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CHARACTERIZATION OF f-DERIVATIONS OF A BCI-ALGEBRA

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Abstract. In this paper we characterize f-derivations of a BCI-algebra as well as its center.

1. Introduction

In [9], K. Iséki gave the concept of BCI-algebras. In [14], L. Tiande and X. Changchang introduced the class of p-semi-simple BCI-algebras. In [12], Y.B. Jun and X.L. Xin introduced the notion of derivation in BCI-algebras, which is defined in a way similar to the notion in ring theory (see [1, 2, 10, 13]), and investigated some properties related to this concept. In [15] J. Zhan and Y.L. Liu introduced the notion of f-derivation in BCI-algebras. In particular, they studied the regular f-derivations in detail and gave a characterization of regular f-derivations and characterized p-semisimple BCI-algebras using the notion of regular f-derivation. In this paper we characterize f-derivations of a BCI-algebra as well as its center.

2. Preliminaries

Definition 2.1. ([9]) Let X be an abstract algebra of type $(2, o)$ with a binary operation $*$ and a constant o. Then X is a BCI-algebra, if the following conditions are satisfied for all $x, y, z \in X$,

$$((x * y) * (x * z)) * (z * y) = o, \tag{1.1}$$

$$(x * (x * y)) * y = o, \tag{1.2}$$

$$x * x = o, \tag{1.3}$$

$$x * y = o = y * x \Rightarrow x = y, \tag{1.4}$$

$$x * o = o \Rightarrow x = o. \tag{1.5}$$

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In any BCI-algebra X, one can define a partial order "$\leq$" by putting $x \leq y$ if and only if $x * y = o$. In a BCI-algebra X, the set $M = \{x \in X : o * x = o\}$ is called the BCK-part of X. If $X = M$, then X is called a BCK-algebra.

Moreover, the following properties hold in every BCI-algebra ([8,9]):

Let $x, y, z \in X$:

$$x * o = x, \tag{1.6}$$

$$(x * y) * z = (x * z) * y, \tag{1.7}$$

$$x \leq y \Rightarrow x * z \leq y * z \text{ and } z * y \leq z * x, \tag{1.8}$$

$$(x * y) * (z * y) \leq x * z, \tag{1.9}$$

$$x * (x * (x * y)) = x * y, \tag{1.10}$$

$$o * (x * y) = (o * x) * (o * y) \quad [5]. \tag{1.11}$$

Definition 2.2. ([3]) Let X be a BCI-algebra. An element $x_o \in X$ is said to be an initial element of X, if $x \leq x_o \Rightarrow x = x_o$.

Definition 2.3. ([3]) Let I_x denote the set of all initial elements of X. We call it the center of X. It is well known that the center I_x of a BCI-algebra X is p-semisimple ([4]).

Definition 2.4. ([3]) Let X be a BCI-algebra with I_x as its center. Let $x_o \in I_x$, then the set $A(x_o) = \{x \in X : x_o \leq x\}$ is known as the branch of X determined by x_o.

Definition 2.5. ([3]) A BCI-algebra X is said to be fully-nonassociative if $o * x \neq x$, for all $x \in X - \{o\}$.

Definition 2.6. ([14]) Let X be a BCI-algebra. If $M = \{o\}$ then X is called a p-semisimple BCI-algebra.

(1.12) Let X be a BCI-algebra. The following properties are equivalent for all $x, y \in X$:
(i) X is p-semi-simple,
(ii) $o * (o * x) = x$,
(iii) $x * y = o \Rightarrow x = y$,
for all $x, y, z \in X$ ([6, 14]).

(1.13) Let X be a BCI-algebra. If $x \leq y$, then x, y are contained in the same branch of X ([3]).

(1.14) Let X be a BCI-algebra and $A(x_o) \subseteq X$. Then $x, y \in A(x_o) \Rightarrow x * y, y * x \in M$ ([3]).

(1.15) Let X be a BCI-algebra with I_x as its center. If $x \in A(x_o)$, $y \in A(y_o)$, then $x * y \in A(x_o * y_o)$, for $x_o, y_o \in I_x$ ([7]).

(1.16) Let X be a BCI-algebra with I_x as its center. Let $x_o, y_o \in I_x$. Then for all $y \in A(y_o)$, $x_o * y = x_o * y_o$ ([7]).

Definition 2.7. ([12]) Let X be a BCI-algebra. By a left-right derivation (briefly, (l,r)-derivation) of X, a self map d of X satisfying the identity $d(x*y) = (d(x)*y)\wedge(x*d(y))$, for all $x,y\in X$. If d satisfies the identity $d(x*y) = (x*d(y))\wedge(d(x)*y))$, for all $x,y\in X$ then we say that d is a right-left derivation (briefly, (r,l)-derivation) of X. Moreover, if d is both (l,r)- and (r,l)-derivation, it is said that d is a derivation of X.

Definition 2.8. ([12]) A self-map d_f of a BCI-algebra X is said to be regular if $d(o) = o$.

Definition 2.9. ([15]) Let X be a BCI-algebra. By a left-right f-derivation (briefly, (l,r)-derivation) of X, a self map d_f of X satisfying the identity $d_f(x*y) = (d_f(x)*f(y))\wedge(f(x)*d_f(y))$, for all $x,y\in X$ is meant, where f is an endomorphism of X. If d_f satisfies the identity $d_f(x*y) = (f(x)*d_f(y))\wedge(d_f(x)*f(y)))$, for all $x,y\in X$ then it is said that d_f is a right-left f-derivation (briefly, (r,l)-f-derivation) of X. Moreover, if d_f is both (l,r)- and (r,l)-f-derivation, it is said that d_f is an f-derivation of X.

Definition 2.10. ([12]) A self-map of a BCI-algebra X is said to be regular if $d_f(o) = o$. If $d_f(o)\neq o$, we call d_f an irregular f-derivation.
(1.17) Let d_f be a regular derivation of a BCI-algebra X. Then, for all $x,y\in X$,
(i) $d_f(x)\leq f(x)$,
(ii) $d_f(x*y) = d_f(x)*f(y)$ ([15]).

Definition 2.11. ([15]) A mapping f of a BCI-algebra X into it self is called an endomorphism of X if $f(x*y) = f(x)*f(y)$. Note that $f(o) = o$. Especially, f is monic if for any $x,y\in X$, $f(x) = f(y)\Rightarrow x = y$.

Definition 2.12. ([11]) A BCI-algebra X is said to be commutative if and only if $x\leq y\Rightarrow y*(y*x) = x$, for all $x,y\in X$.

Theorem 2.9. *([15]) Let d_f be a self map of a BCI-algebra X defined by $d_f(x) = o*(o*f(x)) = f_x$. Then d_f is a (l,r)-f-derivation of X. Moreover, if X is a commutative BCI-algebra then d_f is a (r,l)-f-derivation of X, where f is an endomorphism of X.*

Proposition 2.11. *((iii)[15]) Let d_f be a (l,r)-f-derivation of a BCI-algebra X. Then $d_f(x)\in I_x$, for all $x\in I_x$.*

Proposition 2.12. *((ii)[15]) Let d_f be a (r,l)-f-derivation of a BCI-algebra X. Then $d_f(x)\in I_x$, for all $x\in G(x) = \{x\in X : o*x = x\}$.*

In sequel, we will denote $o*(o*f(x)) = f_x$ and $x\wedge y = y*(y*x)$.

3. Characterization of f-derivations of a BCI-algebra

In this section we characterize f-derivations of BCI-algebras.

Lemma 3.1. *Let f be an endomorphism of a BCI-algebra X and I_x be its center. Then for any $x \in I_x$, $f(x) \in I_x$.*

Proof. Let f be an endomorphism of a BCI-algebra X and I_x be its center. Since for $x \in I_x$, $o * (o * x) \in I_x$. So,

$$\begin{aligned} f(x) &= f(o * (o * x)) = f(o) * f(o * x) \quad \text{(since } f \text{ is an endomorphism)} \\ &= o * (f(o) * f(x)) = o * (o * f(x)) \quad \text{(since } f(o) = o\text{)}. \end{aligned}$$

Because of (1.12)-(ii), $f(x) \in I_x$. This completes the proof. □

Lemma 3.2. *Let X be a BCI-algebra. Then for $x \in X$, $o*(o*x)$ and $o*x \in I_x$.*

Proof. Let $x \in X$. By property 1.2, $o * (o * x) \leq x$. Assume that $y \leq o * (o * x)$ which implies that $y * (o * (o * x)) = o$. Now,

$$\begin{aligned} & y \leq o * (o * x) \\ \Rightarrow\ & y * (o * (o * x)) \leq y * y = o \quad \text{(using (1.8))} \\ \Rightarrow\ & y * (o * (o * x)) = o \quad \text{(using (1.5))} \\ \Rightarrow\ & (y * (o * (o * x))) * y = o * y \\ \Rightarrow\ & (y * y) * (o * (o * x)) = o * y \quad \text{(using (1.7))} \\ \Rightarrow\ & o * (o * (o * x)) = o * y \quad \text{(using (1.3))} \\ \Rightarrow\ & o * x = o * y \quad \text{(using (1.10))} \\ \Rightarrow\ & o * (o * x) = o * (o * y) \leq y \\ \Rightarrow\ & (o * (o * x)) * y = o. \end{aligned}$$

Because of property 1.4, $y * (o * (o * x)) = o = (o * (o * x)) * y \Rightarrow o * (o * x) = y$. So, $y \leq o*(o*x) \Rightarrow y = o*(o*x)$. Thus it follows that $o*(o*x) \in I_x$. So there exists some $x_o \in I_x$ such that $o * (o * x) = x_o$, which implies $o * (o * (o * x)) = o * x_o \Rightarrow o * x = o * x_o$ (using (1.10)). Since for o, $x_o \in I_x$, $o * x_o \in I_x$, therefore $o * x \in I_x$. This completes the proof. □

Theorem 3.1. *Let d_f be a (r, l)-f-derivation of a BCI-algebra X. Then $d_f(x) \in I_x$, for all $x \in I_x$.*

Proof. Let $x \in I_x$. Then $x = o * (o * x)$. Since d_f is a (r, l)-f-derivation of a BCI-algebra X, therefore

$$\begin{aligned} d_f(x) = d_f(o * (o * x)) &= (f(o) * d_f(o * x)) \wedge (d_f(x) * f(o))) \\ &= (d_f(x) * o) * ((d_f(x) * o) * (o * d_f(o * x))) \\ &\qquad (\text{since} f(o) = o \quad \text{and} \quad x \wedge y = y * (y * x)) \\ &\leq o * d_f(o * x) \quad \text{(using (1.2))}. \end{aligned}$$

As d_f is a self map, so for $x \in I_x \subseteq X$, $d_f(x)$ and $d_f(o * x) \in X$. Because of lemma 3.2, it follows that for $d_f(o * x) \in X$, $o * d_f(o * x) \in I_x$. So, $d_f(x) \leq o * d_f(o * x) \Rightarrow d_f(x) = o * d_f(o * x)$. Hence for $x \in I_x$, $d_f(x) \in I_x$. This completes the proof. □

Lemma 3.3. *Let d_f be an f-derivation of a BCI-algebra X. Then $d_f(x) \in I_x$, for all $x \in I_x$.*

Proof. It follows directly from theorem 3.1 and part (iii) of proposition 2.11. □

Theorem 3.2. *Let f be an endomorphism of a BCI-algebra X with center I_x. Then for $x, y \in X$, following identities hold:*

(i) $f(x) * f_x \in M$,
(ii) $f_x * m = f_x$, *for all* $m \in M$,
(iii) $f_x * f_y \in I_x$,
(iv) $f_x * y = f_x * f_y = f_{x*y}$.

Proof. (i) Let f be an endomorphism of X. Then for $x \in X$, $f(x) \in X$. By property (1.2),

$$\begin{aligned}(o * (o * f(x))) * f(x) = o &\Rightarrow (o * (o * f(x))) \le f(x)\\ &\Rightarrow f_x \le f(x) \quad (\text{since } \ o * (o * f(x)) = f_x)\\ &\Rightarrow f(x) * f(x) \le f(x) * f_x \Rightarrow o \le f(x) * f_x\end{aligned}$$

which implies that $f(x) * f_x \in M$. Thus there exists some $m \in M$ such that $f(x) * f_x = m$.

(ii)

$$\begin{aligned}f_x * m &= (o * (o * f(x))) * m = (o * m) * (o * f(x)) \quad (\text{using } (1.7))\\ &= o * (o * f(x)) = f_x \quad (\text{since for } \ m \in M, o * m = o).\end{aligned}$$

(iii) Using (1.11) repeatedly,

$$\begin{aligned}f_x * f_y &= (o * (o * f(x))) * (o * (o * f(y)))\\ &= o * ((o * f(x)) * (o * f(y)))\\ &= o * (o * (f(x) * f(y))).\end{aligned}$$

Since f is an endomorphism, therefore above equation becomes

$$f_x * f_y = o * (o * f(x * y)). \tag{1}$$

Because of lemma 3.2, for $f(x * y) \in X$, $o * (o * f(x * y)) \in I_x$. Thus it follows that $f_x * f_y \in I_x$.

(iv)

$$\begin{aligned}f_x * f(y) &= (o * (o * f(x))) * f(y) = (o * f(y)) * (o * f(x)) \quad (\text{using } (1.7))\\ &= (o * (o * (o * f(y))) * (o * f(x)) \quad (\text{using } (1.10))\\ &= (o * (o * f(x))) * (o * (o * f(y))\\ &= f_x * f_y.\end{aligned} \tag{2}$$

From $o * (o * f(x)) = f_x$ and equation (1) of (iii) it follows

$$f_x * f_y = f_{x*y}. \tag{3}$$

Thus from equations (2) and (3) it follows that $f_x * f(y) = f_x * f_y = f_{x*y}$. This completes the proof. □

Theorem 3.3. *Let f be an endomorphism and $A(x_o)$ be any branch of a BCI-algebra X. If for any $x \in A(x_o)$, $f(x) = x_o$, then f is a regular derivation.*

Proof. Let f be an endomorphism and $A(x_o)$ be any branch of a BCI-algebra X. According to given condition

$$\text{for any } x \in A(x_o), \ \ f(x) = x_o. \tag{1}$$

For $x \in A(x_o)$, $x_o \leq x \Rightarrow x_o * x = o$. By 1.15 for x_o, $x \in A(x_o)$, $x_o * x = o$ and $x * x_o \in M$. So, for some $m \neq o \in M = A(o)$, $x * x_o = m$, otherwise, $x_o * x = o = x * x_o \Rightarrow x = x_o$, a contradiction. Also by 1.16 for o, $x_o \in I_x$ and $m \neq o \in M = A(o)$, $x_o * m = x_o * o = x_o$. Now for distinct $x, y \in X$, we have following two cases:

Case 1: Both x and y belongs to the same branch of X.

Case 2: x and y belongs to different branches of X.

Case 1: Let $x, y \in A(x_o)$. So, $x_o \leq x$ and $x_o \leq y$. Then by (1.15), $x * y \in M = A(o)$. So using (1),

$$f(x * y) = o. \tag{2}$$

Also

$$x_o \leq y \Rightarrow x_o * y = o. \tag{3}$$

Now

$$\begin{aligned}
&(f(x) * y) \wedge (x * f(y)) \\
&\quad = (x_o * y) \wedge (x * y_o) \quad (using\ (1)) \\
&\quad = (x * y_o) * ((x * y_o) * (x_o * y)) \quad (since\ x \wedge y = y * (y * x)) \\
&\quad = (x * y_o) * ((x * (x_o * y)) * y_o) \quad (using\ (1.7)) \\
&\quad \leq x * (x * (x_o * y)) \leq x_o * y = o \quad (using\ (1.9),\ (1.2)\ and\ equation\ (3)) \\
&\Rightarrow (f(x) * y) \wedge (x * f(y)) = o \quad (using\ (1.5)) \\
&\Rightarrow (f(x) * y) \wedge (x * f(y)) = o = f(x * y) \quad (using\ (2)).
\end{aligned}$$

Thus f is an (l, r)-derivation. Now

$$\begin{aligned}
(x * f(y)) \wedge (f(x) * y) &= (x * x_o) \wedge (x_o * y) \\
&= (x * x_o) \wedge o \quad (using\ (3)) \\
&= o * (o * (x * x_o)) \\
&= o * (o * m) \\
&= o \quad (since\ x * x_o = m, o * m = o).
\end{aligned}$$

$$(x * f(y)) \wedge (f(x) * y) = o = f(x * y). \quad (using\ (2))$$

So, f is an (r, l)-derivation. Hence it follows f is a derivation.

Case 2: Let $x \in A(x_o)$ and $y \in A(y_o)$. Then by (1.15), $x * y \in A(x_o * y_o)$. So, using (1)

$$f(x * y) = x_o * y_o. \tag{4}$$

As x_o, $y_o \in I_x$ and $y \in A(y_o)$, therefore by (1.16),

$$x_o * y = x_o * y_o. \tag{5}$$

Now,

$$\begin{aligned}
&(f(x) * y) \wedge (x * f(y)) \\
&= (x_o * y) \wedge (x * y_o) \quad (using\ (1)) \\
&= (x * y_o) * ((x * y_o) * (x_o * y)) \quad (since\ x \wedge y = y * (y * x)) \\
&= (x * y_o) * ((x * (x_o * y)) * y_o) \quad (using\ (1.7)) \\
&\leq x * (x * (x_o * y)) \quad (using\ (1.9))) \\
&\leq x_o * y = x_o * y_o \quad (using\ (1.2)\ and\ (5)).
\end{aligned}$$

Since for x_o, $y_o \in I_x$, $x_o * y_o \in I_x$, therefore $x_o * y_o$ is an initial element. Thus it follows that

$$\begin{aligned}
(f(x) * y) \wedge (x * f(y)) &= x_o * y_o \\
&= f(x * y) \quad (using\ (4)).
\end{aligned}$$

So, f is an (l, r)-derivation.
Also

$$\begin{aligned}
&(x * f(y)) \wedge (f(x) * y) \\
&\quad = (x * y_o) \wedge (x_o * y) \quad (using\ (1)) \\
&\quad = (x_o * y) * ((x_o * y) * (x * y_o)) \\
&\quad = (x_o * y) * ((x_o * (x * y_o)) * y) \quad (using\ (1.7)) \\
&\quad \leq x_o * (x_o * (x * y_o)) \leq x * y_o \quad (using\ (1.9)\ and\ (1.2)) \\
&\Rightarrow (x * f(y)) \wedge (f(x) * y) = (x * y_o) \wedge (x_o * y) \leq x * y_o \\
&\Rightarrow ((x * y_o) \wedge (x_o * y)) * (x_o * y_o) \leq (x * y_o) * (x_o * y_o) \quad (using\ (1.8)) \\
&\Rightarrow ((x * y_o) \wedge (x_o * y)) * (x_o * y_o) \leq x * x_o \quad (using\ (1.9)) \\
&\Rightarrow ((x * y_o) \wedge (x_o * y)) * (x_o * y_o) \leq m \quad (since\ x * x_o = m) \\
&\Rightarrow ((x * y_o) \wedge (x_o * y)) * (x_o * y_o)) * m \leq m * m = o \quad (using\ (1.2)\ and\ (1.8)) \\
&\Rightarrow ((x * y_o) \wedge (x_o * y)) * (x_o * y_o)) * m = o \quad (using\ (1.5)) \\
&\Rightarrow ((x * y_o) \wedge (x_o * y)) * m) * (x_o * y_o) = o \quad (using\ (1.7)) \\
&\Rightarrow ((x * y_o) \wedge (x_o * y)) * m) \leq x_o * y_o \\
&\Rightarrow ((x_o * y) * ((x_o * y) * (x * y_o))) * m) \leq x_o * y_o \quad (since\ x \wedge y = y * (y * x)).
\end{aligned}$$

Since for x_o, $y_o \in I_x$, $x_o * y_o \in I_x$, therefore $x_o * y_o$ is an initial element. Thus it follows that

$$((x_o * y) * ((x_o * y) * (x * y_o))) * m) = x_o * y_o$$

$$\begin{aligned}\Rightarrow x_o * y_o &= ((x_o * y) * m) * ((x_o * y) * (x * y_o)) \quad (using\ (1.7))\\ &= ((x_o * m) * y) * ((x_o * y) * (x * y_o)) \quad (using\ (1.7))\\ &= (x_o * y) * ((x_o * y) * (x * y_o)) \quad (since\ x_o * m = x_o)\\ &= (x * y_o) \wedge (x_o * y) = (x * f(y)) \wedge (f(x) * y)\\ \Rightarrow f(x * y) &= (x * f(y)) \wedge (f(x) * y).\end{aligned}$$

So, f is an (r, l)-derivation. Thus it follows that f is a derivation of X. Since $f(o) = o$, therefore f is a regular f-derivation. This completes the proof. □

Theorem 3.4. *Let f be an endomorphism and d_f be a self map of a BCI-algebra X. If for $x \in X$, $d_f(x) = f(x)$, then d_f is an f-derivation.*

Proof. Let f be an endomorphism and d_f be a self map of a BCI-algebra X. Assume that for any $x \in X$,

$$d_f(x) = f(x). \tag{1}$$

Then

$$d_f(y) = f(y). \tag{2}$$

Since for $x, y \in X$, $x * y \in X$, therefore $d_f(x * y) = f(x * y)$. Thus

$$\begin{aligned}d_f(x * y) = f(x * y) &= f(x) * f(y)\\ &= d_f(x) * f(y) \quad (using\ (1))\\ &= (d_f(x) * f(y)) * o.\end{aligned} \tag{3}$$

Now

$$\begin{aligned}o &= f(x * y) * f(x * y)\\ &= (f(x) * f(y)) * (f(x) * f(y)) \quad (since\ f\ is\ an\ endomorphism)\\ &= (d_f(x) * f(y)) * (f(x) * d_f(y)) \quad (using\ (1)).\end{aligned}$$

So equation (3) becomes

$$\begin{aligned}d_f(x * y) &= (d_f(x) * f(y)) * (d_f(x) * f(y)) * (f(x) * d_f(y))\\ &= (f(x) * d_f(y)) \wedge (d_f(x) * f(y)) \quad (since x \wedge y = y * (y * x)).\end{aligned}$$

So, d_f is an (r, l)-f-derivation. Further,

$$\begin{aligned}d_f(x * y) = f(x * y) = f(x) * f(y) &= f(x) * d_f(y) (using\ (2))\\ &= (f(x) * d_f(y)) * o.\end{aligned} \tag{4}$$

Now

$$\begin{aligned}o = f(x * y) * f(x * y) &= (f(x) * f(y)) * (f(x) * f(y))\\ &= (f(x) * d_f(y)) * (d_f(x) * f(y)) \quad (using\ (1)\ and\ (2)).\end{aligned}$$

So equation (4) becomes

$$\begin{aligned}&d_f(x * y)\\ &= (f(x) * d_f(y)) * (f(x) * d_f(y)) * (d_f(x) * f(y))\\ &= (d_f(x) * f(y)) \wedge (f(x) * d_f(y)) \quad (since\ x \wedge y = y * (y * x)).\end{aligned}$$

Thus d_f is an (l,r)-f-derivation. Hence d_f is an f-derivation. □

Theorem 3.5. *Let d_f be a f-derivation of a BCI-algebra of X. If $d_f(m) = o$, for all $m \in M = A(o)$, then $d_f(x) = f(x_o)$, for all $x \in A(x_o)$.*

Proof. If $d_f(m) = o$, for all $m \in M = A(o)$, then $d_f(o) = o$. So d_f is a regular derivation of X. Because of (1.17)-(i), for all $x \in A(x_o) \subseteq X$,

$$d_f(x) \leq f(x). \tag{1}$$

For any $x \in A(x_o)$, $x_o \leq x \Rightarrow x_o * x = o$. So,

$$\begin{aligned} &o = f(o) = f(x_o * x) = f(x_o) * f(x) \quad (\textit{since } f \textit{ is an endomorphism.}) \\ &\Rightarrow f(x_o) \leq f(x). \end{aligned} \tag{2}$$

By lemma 3.1, for $x_o \in I_x$, $f(x_o) \in I_x$, so $f(x_o)$ is an initial element. Thus from (2) it follows that $f(x) \in A(f(x_o)$. Because of (1.13), from (1) and (2) it follows that $f(x_o)$, $f(x)$ and $d_f(x)$ belong to the same branch. So

$$f(x_o) \leq d_f(x) \Rightarrow f(x_o) * d_f(x) = o. \tag{3}$$

For $x \in A(x_o)$, $x_o \leq x \Rightarrow x_o * x = o$. By (1.14) for x_o, $x \in A(x_o)$, $x_o * x = o$ and $x * x_o \in M$. So, for some $m \neq o \in M = A(o)$, $x * x_o = m$, otherwise, $x_o * x = o = x * x_o \Rightarrow x = x_o$, a contradiction. Now by (1.17)-(ii), for x_o, $x \in A(x_o) \subseteq X$,

$$\begin{aligned} &d_f(x) * f(x_o) = d_f(x * x_o) = d_f(m) = o \quad (\textit{using given condition}) \\ &\Rightarrow d_f(x) \leq f(x_o) \Rightarrow d_f(x) * f(x_o) = o. \end{aligned} \tag{4}$$

Because of (1.4), from (3) and (4) it follows that $d_f(x) = f(x_o)$. This completes the proof. □

Theorem 3.6. *Let d_f be a f-derivation of a commutative BCI-algebra X, where f is an endomorphism of X. Then $x \leq y \Rightarrow d_f(x) \leq d_f(y)$.*

Proof. Let d_f be a f-derivation of a commutative BCI-algebra X, where f is an endomorphism of X. Since X is a commutative BCI-algebra, therefore $x \leq y \Rightarrow y * (y * x) = x$. So,

$$\begin{aligned} d_f(x) &= d_f(y * (y * x)) \\ &= (d_f(y) * f(y * x)) \wedge (f(y) * d_f(y * x)) \\ &= (f(y) * d_f(y * x)) * ((f(y) * d_f(y * x)) * (d_f(y) * f(y * x))) \\ &\leq d_f(y) * f(y * x) \end{aligned}$$

$$\Rightarrow d_f(x) \leq d_f(y) * (f(y) * f(x)) \quad (A)(\textit{since } f \textit{ is an endomorphism}).$$

Since $x \leq y \Rightarrow x * y = o$, therefore $o = f(o) = f(x * y) = f(x) * f(y) \Rightarrow f(x) \leq f(y)$. By (1.13), $f(x)$ and $f(y)$ belong to the same branch of X and by (1.14), $f(x)*f(y)$ and $f(y)*f(x) \in M$. As $f(x)*f(y) = o$, so $f(y)*f(x) \neq o$, otherwise because of property (1.4), $f(x) * f(y) = o = f(y) * f(x) \Rightarrow f(x) = f(y)$, a contradiction. Thus there exists some $m \neq o \in M$ such that $f(y) * f(x) = m$. Hence inequality (A) becomes

$$\begin{aligned}
&d_f(x) \leq d_f(y) * m \\
&\Rightarrow d_f(x) * d_f(y) \leq (d_f(y) * m) * d_f(y) \quad (using\ (1.8)) \\
&\Rightarrow d_f(x) * d_f(y) \leq (d_f(y) * d_f(y)) * m \quad (using (1.7)) \\
&\Rightarrow d_f(x) * d_f(y) \leq o * m \quad (using\ (1.3)) \\
&\Rightarrow d_f(x) * d_f(y) \leq o \quad (since m \in M) \\
&\Rightarrow d_f(x) * d_f(y) = o \quad (using\ (1.5))
\end{aligned}$$

which shows that $d_f(x) \leq d_f(y)$. This completes the proof. □

Theorem 3.7. *Let d_f be a f-derivation of a BCI-algebra X, where f is an endomorphism of X. If for all $x \in A(x_o)$, $y \in A(y_o)$, $f(x) \in A(x_o)$ then $d_f(x) = y_o \Rightarrow d_f(y) \in A(x_o)$.*

Proof. Assume that for all $x \in A(x_o)$, $d_f(x) = y_o$. Since d_f is a f-derivation of BCI-algebra X, therefore d_f is a (l, r)-f-derivation. So for all $x \in A(x_o)$, $y \in A(y_o)$,

$$\begin{aligned}
&d_f(x * y) \\
&= (d_f(x) * f(y)) \wedge (f(x) * d_f(y)) \\
&= (f(x) * d_f(y)) * ((f(x) * d_f(y)) * (d_f(x) * f(y))) \ \ (since\ x \wedge y = y * (y * x)) \\
&\leq d_f(x) * f(y) \quad (using\ (1.2)) \\
&\Rightarrow d_f(x * y) \leq y_o * f(y) \quad (since\ d_f(x) = y_o).
\end{aligned}$$

According to given condition $y \in A(y_o)$,$f(y) \in A(y_o)$. So by definition 2.4, $f(y) \in A(y_o) \Rightarrow y_o \leq f(y)$. Now $y_o \leq f(y) \Rightarrow y_o * f(y) = o$. So above inequality becomes

$$d_f(x * y) \leq o \Rightarrow d_f(x * y) = o. \tag{1}$$

As d_f is also a (r, l)-f-derivation. So for all $x \in A(x_o)$, $y \in A(y_o)$,

$$\begin{aligned}
&d_f(x * y) = (f(x) * d_f(y)) \wedge (d_f(x) * f(y)) \\
&d_f(x * y) = (d_f(x) * f(y)) * ((d_f(x) * f(y))) * (f(x) * d_f(y))) \\
&\qquad\qquad\qquad\qquad\qquad\qquad (since\ x \wedge y = y * (y * x)) \\
&\qquad\quad \leq f(x) * d_f(y) \quad (using\ (1.2)).
\end{aligned} \tag{2}$$

Since $d_f$is both (l, r)- and (r, l)-f-derivation. So from (1) and (2), it follows that

$$\begin{aligned}
&o \leq f(x) * d_f(y) \\
&\Rightarrow o * f(x) \leq (f(x) * d_f(y)) * f(x) \quad (using\ (1.8))
\end{aligned}$$

$\Rightarrow o * f(x) \leq (f(x) * f(x)) * d_f(y) \quad (using\ (1.7))$
$\Rightarrow o * f(x) \leq o * d_f(y) \quad (using\ (1.3))$
$\Rightarrow o * (o * d_f y)) \leq o * (o * f(x)) = f(x_o) \quad (using\ theorem 4.1 - (iii))$
$\Rightarrow o * (o * d_f(y)) = f(x_o) \quad (since\ f(x_o) \in I_x)$
$\Rightarrow f(x_o) \leq d_f(y) \quad (using(1.2)).$

By (1.13) both $d_f(y)$ and $f(x_o)$ belong to the same branch of X. Since $f(x_o) \in A(x_o)$, therefore $d_f(y) \in A(x_o)$. This completes the proof. □

Theorem 3.8. *Let d_f be a f-derivation of a BCI-algebra X. If for distinct $x, y \in I_x$, $d_f(x) = f(y) \Rightarrow d_f(y) = f(x)$, then d_f is not regular.*

Proof. Assume that for distinct $x, y \in I_x$,

(1) $$d_f(x) = f(y) \Rightarrow d_f(y) = f(x).$$

Since d_f is a f-derivation of X, therefore d_f is a (l, r)-f-derivation as well as (r, l)-f-derivation. When d_f is a (r, l)-f-derivation, then

$$\begin{aligned} d_f(x * y) &= (d_f(x) * f(y)) \wedge (f(x) * d_f(y)) \\ &= (f(x) * d_f(y)) * ((f(x) * d_f(y)) * (d_f(x) * f(y))) \\ &\leq d_f(x) * f(y) = f(y) * f(y) = o \quad (using\ d_f(x) = f(y)\ and\ (1.3)) \\ &\Rightarrow d_f(x * y) = o \quad (using(1.5)). \end{aligned}$$

Since for $x, y \in I_x$, $x * y \in I_x$ and $x * y \neq o$, otherwise by (1.12)-(ii), $x * y = o \Rightarrow x = y$, a contradiction, therefore by our assumption

$$d_f(x * y) = o = f(o) \Rightarrow d_f(o) = f(x * y).$$

Thus it follows that $d_f(o) \neq o$ as $x * y \neq o$, so d is not regular. This completes the proof. □

Note that Lemma 3.3 of this paper generalizes the part (iii) of proposition 2.11[15] and from Lemma 3.2, it follows that $G(X) \subseteq I_x$, thus part (ii) of proposition 2.12[15] becomes a corollary of our Lemma 3.1. Now we generalize the first half of theorem 2.9 and first part of proposition 3.3 of [15] vide theorem 3.9 and theorem 3.10 respectively.

Theorem 3.9. *A self map d_f of a BCI-algebra X defined as $d_f(x) = o * (o * f(x)) = f_x$, for all $x \in X$, is an f-derivation of X, where f is an endomorphism of X.*

Proof. Let d_f be a self map of a BCI-algebra X, where f is an endomorphism of X, defined as follows:

(1) $$d_f(x) = o * (o * f(x)) = f_x$$

for all $x \in X$. As for $x, y \in X$, $x*y \in X$, therefore $d_f(x*y) = o*(o*f(x*y)) = f_{x*y}$ which implies that

(2) $$d_f(x * y) = f_x * f_y \quad (using\ theorem\ 3.2, (iv)).$$

Now

$$
\begin{aligned}
&(f(x) * d_f(y)) \wedge (d_f(x) * f(y)) \\
&= (f(x) * f_y) \wedge (f_x * f(y)) \quad (using\ equation\ (1)) \\
&= (f_x * f(y)) * ((f_x * f(y)) * (f(x) * f_y)) \quad (since\ x \wedge y = y * (y * x)) \\
&= (f_x * f(y)) * ((f_x * (f(x) * f_y)) * f(y)) \quad (using\ (1.7)) \\
&\leq f_x * (f_x * (f(x) * f_y)) \quad (using\ (1.9)) \\
&\leq f(x) * f_y \quad (using\ (1.2)) \\
&\Rightarrow (f(x) * d_f(y)) \wedge (d_f(x) * f(y)) = (f(x) * f_y) \wedge (f_x * f(y)) \leq f(x) * fy \\
&\Rightarrow ((f(x) * f_y) \wedge (f_x * f(y))) * (f_x * f_y) \\
&\quad \leq (f(x) * f_y) * (f_x * f_y) \quad (using\ (1.8)) \\
&\Rightarrow \leq f(x) * f_x \quad (using\ (1.9)) \\
&\Rightarrow ((f(x) * f_y) \wedge (f_x * f(y))) * (f_x * f_y) \leq m \quad (By\ Theorem 3.2, (i)) \\
&\Rightarrow (((f(x) * f_y) \wedge (f_x * f(y))) * (f_x * f_y)) * m \leq m * m \quad (using\ (1.8)) \\
&\Rightarrow (((f(x) * f_y) \wedge (f_x * f(y))) * m) * (f_x * f_y) \leq o \quad (using\ (1.3)) \\
&\Rightarrow (((f(x) * f_y) \wedge (f_x * f(y))) * m) * (f_x * f_y) = o \quad (using\ (1.5)) \\
&\Rightarrow ((f(x) * f_y) \wedge (f_x * f(y))) * m \leq (f_x * f_y).
\end{aligned}
$$

Since $f_x * f_y \in I_x$, therefore $f_x * f_y$ is an initial element. Thus it follows that

$$
\begin{aligned}
&((f(x) * f_y) \wedge (f_x * f(y))) * m = (f_x * f_y) \\
&\Rightarrow ((f_x * f(y)) * ((f_x * f(y)) * (f(x) * f_y))) * m \\
&\quad = f_x * f_y (since\ x \wedge y = y * (y * x)) \\
&\Rightarrow ((f_x * f(y)) * m) * ((f_x * f(y)) * (f(x) * f_y)) = f_x * f_x \quad (using\ (1.7)) \\
&\Rightarrow ((f_x * m) * f(y)) * ((f_x * f(y)) * (f(x) * f_y)) = f_x * f_y \quad (using\ (1.7)) \\
&\Rightarrow ((f_x * f(y))) * ((f_x * f(y)) * (f(x) * f_y)) \\
&\quad = f_x * f_y \quad (using\ Theorem\ 3.2, (ii)) \\
&\Rightarrow (f(x) * d_f(y)) \wedge (d_f(x) * f(y) = f_x * f_y = d_f(x * y) \quad (using\ equation\ (2)).
\end{aligned}
$$

Thus d_f is an (r, l)-f-derivation.
From first half of theorem 2.9[15], it follows that d_f is a (l, r)-f-derivation of X. Thus it follows d_f is both (l, r)- and (r, l)-f-derivations of X, therefore d_f is a f-derivation of X. □

Theorem 3.10. *Let X be a BCI-algebra and let d_f be a regular (r, l)-f-derivation of X. Then both $f(x)$ and $d_f(x)$ belongs to the same branch of X.*

Proof. Let $x \in X$. Then by property (1.2), $o*(o*x) \leq x$. As shown in Lemma 3.2, $o*(o*x) = x_o$, for some $x_o \in I_x$. So for any $x \in X$, $o*(o*x) \leq x \Rightarrow x_o \leq x$

which implies $x \in A(x_o)$ and $x_o * x = o$. Since d_f is a regular (r,l)-f-derivation of X, therefore

$$\begin{aligned} o = d_f(o) &= d_f(x_o * x) = (f(x_o) * d_f(x)) \wedge (d_f(x_o) * f(x)) \\ &= (d_f(x_o) * f(x)) * ((d_f(x_o) * f(x)) * ((f(x_o) * d_f(x))) \\ &\qquad\qquad\qquad\qquad (\textit{since } x \wedge y = y * (y * x)) \\ &\leq f(x_o) * d_f(x) \quad (\textit{using } (1.2)) \\ &\Rightarrow f(x_o) * (f(x_o) * d_f(x))) \leq f(x_o) * o = f(x_o) \quad (\textit{using } (1.8) \textit{ and } (1.6)). \end{aligned}$$

Because of lemma 3.1, for $x_o \in I_x$, $f(x_o) \in I_x$. Thus it follows that $f(x_o)$ is an initial element. So,

$$f(x_o) * (f(x_o) * d_f(x))) \leq f(x_o) \Rightarrow f(x_o) * (f(x_o) * d_f(x))) = f(x_o).$$

Now by property (1.2),

$$(A) \qquad f(x_o) = f(x_o) * (f(x_o) * df(x)) \leq d_f(x).$$

Also,

$$(B) \qquad o = f(x_o * x) = f(x_o) * f(x) \Rightarrow f(x_o) \leq f(x).$$

Because of (1.13), from inequalities (A) and (B), it follows $f(x)$ and $d_f(x)$ belongs to the same branch of X. This completes the proof. □

4. Characterization of f- f-derivations of the center of a BCI-algebras

In this section we characterize the role of f-derivations of the center of BCI-algebra.

Lemma 4.1. *Let f be an endomorphism of a BCI-algebra X. Then,*
(i) for all $x \in A(x_o)$, $f(x_o)$ and $f(x)$ belong to the same branch of X.
(ii) for all $x \in A(x_o)$, $o * f(x) \in I_x$.
(iii) for all $x \in A(x_o)$, $f(x_o) = o * (o * f(x))$.
(iv) for all $m \in A(o)$, $f(m) \in A(o) = M$.

Proof. (i) Let f be an endomorphism of a BCI-algebra X. Then, for $x \in A(x_o)$, $x_o \leq x \Rightarrow x_o * x = o$. So that,

$$(1) \qquad o = f(o) = f(x_o * x) = f(x_o) * f(x) \Rightarrow f(x_o) \leq f(x).$$

Because of (1.13), from inequality (1) it follows that $f(x_o)$ and $f(x)$ belong to the same branch of X.

(ii) From the proof of part (i) it follows that for $x \in A(x_o)$, $f(x_o) \leq f(x)$. Because of (1.8) it implies that $o * f(x) \leq o * f(x_o)$. Since for $x_o \in I_x$, $f(x_o) \in I_x \subseteq X$, therefore by lemma 3.2, $o * f(x_o) \in I_x$. Thus it follows that $o * f(x_o)$ is an initial point. So, $o * f(x) = o * f(x_o)$. Hence $o * f(x) \in I_x$.

(iii) From the proof of part (ii) it follows that

$o * f(x) = o * f(x_o)$
$\Rightarrow o * (o * f(x)) = o * (o * f(x_o))$
$\Rightarrow o * (o * f(x)) = f(x_o) \quad (since\ f(x_o) \in I_x,\ therefore\ using\ (1.12) - (ii).$

(iv) From the proof of part (i) it follows that for all $m \in A(o) = M$, $f(o) \leq f(m) \Rightarrow o \leq f(m) \Rightarrow f(m) \in A(o) = M$. This completes the proof.

□

Theorem 4.2. *Let d_f be an irregular f-derivation of a BCI-algebra X, where f is an endomorphism of X. Then the center I_x of X is not fully non-associative.*

Proof. Let d_f be an irregular f-derivation and I_x be the center of a BCI-algebra X. Since d_f is a f-derivation of X, therefore d_f is a (l, r)-f-derivation. So,

$$\begin{aligned} d_f(o) &= d_f(o * o) = (d_f(o) * f(o)) \wedge (f(o) * d_f(o)) \\ &= (d_f(o) * o) \wedge (o * d_f(o)) \quad (using\ f(o) = o) \\ &= d_f(o) \wedge (o * d_f(o)) \quad (using\ (1.6)) \\ &= (o * d_f(o)) * ((o * d_f(o)) * d_f(o)) \quad (since\ x \wedge y = y * (y * x)) \\ &\leq o * (o * d_f(o)) \leq d_f(o) \quad (using\ (1.9)\ and\ (1.2)) \\ &\Rightarrow d_f(o) = o * (o * d_f(o)) \quad (1). \end{aligned}$$

Because of (1.12)-(ii), it implies that $d_f(o) \in I_x$. Since d_f is an irregular f-derivation therefore $d_f(o) \neq o$. Thus there exists some $x_o \neq o \in I_x$ such that

$$d_f(o) = x_o. \tag{2}$$

Since d_f is a f-derivation of X, therefore d_f is a (r, l)-f-derivation. So,

$$\begin{aligned} d_f(o) &= d_f(o * o) = (f(o) * d_f(o)) \wedge (d_f(o) * g(o))) \\ &= (o * d_f(o)) \wedge (d_f(o) * o) \quad (since\ f\ is\ an\ endomorphism) \\ &= (o * d_f(o)) \wedge d_f(o) \\ &= d_f(o) * (d_f(o) * (o * d_f(o))) \leq o * d_f(o). \end{aligned}$$

By lemma 3.2, for $d_f(o) \in I_x \subseteq X$, $o * d_f(o) \in I_x$. Thus $o * d_f(o)$ is an initial point. Hence,

$$d_f(o) = o * d_f(o). \tag{3}$$

Since d_f is a f-derivation, therefore from equations (1), (2) and (3),

$$x_o = o * x_o \Rightarrow o * x_o = o * (o * x_o).$$

Because of (1.12)-(ii), it implies $x_o = o * x_o$. Hence it follows the center I_x of X is not fully non-associative. □

Theorem 4.3. *Let d_f be a f-derivation of a BCI-algebra X. If its center I_x is fully non-associative and for all $x \in A(x_o)$, $f(x) \in A(x_o)$ then d_f is regular.*

Proof. Contrarily assume that d_f is not regular. So, $d_f(o) \neq o$. By lemma 3.3, $d_f(o) \in I_x$, thus there exists some $x_o \neq o \in I_x$ such that

$$d_f(o) = x_o. \tag{1}$$

Since d_f is a derivation, therefore d_f is a (l, r)-derivation as well as (r, l)-f-derivation. When d_f is a (l, r)-derivation then

$$\begin{aligned} & d_f(o * x_o) \\ & = (d_f(o) * f(x_o)) \wedge (f(o) * d_f(x_o)) \\ & = (d_f(o) * f(x_o)) \wedge (o * d_f(x_o)) \\ & = (o * d_f(x_o)) * ((o * d_f(x_o)) * (d_f(o) * f(x_o))) \quad (since\ x \wedge y = y * (y * x)) \\ & \leq d_f(o) * f(x_o). \end{aligned}$$

By lemma 3.3 and 3.1, for o, $x_o \in I_x$, $d_f(o)$ and $f(x_o) \in I_x$. Thus $d_f(o)*f(x_o) \in I_x$. So, Above inequality becomes

$$\begin{aligned} & d_f(o * x_o) = d_f(o) * f(x_o) \quad (since\ d_f(o) * f(x_o) \in I_x) \\ & \Rightarrow d_f(o * x_o) = x_o * f(x_o) \quad (using\ (1)). \end{aligned} \tag{2}$$

Since $f(x) \in A(x_o)$, therefore by lemma 3.1, $f(x_o) \in A(x_o)$ which implies that

$$x_o \leq f(x_o) \Rightarrow x_o = f(x_o) \quad (since\ f(x_o) \in I_x).$$

So, equation (2) implies

$$d_f(o * x_o) = x_o * x_o = o. \tag{3}$$

Since d_f is also a (r, l)-f-derivation. So for all $x \in A(x_o)$,

$$\begin{aligned} & d_f(o * x_o) \\ & = (f(o) * d_f(x_o)) \wedge (d_f(o) * f(x_o)) \\ & = (d_f(o) * f(x_o)) * ((d_f(o) * f(x_o))) * (o * d_f(x_o)))(since\ x \wedge y = y * (y * x)) \\ & \leq o * d_f(x_o)(using(1.2)) \end{aligned}$$

$$\Rightarrow d_f(o * x_o) = o * d_f(x_o) \quad (since\ o * f(x_o) \in I_x). \tag{4}$$

As d_f is a f-derivation. So from (3) and (4), it follows that

$$o = o * d_f(x_o) \Rightarrow o \leq d_f(x_o) \Rightarrow o = d_f(x_o) \quad (since\ d_f(x_o) \in I_x).$$

Since I_x is fully non-associative, therefore there exists some $y_o \neq x_o \in I_x$ such that $o * y_o = x_o$. So,

$$\begin{aligned} o & = d_f(x_o) = d_f(o * y_o) = (d_f(o) * f(y_o)) \wedge (f(o) * d_f(y_o)) \\ & = d_f(o * y_o) = (d_f(o) * f(y_o)) \wedge (o * d_f(y_o)) \\ & = (o * d_f(y_o)) * ((o * df(y_o)) * (d_f(o) * f(y_o))) \quad (since\ x \wedge y = y * (y * x)) \\ & \leq d_f(o) * f(y_o) \Rightarrow o = d_f(o) * f(y_o) \quad (since d_f(o) * f(y_o) \in I_x) \end{aligned}$$

$$\Rightarrow o = x_o * f(y_o) \quad (using\ (1)). \tag{5}$$

According to given condition $f(y_o) \in A(y_o)$. So, $y_o \leq f(y_o)$. Since $f(y_o) \in I_x$. So, $f(y_o)$ is an initial element. Hence $y_o = f(y_o)$. Thus equation (5) becomes

$$o = x_o * y_o \Rightarrow x_o \leq y_o \Rightarrow x_o = y_o \quad (using\ (1.12) - (iii))$$

a contradiction. Thus our assumption is wrong. Hence d_f is regular.

□

Theorem 4.4. *Let d_f be a f-derivation of a BCI-algebra X. If its center I_x is fully non-associative and for all $x \in A(x_o)$, $f(x) \in A(x_o)$ then for all $x \in A(x_o)$, $d_f(x) \in A(x_o)$.*

Proof. Let d_f be a f-derivation of a BCI-algebra X. If its center I_x is fully non-associative and for all $x \in A(x_o)$, $f(x) \in A(x_o)$ then by theorem 3.2, d_f is regular, so $d_f(o) = o$. For $x \in A(x_o)$, $x_o \leq x \Rightarrow x_o * x = o$. Since d_f is a regular f-derivation of X, therefore d_f is a (l, r)-f-derivation. So,

$$\begin{aligned}
o = d_f(o) &= d_f(x_o * x) = (d_f((x_o)) * f(x)) \wedge (f((x_o)) * d_f(x)) \\
&= (f((x_o)) * d_f(x)) * ((f((x_o)) * d_f(x)) * (d_f((x_o)) * f(x))) \\
&\qquad\qquad (since\ x \wedge y = y * (y * x)) \\
&\leq d_f((x_o)) * f(x) \quad (using (1.2)) \\
&\Rightarrow o * d_f(x_o) \leq (d_f(x_o) * f(x)) * d_f(x_o) \quad (using\ (1.8)) \\
&\Rightarrow o * d_f(x_o) \leq (d_f(x_o) * d_f(x_o)) * f(x) \quad (using (1.7)) \\
&\Rightarrow o * d_f(x_o) \leq o * f(x) \\
&\Rightarrow o * (o * f(x))) \leq o * (o * d_f(x_o)) \\
&\Rightarrow o * (o * f(x))) \leq o * (o * d_f(x_o) \leq d_f(x_o) \quad (using\ (1.2)) \\
&\Rightarrow o * (o * f(x))) = d_f(x_o) \quad (since\ d_f(x_o) \in I_x) \\
&\Rightarrow d_f(x_o) = o * (o * f(x)) \leq f(x) \quad (using (1.2))
\end{aligned}$$

By (1.13) both $d_f(x_o)$ and $f(x)$ belong to the same branch of X. Since $f(x) \in A(x_o)$, therefore $d_f(x_o) \in A(x_o)$. Since d_f is also a (r, l)-f-derivation. Therefore

$$\begin{aligned}
o = d_f(o) &= d_f(xo * x) = (f(x_o) * d_f(x)) \wedge (d_f(x_o) * f(x)) \\
&= (d_f(x_o) * f(x)) * ((d_f(x_o) * f(x)) * (f(x_o) * d_f(x))) \\
&\qquad\qquad (since\ x \wedge y = y * (y * x)) \\
&\leq f(x_o) * d_f(x) \quad (2)(using\ (1.2)) \\
&\Rightarrow o * f(x_o) \leq (f(x_o) * d_f(x)) * f(x_o) \quad (using(1.8)) \\
&\Rightarrow o * f(x_o) \leq (f(x_o) * f(x_o)) * d_f(x) \quad (using(1.7)) \\
&\Rightarrow o * f(x_o) \leq o * d_f(x) \Rightarrow o * (o * d_f(x)) \leq o * (o * f(x_o)) \quad (using\ (1.8)) \\
&\Rightarrow o * (o * d_f(x)) \leq f(x_o) \\
&\Rightarrow o * (o * d_f(x)) = f(x_o) \\
&\Rightarrow f(x_o) \leq d_f(x) \quad (using\ lemma\ 3.1\ and\ (1.2)).
\end{aligned}$$

By (1.13) both $d_f(x)$ and $f(x_o)$ belong to the same branch of X. Since $f(x_o) \in A(x_o)$, therefore $d_f(x) \in A(x_o)$. This completes the proof. □

Theorem 4.5. *Let d_f be a regular f-derivation and f be an endomorphism of a BCI-algebra X with center I_x. Then for $x_o \in I_x$, $d_f(x_o) = f(x_o)$.*

Proof. Let d_f be a regular f-derivation and f be an endomorphism of a BCI-algebra X with center I_x. Since d_f is a regular f-derivation of X, therefore $d_f(o) = o$. As d_f is a f-derivation, so d_f is a (l,r)-f-derivation. Also for $x_o \in I_x$, $o*(o*x_o)) = x_o$, so

$$\begin{aligned}
d_f(x_o) &= d_f(o*(o*x_o)) = (d_f(o)*f(o*x_o)) \wedge (f(o)*d_f(o*xo)) \\
&= (o*f(o*x_o)) \wedge (o*d_f(o*x_o)))(since d_f(o) = o = f(o)) \\
&= (o*(f(o)*f(x_o)) \wedge (o*d_f(o*x_o))) \\
&= (o*(o*f(x_o)) \wedge (o*d_f(o*x_o))) \\
&= f(x_o) \wedge (o*d_f(o*x_o))) \quad (using\ lemma\ 3.1, (iii)) \\
&= (o*d_f(o*x_o))*((o*d_f(o*x_o))*f(x_o))(since\ x \wedge y = y*(y*x)) \\
&\leq f(x_o) \quad (using (1.2)).
\end{aligned}$$

Since $f(x_o) \in I_x$, therefore $f(x_o)$ is an initial element. So,

$$(1) \qquad d_f(x_o) = f(x_o).$$

As d_f is also a (r,l)-f-derivation. So,

$$\begin{aligned}
&d_f(x_o) \\
&= d_f(o*(o*x_o)) = (f(o)*d_f(o*x_o)) \wedge (d_f(o)*f(o*x_o))) \\
&= (o*d_f(o*x_o)) \wedge (o*(f(o)*f(x_o))) \quad (since\ f\ is\ an\ endomorphism) \\
&= (o*d_f(o*x_o)) \wedge (o*(o*f(x_o))) \\
&= (o*d_f(o*x_o)) \wedge f(x_o) \quad (since\ f(x_o) \in I_x) \\
&= f(x_o)*(f(x_o)*(o*d_f(o*x_o))) \\
&\leq o*d_f(o*x_o) \quad (using\ (1.2)) \\
&= o*((f(o)*d_f(x_o)) \wedge (d_f(o)*f(x_o))) \\
&= o*((o*d_f(x_o)) \wedge (o*f(x_o))) \quad (since\ f(o) = o) \\
&= o*((o*f(x_o))*((o*f(x_o))*(o*d_f(x_o))) \quad (since\ x \wedge y = y*(y*x)) \\
&= (o*(o*f(x_o))*(o*((o*f(x_o))*(o*d_f(x_o))) \quad (using\ (1.11)) \\
&= f(x_o)*(o*(o*f(x_o))*(o*d_f(x_o))) \\
&= f(x_o)*((o*(o*f(x_o))*(o*(o*d_f(x_o))) \quad (using\ (1.11)) \\
&= f(x_o)*(f(x_o)*(o*(o*d_f(x_o))).
\end{aligned}$$

$$(2) \qquad \begin{aligned}
&d_f(x_o)) \leq o*(o*d_f(x_o)) \leq d_f(x_o) \quad (using\ (1.2)\ repeatedly) \\
&\Rightarrow d_f(x_o) = o*(o*d_f(x_o)(using(1.4)).
\end{aligned}$$

Since d_f is a f-derivation of X, therefore d_f is both (l,r)- and (r,l)-f-derivation. So, from (1) and (2) it follows $f(x_o) = o*(o*d_f(x_o)) = d_f(x_o)$(using theorem 4.1). This completes the proof. □

Theorem 4.6. *Let d_f be a f-derivation of a BCI-algebra X. Then for distinct $x, y \in I_x$, $d_f(x) = f(y) \Longleftrightarrow d_f(y) = f(x)$.*

Proof. Let $x, y \in I_x$ and $d_f(x) = f(y)$. Since d_f is a f-derivation of X, therefore d_f is a (l,r)-f-derivation as well as (r,l)-f-derivation. When d_f is a (r,l)-f-derivation, then

$$\begin{aligned} d_f(x*y) &= (d_f(x)*f(y)) \wedge (f(x)*d_f(y)) \\ &= (f(x)*d_f(y))*((f(x)*d_f(y))*(d_f(x)*f(y))) \\ &\leq d_f(x)*f(y) \\ &= f(y)*f(y) = o \quad (since\ d_f(x) = f(y)\ and\ (1.3)), \end{aligned}$$

$$d_f(x*y) = o \quad (using\ (1.5)).$$

Also, when d_f is a (l,r)-f-derivation, then

$$\begin{aligned} d_f(x*y) &= (f(x))*d_f(y)) \wedge (d_f(x)*f(y))) \\ &= (d_f(x)*f(y))*((d_f(x)*f(y))*(f(x))*d_f(y)) \\ &\leq f(x)*d_f(y). \end{aligned}$$

Thus it follows that

$$o = d_f(x*y) \leq f(x)*d_f(y) \Rightarrow f(x)*(f(x)*d_f(y)) \leq f(x).$$

Since by lemma 3.1 for any $x \in I_x$, $f(x) \in I_x$, therefore $f(x)$ is an initial element. So, above inequality implies $f(x)*(f(x)*d_f(y)) = f(x)$. Because of property 1.2, it implies $f(x) \leq d_f(y)$. By theorem 3.1, for any $y \in I_x$, $d_f(y) \in I_x$. So, $d_f(y)$ is an initial element. Thus $f(x) = d_f(y)$. Likewise we can prove that $d_f(y) = f(x) \Rightarrow d_f(x) = f(y)$. This completes the proof. □

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