $Bw \neq z = Tw$, then

$$
\begin{aligned}
d(Az, Bw) &\le k \max\{d(Sz, Tw).d(Sz, Bw), d(Sz, Tw).d(Tw, Az),\\
&\qquad d(Sz, Az).d(Sz, Bw), d(Tw, Bw).d(Tw, Az)\}\\
&\quad \div \max\{d(Sz, Bw), d(Tw, Az)\}\\
&= k \max\{0, 0, 0, 0\} \div \max\{d(z, Bw), 0\}\\
&= 0.
\end{aligned}
$$

Hence $Bw = Tw = z$. Since B and T are weak compatible mappings of type (A), $d(BTw, TTw) = 0$ implies $BTw = TTw$ i.e. $Bz = BTw = TTw = Tz$.

Further, we claim that $Bz = z$ for if $Bz \neq z$, then

$$
d(Az, Bz) \le k \max\{d^2(z, Tz), d^2(z, Tz), 0, 0\} \div \max\{d(z, Bz), d(Tz, z)\}
$$

and $d(z, Bz) < d(z, Bz)$, a contradiction. Hence $z = Bz = Tz$.

Thus z is a common fixed point of A, B, S and T.

Now suppose that $y_{2n} = y_{2n+2}$ for some n. Then condition (iii) implies that $y_{2n} = y_{2n+1} = y_{2n+2}$ or equivalently

$$
Tx_{2n} = Ax_{2n-1} = Sx_{2n+1} = Bx_{2n} = Tx_{2n+2} = Ax_{2n+1} = z \text{ (say)}.
$$

Put $x = x_{2n}$ and $y = x_{2n+1}$. Then $Tx = Sy = Bx = Ay = z$.
Since $\{A, S\}$ is a pair of weak compatible mappings of type (A), so, Proposition 2.1 implies, $SAy = S^2y$ implies $Sz = Az$.

Similarly, $Tz = Bz$ as $\{B, T\}$ is also a pair of weak compatible mappings of type(A). Suppose $Az \neq Bz$. Then by use of condition (iii) we have $d(Az, Bz) \leq kd(Az, Bz)$ which is a contradiction and thus $Az = Bz$.

Finally, suppose $Az \neq z = Bz$ and again using condition (iii) which implies $d(Az, Bx) \leq kd(Az, Bx)$ a contradiction and so $Az = z = Sz$.

Similarly $Bz = z = Tz$, and again A, B, S and T have a common fixed point. Similarly if $y_{2n-1} = y_{2n+1}$. Then we have again a common fixed point for A, B, S and T. Uniqueness of z follows easily from condition (iii). Similarly, we can also complete the proof when T is continuous instead of S is continuous. □

Remark 3.1. Theorem 3.1 still holds if one of A, B, S and T is continuous instead of S or T is continuous.

(i) We list here some of the results in which it is possible to replace weak commutativity or compatibility or compatibility of type(A) by weak compatibility of type(A).

Our theorem includes the corresponding theorems of Chang [2], Fisher [3]-[5], Hadjic [6], Khan and Imdad [12], Khan and Fisher [13], Kubiak [14], Murthy [15], Singh [19], Singh and Tiwari [18], Naidu and Prasad [16].

(ii) Our theorem includes Jungck [9] where $\{A, S\}$ and $\{B, T\}$ are compatible pairs and condition (iii) is repalaced by $d(Ax, By) \leq \delta \max \alpha_{x,y}$ for $x, y \in X$, $\delta \in (0, 1)$, where

$$\alpha_{x,y} = \{d(Ax, Sx), d(By, Ty), d(Sx, Ty), \frac{1}{2}[d(Ax, Ty) + d(By, Sx)]\}.$$

In the above result Jungck assumed S and T are continuous but result still holds if one of A, B, S and T is continuous.

(iii) Our theorem also includes Kang, Cho and Jungck [11], if condition (ii) is replaced by A and S, B and T are compatible mappings and condition (iii) is replaced by

$$d(Ax, By) \leq \phi(d(Ax, Sx), d(By, Ty), d(Sx, Ty), d(Ax, Ty), d(By, Sx))$$

where ϕ is (i) non-decreasing and upper semi-continuous is each coordinate variable and
(ii) for each $t > 0$, $\gamma(t) = \max\{(0, 0, t, t, t), \phi(t, t, t, 2t, 0), \phi(t, t, t, 0, 2t)\} < t$.

The following examples illustrate our main theorem:

Example 1. Let $X = [0, 1]$ with usual metric $d(x, y) = |x - y|$ and we define: A, B, S and $T : [0, 1] \to [0, 1]$ by

$$A(x) = \begin{cases} \frac{1}{4} & x = 0 \\ \frac{x}{2} & x \neq 0 \end{cases} \qquad S(x) = \begin{cases} \frac{1}{4} & x = 0 \\ \frac{3x}{2} & x \neq 0 \end{cases}$$

$$B(x) = \begin{cases} \frac{1}{3} & x = 0 \\ \frac{x}{2} & x \neq 0 \end{cases} \qquad T(x) = \begin{cases} \frac{1}{6} & x = 0 \\ x & x \neq 0 \end{cases}$$

Then we have the following:

(i) $A(X) = \left[\frac{1}{4}, \frac{1}{2}\right] \subset \left[\frac{1}{6}, \frac{2}{3}\right] = T(X)$ and $B(X) = \left[\frac{1}{3}, \frac{1}{2}\right] \subset \left[\frac{1}{4}, \frac{3}{4}\right] = S(X)$.

(ii) $\{A, S\}$ and $\{B, T\}$ are weak compatible pairs of type(A) by choosing a non-zero sequence $\{x_n\} \subset X$ which converges to zero. Also we can see that $\{A, S\}$ and $\{B, T\}$ are not weakly commuting pairs.

$$d(AS(0), SA(0)) = |AS(0) - SA(0)| = \left|\frac{1}{8} - \frac{3}{16}\right| = \frac{1}{16} > 0 = d(A(0), S(0))$$

and

$$d(BT(0), TB(0)) = |BT(0) - TB(0)| = \left|\frac{1}{12} - \frac{1}{3}\right| = \frac{1}{4} > \frac{1}{6} = d(B(0), T(0)).$$

(iii) Putting $x = 0$, $y = 0$ and $\frac{1}{24} < k < 1$, then the condition (iii) of Theorem 3.1 verified. Hence we see that all the conditions of the Theorem 3.1 verified except the continuity of S or T and there is no common fixed point of A, B, S and T. Therefore we have seen that the continuity of S or T is necessary in our theorem.

The following example illustrates that weak compatible mappings of type(A) is also necessary in our theorem.

Example 2. Let $X = [0, \infty)$ with usual metric $d(x, y) = |x - y|$ and we define $A = B$, $S = T : X \to X$ by setting $A(x) = \frac{x}{4} + 1$ and $S(x) = x + 1$. Choose that sequence $\{x_n\} \subset X$ such that $x_n \to 0$ as $n \to \infty$. Thus $Ax_n,\ Sx_n \to 1$ if and only if $x_n \to 0$.

We assert that A and S is not a weak compatible mappings of type (A). To see this we have

$$\begin{aligned} d(ASx_n, SSx_n) &= |ASx_n - SSx_n| \\ &= \left|\frac{1}{4}(x_n + 1) + 1 - (x_n + 1) - 1\right| \to \frac{3}{4} \neq 0 \end{aligned}$$

as $x_n \to 0$. All the conditions of Theorem 3.1 satisfied except weak compatibility of type (A) of $\{A, S\}$ and hence A and S do not have common fixed point in X.

Example 3. Let $X = \{0, 1, 2^{-1}, 2^{-2}, 2^{-3}, \cdots\}$ with the usual metric $d(x, y) = |x - y|$ and define A, B, S, and $T : X \to X$ by $A(0) = B(0) = S(0) = T(0) = 0$.

$$A(2^{-n}) = \begin{cases} 2^{-(n+2)} & n \text{ is even} \\ 2^{-(n+3)} & n \text{ is odd} \end{cases} \quad B(2^{-n}) = \begin{cases} 2^{-(n+1)} & n \text{ is even} \\ 2^{-(n+4)} & n \text{ is odd} \end{cases}$$

$$S(2^{-n}) = \begin{cases} 2^{-(n+5)} & n \text{ is even} \\ 2^{-(n+2)} & n \text{ is odd} \end{cases} \quad T(2^{-n}) = \begin{cases} 2^{-(n+4)} & n \text{ is even} \\ 2^{-(n+3)} & n \text{ is odd} \end{cases}$$

Then we have the following:
(i) $A(X) \subset T(X)$ and $B(X) \subset S(X)$
(ii) S or T is continuous.
(iii) $\{A, S\}$ and $\{B, T\}$ are weak compatible pairs of type(A).
(iii) If $x = 2^{-1}$ and $y = 2^{-2}$ in condition (iii) of the Theorem 3.1.
Then we have,

$$\begin{aligned}|2^{-4} - 2^{-3}| \leq k \max\{&|2^{-3} - 2^{-6}|.|2^{-3} - 2^{-3}|, |2^{-3} - 2^{-6}|.|2^{-6} - 2^{-4}|,\\ &|2^{-3} - 2^{-4}|.|2^{-3} - 2^{-3}|, |2^{-6} - 2^{-3}|.|2^{-6} - 2^{-4}|\}\\ &\div \max\{|2^{-3} - 2^{-3}|, |2^{-6} - 2^{-4}|\}\\ 2^{-4} &\leq k \max\{0, 21(2^{-12}), 21(2^{-12})\} \div \max\{0, 3(2^{-6})\}\\ 2^{-4} &\leq 21k.21(2^{-12}) \div 3(2^{-6})\\ 2^{-4} &\leq 7k.2^{-6}\\ 1 &\leq 7k.4^{-1}\end{aligned}$$

for $4(7^{-1}) \leq k < 1$ satisfied condition (iii) of the Theorem 3.1. Hence all the conditions of the Theorem 3.1 verified and zero is the only common fixed point of A, B, S and T.

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