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COMPLEXITY ANALYSIS OF IPM FOR $P_*(\kappa)$ LCPS BASED ON ELIGIBLE KERNEL FUNCTIONS

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Abstract. In this paper we propose new large-update primal-dual interior point algorithms for $P_*(\kappa)$ linear complementarity problems(LCPs). New search directions and proximity measures are proposed based on the kernel function $\psi(t) = \frac{t^{p+1}-1}{p+1} + \frac{e^{\frac{1}{t}}-e}{e}$, $p \in [0, 1]$. We showed that if a strictly feasible starting point is available, then the algorithm has $O((1 + 2\kappa)(\log n)^2 n^{\frac{1}{p+1}} \log \frac{n}{\varepsilon})$ complexity bound.

1. Introduction

In this paper we consider the following linear complementarity problem (LCP) :

$$(1) \qquad s = Mx + q,\ xs = 0,\ x \geq 0,\ s \geq 0,$$

where $M \in R^{n\times n}$ is a $P_*(\kappa)$ matrix and $x, s, q \in R^n$, and xs denotes the componentwise product of vectors x and s.

LCPs have many applications in mathematical programming and equilibrium problems. The reader can refer to [3] for the basic theory, algorithms and applications.

The primal-dual IPM for linear optimization(LO) problem was first introduced in [5] and [9]. They analyzed the polynomial complexity of the algorithm. Later on, Kojima et al. generalized their algorithms to monotone LCPs([7]), i.e. $P_*(0)$ LCPs and to $P_*(\kappa)$ LCPs([6]). Since then an interior point algorithm's quality is measured by the fact whether it can be generalized to $P_*(\kappa)$ LCPs or not([4]). Most of polynomial time interior point algorithms are based on the logarithmic barrier functions, e.g. see [12] . Peng et al.([11]) introduced self-regular barrier functions and obtained the best complexity result for large-update primal-dual IPMs for LO with some specific self regular barrier function. Recently, Bai et al.([1]) proposed a new class of kernel functions

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which are called eligible and they obtained polynomial complexity for LO and greatly simplified the analysis.

In this paper we propose new large-update primal-dual interior point algorithms for $P_*(\kappa)$ LCPs and show that the algorithm has $O((1+2\kappa)(\log n)^2 n^{\frac{1}{p+1}} \log \frac{n}{\varepsilon})$ complexity bound. Since we define a neighborhood and use a search direction based on the kernel functions which are neither self-regular nor logarithmic barrier, the analysis is different from the ones in [4], [6], [7], [8], and [10].

This paper is organized as follows. In Section 2 we recall basic concepts. In Section 3 we define the kernel function and its properties. In Section 4 we give complexity analysis of the algorithm.

We use the following notations throughout the paper : R^n_+ denotes the set of n dimensional nonnegative vectors and R^n_{++}, the set of n dimensional positive vectors. For $x = (x_1, x_2, \cdots, x_n)^T \in R^n$, $x_{min} = \min\{x_1, x_2, \cdots, x_n\}$, i.e. the minimal component of x, $\|x\|$ is the 2-norm of x, and X is the diagonal matrix from vector x , i.e. $X = diag(x)$. xs denotes the componentwise product (Hadamard product) of vectors x and s. $x^T s$ is the scalar product of the vectors x and s. e is the n-dimensional vector of ones and I is the n-dimensional identity matrix. J is the index set, i.e. $J = \{1, 2, \cdots, n\}$. We write $f(x) = O(g(x))$ if $\mid f(x) \mid \leq k \mid g(x) \mid$ for some positive constant k and $f(x) = \Theta(g(x))$ if $k_1 \mid g(x) \mid \leq \mid f(x) \mid \leq k_2 \mid g(x) \mid$ for some positive constants k_1 and k_2.

2. Preliminaries

In this section we give some basic definitions and the algorithm.

Definition 2.1 ([6])**.** Let $\kappa \geq 0$. A matrix $M \in R^{n\times n}$ is called a $P_*(\kappa)$ *matrix* if

$$(1+4\kappa)\sum_{i\in J_+(x)} x_i(Mx)_i + \sum_{i\in J_-(x)} x_i(Mx)_i \geq 0 \ ,$$

for all $x \in R^n$, where $J_+(x) = \{i \in J : x_i(Mx)_i \geq 0\}$ *and* $J_-(x) = \{i \in J : x_i(Mx)_i < 0\}$.

Note that PSD, the class of positive semidefinite matrices, is the special case of $P_*(\kappa)$ matrices, i.e. $P_*(0)$. We denote the strictly feasible set of LCP (1) by $\mathcal{F}^o$, i.e.,

$$\mathcal{F}^o := \{(x,s) \in R^{2n}_{++} : s = Mx + q\}.$$

Definition 2.2. A $(x, s) \in \mathcal{F}^o$ is an *ε-approximate solution* if and only if $x^T s \leq \varepsilon$ for $\varepsilon > 0$.

Definition 2.3. $\psi : R_+ \to R_+$ is called a *kernel function* if it is twice differentiable and the following conditions are satisfied :

(i) $\psi^{'}(1) = \psi(1) = 0$,

(ii) $\psi^{''}(t) > 0$, for all $t > 0$,
(iii) $\lim_{t\to 0^+}\psi(t) = \lim_{t\to\infty}\psi(t) = \infty$.

Definition 2.4. A function $\psi(\in \mathcal{C}^3) : (0,\infty) \to R$ is *eligible* if it satisfies the following conditions:

(i) $t\psi^{''}(t) + \psi^{'}(t) > 0,\ t > 0.$
(ii) $\psi^{'''}(t) < 0,\ t > 0,$
(iii) $2\psi^{''}(t)^2 - \psi^{'}(t)\psi^{'''}(t) > 0,\ 0 < t \le 1.$
(iv) $\psi^{''}(t)\psi^{'}(\beta t) - \beta\psi^{'}(t)\psi^{''}(\beta t) > 0,\ t > 1,\ \beta > 1.$

Definition 2.5. A function $f : D(\subset R) \to R$ is *exponentially convex* if and only if $f(\sqrt{x_1x_2}) \le \frac{1}{2}(f(x_1) + f(x_2))$ for all $x_1,\ x_2 \in D$.

Lemma 2.6 (Lemma 4.1 in [6])**.** *Let $M \in R^{n\times n}$ be a $P_*(\kappa)$ matrix and $x,\ s \in R^n_{++}$. Then for all $a \in R^n$ the system*

$$\begin{cases} -M\Delta x + \Delta s = 0, \\ S\Delta x + X\Delta s = a \end{cases}$$

has a unique solution $(\Delta x, \Delta s)$.

To find an ε-approximate solution for (1) we perturb the complementarity condition, and we get the following parameterized system :

$$s = Mx + q,\ xs = \mu e,\ x > 0,\ s > 0, \tag{2}$$

where $\mu > 0$. Without loss of generality, we assume that (1) is strictly feasible, i.e. there exists (x^0, s^0) such that $s^0 = Mx^0 + q$, $x^0 > 0$, $s^0 > 0$, and moreover, we have an initial strictly feasible point with $\Psi(x^0, s^0, \mu^0) \le \tau$ for some $\mu^0 > 0$. For this the reader refers to [6]. Since M is a $P_*(\kappa)$ matrix and (1) is strictly feasible, (2) has a unique solution for any $\mu > 0$. We denote the solution of (2) as $(x(\mu), s(\mu))$ for given $\mu > 0$. We call the solution set $\{(x(\mu), s(\mu)) \mid \ \mu > 0\}$ the *central path* for system (1). Note that the sequence $(x(\mu), s(\mu))$ approaches to the solution (x, s) of the system (1) as $\mu \to 0$ ([6]). IPMs follow the central path approximately. For the convenience we define the following notations:

$$d = \sqrt{\frac{x}{s}},\ v = \sqrt{\frac{xs}{\mu}},\ d_x = \frac{v\Delta x}{x},\ d_s = \frac{v\Delta s}{s}. \tag{3}$$

Using (3), we can write the Newton system as follows :

$$-\bar{M}d_x + d_s = 0,\ d_x + d_s = v^{-1} - v, \tag{4}$$

where $\bar{M} = DMD$ and $D = diag(d)$.

Note that $v^{-1} - v$ in (4) is the negative gradient of the logarithmic barrier function $\Psi_l(v) = \sum_{i=1}^n \psi_l(v_i)$, $\psi_l(t) = ((t^2-1)/2 - \log\ t)$. In this paper we replace the centering equation by

$$d_x + d_s = -\nabla\Psi(v), \tag{5}$$

where $\Psi(v) = \sum_{i=1}^{n} \psi(v_i)$,

$$\psi(t) = \frac{t^{p+1}-1}{p+1} + \frac{e^{\frac{1}{t}}-e}{e},\ p \in [0,1]. \tag{6}$$

Then we have the modified Newton system as follows :

$$-M\Delta x + \Delta s = 0,\ S\Delta x + X\Delta s = -\mu v \nabla \Psi(v). \tag{7}$$

Since M is a $P_*(\kappa)$ matrix and (1) is strictly feasible, this system uniquely defines a search direction $(\Delta x, \Delta s)$ by Lemma 2.6. Throughout the paper, we assume that a proximity parameter τ and a barrier update parameter θ are given and $\tau = O(n)$ and $0 < \theta < 1$, fixed. The algorithm works as follows. We assume that a strictly feasible point (x, s) is given which is in a $\tau-$neighborhood of the given $\mu-$center. Then after decreasing μ to $\mu_+ = (1-\theta)\mu$, for some fixed $\theta \in (0, 1)$, we solve the modified Newton system (7) to obtain the unique search direction. The positivity condition of a new iterate is ensured with the right choice of the step size α which is defined by some line search rule. This procedure is repeated until we find a new iterate (x_+, s_+) which is in a $\tau-$neighborhood of the μ_+-center and then we let $\mu := \mu_+$ and $(x, s) := (x_+, s_+)$. Then μ is again reduced by the factor $1-\theta$ and we solve the modified Newton system targeting at the new μ_+-center, and so on. This process is repeated until μ is small enough, e.g. $n\mu \le \varepsilon$.

Algorithm

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Input:
    A threshold parameter τ > 1;
    an accuracy parameter ε > 0;
    a fixed barrier update parameter θ, 0 < θ < 1;
    starting point (x^0, s^0) and μ^0 > 0 such that Ψ(x^0, s^0, μ^0) ≤ τ;
begin
    x := x^0; s := s^0; μ := μ^0;
    while nμ ≥ ε do
    begin
        μ := (1 − θ)μ;
        while Ψ(v) > τ do
        begin
            solve (7) for Δx and Δs;
            determine a step size α from (17);
            x := x + αΔx;
            s := s + αΔs;
        end
    end
end
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3. The kernel function and its properties

For $\psi(t)$ we have

$$\psi^{'}(t) = t^p - \frac{e^{\frac{1}{t}-1}}{t^2}, \quad \psi^{''}(t) = pt^{p-1} + \frac{1+2t}{t^4}e^{\frac{1}{t}-1}, \qquad (8)$$
$$\psi^{'''}(t) = p(p-1)t^{p-2} - \frac{1+6t+6t^2}{t^6}e^{\frac{1}{t}-1}.$$

Since $\psi^{''}(t) > 0$, $\psi(t)$ is strictly convex. Note that for $p \in [0,1]$, $\psi(1) = \psi^{'}(1) = 0$. Since $\psi(1) = \psi^{'}(1) = 0$, $\psi(t) = \int_1^t \int_1^\xi \psi^{''}(\varsigma)d\varsigma d\xi$. We define the norm-based proximity measure $\delta(v)$ as follows :

$$\delta(v) = \frac{1}{2} \parallel \nabla\Psi(v) \parallel = \frac{1}{2} \parallel d_x + d_s \parallel . \qquad (9)$$

Note that since $\Psi(v)$ is strictly convex and minimal at $v = e$, we have $\Psi(v) = 0 \;\Leftrightarrow\; \delta(v) = 0 \;\Leftrightarrow\; v = e$. For the notational convenience we denote $\delta(v)$ by δ. In the following lemma we give properties of the kernel function $\psi(t)$.

Lemma 3.1. *Kernel function $\psi(t)$ in (6) satisfies the following properties.*

(i) $t\psi^{''}(t) + \psi^{'}(t) > 0,\ t > 0.$
(ii) $\psi^{'''}(t) < 0,\ t > 0,$
(iii) $2\psi^{''}(t)^2 - \psi^{'}(t)\psi^{'''}(t) > 0,\ 0 < t \le 1.$
(iv) $\psi^{''}(t)\psi^{'}(\beta t) - \beta\psi^{'}(t)\psi^{''}(\beta t) > 0,\ t > 1,\ \beta > 1.$
(v) $\psi(t) \le \frac{t^{p+1}}{p+1},\ t \ge 1.$

Proof. (*i*): From (8), $t\psi^{''}(t) + \psi^{'}(t) = (pt^p + \frac{(1+2t)}{t^3}e^{\frac{1}{t}-1}) + (t^p - \frac{e^{\frac{1}{t}-1}}{t^2}) = (p+1)t^p + \frac{(1+t)}{t^3}e^{\frac{1}{t}-1} > 0$, for $t > 0$.
(*ii*) By (8), obvious.
(*iii*): From (8), $2\psi^{''}(t)^2 - \psi^{'}(t)\psi^{'''}(t) = 2(pt^{p-1} + \frac{1+2t}{t^4}e^{\frac{1}{t}-1})^2 - (t^p - \frac{e^{\frac{1}{t}-1}}{t^2})(p(p-1)t^{p-2} - \frac{1+6t+6t^2}{t^6}e^{\frac{1}{t}-1}) = p(p+1)t^{2p-2} + (\frac{p(p-1)}{t^{4-p}} + \frac{4p(1+2t)}{t^{5-p}} + \frac{1+6t+6t^2}{t^{6-p}})e^{\frac{1}{t}-1} + \frac{1+2t+2t^2}{t^8}e^{2(\frac{1}{t}-1)} > p(p+1)t^{2p-2} + (\frac{4p(1+2t)}{t^{5-p}} + \frac{p^2+6t+6t^2}{t^{4-p}})e^{\frac{1}{t}-1} + \frac{1+2t+2t^2}{t^8}e^{2(\frac{1}{t}-1)} > 0$, since $\frac{1}{t^{6-p}} > \frac{1}{t^{4-p}}$ for $0 < t \le 1$ and $p \in [0,1]$.
(*iv*): From (8), $\psi^{''}(t)\psi^{'}(\beta t) - \beta\psi^{'}(t)\psi^{''}(\beta t) = (\beta^p e^{\frac{1}{t}-1} - \frac{1}{\beta^3}e^{\frac{1}{\beta t}-1})\frac{1}{t^{4-p}} + (2+p)(\beta^p e^{\frac{1}{t}-1} - \frac{1}{\beta^2}e^{\frac{1}{\beta t}-1})\frac{1}{t^{3-p}} + \frac{1}{\beta^2 t^6}e^{\frac{1}{t}-1}e^{\frac{1}{\beta t}-1}(\frac{1}{\beta}-1) > (3+p)(\beta^p - \frac{1}{\beta^2})\frac{e^{\frac{1}{t}-1}e^{\frac{1}{\beta t}-1}}{t^6} > 0$, since $\frac{1}{t^{4-p}}, \frac{1}{t^{3-p}} > \frac{1}{t^6}$ and $e^{\frac{1}{t}-1}, e^{\frac{1}{\beta t}-1} > e^{\frac{1}{t}-1}e^{\frac{1}{\beta t}-1}$ for $p \in [0,1]$, $t > 1$ and $\beta > 1$.
(*v*): Since $e^{\frac{1}{t}} - e \le 0$ for $t \ge 1$, $\psi(t) = \frac{t^{p+1}-1}{p+1} + \frac{e^{\frac{1}{t}}-e}{e} \le \frac{t^{p+1}}{p+1}$, $t \ge 1$. □

By Lemma 3.1 (*i*) and Lemma 1 in [11], $\psi(t)$ is exponentially convex. Let $\varrho : [0,\infty) \to [1,\infty)$ be the inverse function of $\psi(t)$ for $t \ge 1$, $\rho : [0,\infty) \to (0,1]$ the inverse function of $-\frac{1}{2}\psi^{'}(t)$ for $t \in (0,1]$. We denote the barrier term of $\psi(t)$

as $\psi_b(t) = \frac{e^{\frac{1}{t}} - e}{e}$. Let $\underline{\rho} : [0, \infty) \to (0, 1]$ be the inverse function of the restriction of $-\psi_b'(t)$ to the interval $(0, 1]$. Then we obtain the following lemma.

Lemma 3.2. *We have*

(i) $((p+1)s+1)^{\frac{1}{p+1}} \le \varrho(s) \le 1 + s + \sqrt{s^2 + 2s}$, $s \ge 0$.
(ii) $\rho(s) \ge \underline{\rho}(1 + 2s)$, $s \ge 0$.

Proof. (i): Let $\psi(t) = s$ for $t \ge 1$. Then $s = \psi(t) = \frac{t^{p+1}-1}{p+1} + \psi_b(t) \le \frac{t^{p+1}-1}{p+1}$, for $t \ge 1$. Thus we have $t = \varrho(s) \ge ((p+1)s+1)^{\frac{1}{p+1}}$. For the second inequality, we first want to show that $s = \psi(t) \ge \frac{(t-1)^2}{2t}$, $t \ge 1$. It suffices to show that $2t\psi(t) \ge (t-1)^2$. Let $f(t) = 2t\psi(t) - (t-1)^2$, $t \ge 1$. Then $f(1) = 0$ and $f'(t) = 2\psi(t) + 2t(t^p - 1) + 2(1 - \frac{e^{\frac{1}{t}-1}}{t}) \ge 0$, for $t \ge 1$. Thus we have $f(t) = 2t\psi(t) - (t-1)^2 \ge 0$, for $t \ge 1$. So we have $t^2 - 2(1+s)t + 1 \le 0$ and this implies that $1 + s - \sqrt{s^2 + 2s} \le \varrho(s) = t \le 1 + s + \sqrt{s^2 + 2s}$, for $t \ge 1$. Hence we have $((p+1)s+1)^{\frac{1}{p+1}} \le \varrho(s) = t \le 1 + s + \sqrt{s^2 + 2s}$, for $t \ge 1$.
(ii): Let $t = \rho(s)$. Then by the definition of ρ, $s = -\frac{1}{2}\psi'(t)$ and $-2s = \psi'(t) = t^p + \psi_b'(t)$, for $t \le 1$. Since $t \le 1$, we have

$$-\psi_b'(t) = t^p + 2s \le 1 + 2s = -\psi_b'(\underline{\rho}(1 + 2s)). \tag{10}$$

Since $-\psi_b''(t) = -\frac{1+2t}{t^4}\ e^{\frac{1}{t}-1} < 0$, $-\psi_b'(t)$ is monotonically decreasing in t. Hence by (10), we have $t = \rho(s) \ge \underline{\rho}(1 + 2s)$. □

By the definition of $\underline{\rho}$, we have $\underline{\rho}(s) = t$ and $\frac{e^{\frac{1}{t}-1}}{t^2} = s$ for $0 < t \le 1$. It follows that $e^{\frac{1}{t}-1} = st^2 \le s$. Hence $\underline{\rho}(s) = t \ge \frac{1}{1+\log s}$. Thus, by Lemma 3.2 (ii),

$$\rho(s) \ge \underline{\rho}(1 + 2s) \ge \frac{1}{1 + \log(1 + 2s)}. \tag{11}$$

4. Complexity analysis

In this section we analyze the complexity of the algorithm. Since M is a $P_*(\kappa)$ matrix and $M\Delta x = \Delta s$ from (7), for $\Delta x \in R^n$ we have

$$(1 + 4\kappa) \sum_{i \in J_+} \Delta x_i \Delta s_i + \sum_{i \in J_-} \Delta x_i \Delta s_i \ge 0,$$

where $J_+ = \{\ i \in J\ :\ \Delta x_i \Delta s_i \ge 0\ \}$, $J_- = J - J_+$ and Δx_i, Δs_i denote the i-th components of the vectors Δx and Δs, respectively. Since $d_x d_s = \frac{v^2 \Delta x \Delta s}{xs} = \frac{\Delta x \Delta s}{\mu}$ and $\mu > 0$,

$$(1 + 4\kappa) \sum_{i \in J_+} [d_x]_i [d_s]_i + \sum_{i \in J_-} [d_x]_i [d_s]_i \ge 0. \tag{12}$$

For notational convenience we let $\sigma_+ = \sum_{i \in J_+} [d_x]_i [d_s]_i$, $\sigma_- = -\sum_{i \in J_-} [d_x]_i [d_s]_i$. In the following we cite technical lemmas in [2] without proof.

Lemma 4.1 (Lemma 4.2 in [2]). *$\sum_{i=1}^n([d_x]_i^2+[d_s]_i^2)\leq 4(1+2\kappa)\delta^2$, $\|d_x\|\leq 2\sqrt{1+2\kappa}\ \delta$, and $\|d_s\|\leq 2\sqrt{1+2\kappa}\ \delta$.*

After a damped step for fixed μ we have

$$x_+=x+\alpha\Delta x,\ s_+=s+\alpha\Delta s.$$

Then by (3), we have $x_+=x\left(e+\alpha\frac{\Delta x}{x}\right)=x\left(e+\alpha\frac{d_x}{v}\right)=\frac{x}{v}(v+\alpha d_x),\ s_+=s\left(e+\alpha\frac{\Delta s}{s}\right)=s\left(e+\alpha\frac{d_s}{v}\right)=\frac{s}{v}(v+\alpha d_s)$. Then we get $v_+^2=\frac{x_+s_+}{\mu}=(v+\alpha d_x)(v+\alpha d_s)$. Throughout the paper we assume that the step size α is such that the coordinates of the vectors $v+\alpha d_x$ and $v+\alpha d_s$ are positive. Since $\psi(v)$ is exponentially convexity, we have

$$\Psi(v_+)=\Psi(\sqrt{(v+\alpha d_x)(v+\alpha d_s)}\ )\ \leq\ \frac{1}{2}\ (\Psi(v+\alpha d_x)+\Psi(v+\alpha d_s)\ ).$$

For given $\mu>0$ by letting $f(\alpha)$ be the difference of the new and old proximity measures, i.e.

$$f(\alpha)=\Psi(v_+)-\Psi(v).$$

Then we have

$$f(\alpha)\leq f_1(\alpha),$$

where $f_1(\alpha):=\frac{1}{2}(\Psi(v+\alpha d_x)+\Psi(v+\alpha d_s))-\Psi(v)$. Note that $f(0)=f_1(0)=0$. By taking the derivative of $f_1(\alpha)$ with respect to α, we have $f_1^{'}(\alpha)=\frac{1}{2}\sum_{i=1}^n(\psi^{'}(v_i+\alpha[d_x]_i)[d_x]_i+\psi^{'}(v_i+\alpha[d_s]_i)[d_s]_i)$. From (5) and the definition of δ,

$$f_1^{'}(0)=\frac{1}{2}\nabla\Psi(v)^T(d_x+d_s)=-\frac{1}{2}\nabla\Psi(v)^T\nabla\Psi(v)=-2\delta^2. \tag{13}$$

By taking the derivative of $f_1^{'}(\alpha)$ with respect to α, we have

$$f_1^{''}(\alpha)=\frac{1}{2}\sum_{i=1}^n(\psi^{''}(v_i+\alpha[d_x]_i)[d_x]_i^2+\psi^{''}(v_i+\alpha[d_s]_i)[d_s]_i^2). \tag{14}$$

To compute the upper bound for the difference of the new and old proximity measures, we need the following technical lemmas.

Lemma 4.2 (Lemma 4.3 in [2]). *$f_1^{''}(\alpha)\leq 2(1+2\kappa)\ \delta^2\psi^{''}(v_{min}-2\alpha\sqrt{1+2\kappa}\ \delta)$.*

Lemma 4.3 (Lemma 4.4 in [2]). *$f_1^{'}(\alpha)\leq 0$ if α is satisfying*

$$-\psi'(v_{min}-2\alpha\delta\sqrt{1+2\kappa})+\psi^{'}(v_{min})\leq\frac{2\delta}{\sqrt{1+2\kappa}}. \tag{15}$$

In the following lemma, we compute the feasible step size α such that the proximity measure is decreasing when we take a new iterate for fixed μ.

Lemma 4.4 (Lemma 4.5 in [2]). *Let $\rho:[0,\infty)\rightarrow(0,1]$ denote the inverse function of the restriction of $-\frac{1}{2}\psi^{'}(t)$ to the interval $(0,1]$. Then the largest step size α which satisfies (15) is given by*

$$\bar{\alpha}:=\frac{1}{2\delta\sqrt{1+2\kappa}}\left(\rho(\delta)-\rho\left(\left(1+\frac{1}{\sqrt{1+2\kappa}}\right)\delta\right)\right). \tag{16}$$

In the following lemma we compute the lower bound for $\bar{\alpha}$ in Lemma 4.4.

Lemma 4.5. *Let ρ and $\bar{\alpha}$ be as defined in Lemma 4.4. Then we have*

$$\bar{\alpha} \geq \frac{1}{1+2\kappa}\ \frac{1}{\psi''(\rho((1+\frac{1}{\sqrt{1+2\kappa}})\delta))}.$$

Proof. By the definition of ρ, $-\psi^{'}(\rho(\delta)) = 2\delta$. By taking the derivative with respect to δ, we get $-\psi^{''}(\rho(\delta))\rho^{'}(\delta) = 2$. So we have $\rho^{'}(\delta) = -\frac{2}{\psi''(\rho(\delta))} < 0$ since $\psi^{''} > 0$. Hence ρ is monotonically decreasing. By (16) and the fundamental theorem of calculus, we have

$$\begin{aligned}\bar{\alpha} &= \frac{1}{2\delta\sqrt{1+2\kappa}}(\rho(\delta) - \rho((1+\frac{1}{\sqrt{1+2\kappa}})\delta))\\ &= \frac{1}{2\delta\sqrt{1+2\kappa}}\int_{(1+\frac{1}{\sqrt{1+2\kappa}})\delta}^{\delta}\rho^{'}(\xi)d\xi\\ &= \frac{1}{\delta\sqrt{1+2\kappa}}\int_{\delta}^{(1+\frac{1}{\sqrt{1+2\kappa}})\delta}\frac{d\xi}{\psi''(\rho(\xi))}.\end{aligned}$$

Since $\delta \leq \xi \leq (1+\frac{1}{\sqrt{1+2\kappa}})\delta$ and ρ is monotonically decreasing,

$$\rho(\xi) \geq \rho((1+\frac{1}{\sqrt{1+2\kappa}})\delta).$$

Since $\psi^{''}$ is monotonically decreasing, $\psi^{''}(\rho(\xi)) \leq \psi^{''}(\rho((1+\frac{1}{\sqrt{1+2\kappa}})\delta))$. Hence $\frac{1}{\psi''(\rho(\xi))} \geq \frac{1}{\psi''(\rho((1+\frac{1}{\sqrt{1+2\kappa}})\delta))}$. Therefore we have

$$\bar{\alpha} = \frac{1}{1+2\kappa}\ \frac{1}{\psi''(\rho((1+\frac{1}{\sqrt{1+2\kappa}})\delta))}.$$

□

Define

$$\tilde{\alpha} = \frac{1}{1+2\kappa}\ \frac{1}{\psi''(\rho((1+\frac{1}{\sqrt{1+2\kappa}})\delta))}. \tag{17}$$

Then we will use $\tilde{\alpha}$ as the default step size in our Algorithm. Also by Lemma 4.5, $\bar{\alpha} \geq \tilde{\alpha}$. In the following, we want to evaluate the decrease of the proximity function value. We cite the following result in [11] without proof.

Lemma 4.6 (Lemma 3.12 in [11])**.** *Let $h(t)$ be a twice differentiable convex function with $h(0) = 0$, $h^{'}(0) < 0$ and let $h(t)$ attains its (global) minimum at $t^* > 0$. If $h^{''}(t)$ is increasing for $t \in [0, t^*]$, then $h(t) \leq \frac{th^{'}(0)}{2},\ 0 \leq t \leq t^*$.*

Lemma 4.7 (Lemma 4.8 in [2])**.** *If the step size α is such that $\alpha \leq \bar{\alpha}$, then $f(\alpha) \leq -\alpha\delta^2$.*

In the following theorem we have the upper bound for the difference $f(\alpha)$ between new and old proximity measures.

Theorem 4.8. *Let $\tilde{\alpha}$ be a step size as defined in* (17)*. Then we have*

$$f(\tilde{\alpha}) \leq -\frac{1}{1+2\kappa} \frac{\delta^2}{\psi''(\rho((1+\frac{1}{\sqrt{1+2\kappa}})\delta))}. \tag{18}$$

Proof. By Lemma 4.5, $\tilde{\alpha} \leq \bar{\alpha}$. By Lemma 4.7, we get the result. □

Lemma 4.9. *The right hand side in* (18) *is monotonically decreasing in δ.*

Proof. Let $t = \rho(a\delta)$ where $a = 1 + \frac{1}{\sqrt{1+2\kappa}}$. Then $0 < t \leq 1$ and $-\psi'(\rho(a\delta)) = 2a\delta$, i.e. $\frac{1}{2}\psi'(t) = -\frac{1}{2}\psi'(\rho(a\delta)) = a\delta$. Then

$$\frac{1}{1+2\kappa} \frac{\delta^2}{\psi''(\rho((1+\frac{1}{\sqrt{1+2\kappa}})\delta))} = \frac{1}{4a^2(1+2\kappa)} \frac{\psi'(t)^2}{\psi''(t)}.$$

Define

$$g(t) = \frac{1}{4a^2(1+2\kappa)} \frac{\psi'(t)^2}{\psi''(t)}.$$

Since ρ is monotonically decreasing, t is monotonically decreasing if δ increases. Hence the right hand in (18) is monotonically decreasing in δ if and only if the function $g(t)$ is monotonically decreasing for $0 < t \leq 1$. Note that $g(1) = 0$ and $g'(t) = \frac{1}{4a^2(1+2\kappa)} \frac{\psi'(t)\{2\psi''(t)^2 - \psi'(t)\psi'''(t)\}}{\psi''(t)^2}$. Since $\psi'(1) = 0$ and $\psi'' > 0$, $\psi'(t) \leq 0$ for $0 < t \leq 1$. By Lemma 3.1 (*iii*), $g(t)$ is monotonically decreasing for $0 < t \leq 1$. Hence the lemma is proved. □

Note that at the start of outer iteration of the algorithm, just before the update of μ with the factor $1-\theta$, we have $\Psi(v) \leq \tau$. Due to the update of μ the vector v is divided by the factor $\sqrt{1-\theta}$, with $0 < \theta < 1$, which in general leads to an increase in the value of $\Psi(v)$. Then, during the subsequent inner iterations, $\Psi(v)$ decreases until it passes the threshold τ again. Hence, during the process of the algorithm the largest values of $\Psi(v)$ occur just after the updates of μ.

In the following lemma we obtain an upper bound for $\Psi(v)$.

Lemma 4.10. *If $\Psi(v) \leq \tau$ for $0 < \theta < 1$, then we have*

$$\psi(\frac{v}{\sqrt{1-\theta}}) \leq \frac{n}{(p+1)(1-\theta)^{\frac{p+1}{2}}} \left(1 + \frac{\tau}{n} + \sqrt{\left(\frac{\tau}{n}\right)^2 + \frac{2\tau}{n}}\right)^{p+1}.$$

Proof. By the definition of ϱ and $\frac{1}{\sqrt{1-\theta}} \geq 1$, $\frac{1}{\sqrt{1-\theta}}\varrho\left(\frac{\Psi(v)}{n}\right) \geq 1$. By Theorem 3.2 in [1], Lemma 3.1 (v), and Lemma 3.2 (i), we have

$$\begin{aligned}
\psi(\frac{v}{\sqrt{1-\theta}}) &\leq n\psi\left(\frac{\varrho\left(\frac{\Psi(v)}{n}\right)}{\sqrt{1-\theta}}\right)\\
&\leq n\frac{(\varrho(\frac{\Psi(v)}{n}))^{p+1}}{(p+1)(1-\theta)^{\frac{p+1}{2}}}\\
&\leq \frac{n}{(p+1)(1-\theta)^{\frac{p+1}{2}}}\left(1+\frac{\tau}{n}+\sqrt{\left(\frac{\tau}{n}\right)^2+\frac{2\tau}{n}}\right)^{p+1}.
\end{aligned}$$

□

For notational convenience we denote the value of $\Psi(v)$ after the μ-update as Ψ_0, then

$$\Psi_0 \leq \frac{n}{(p+1)(1-\theta)^{\frac{p+1}{2}}}\left(1+\frac{\tau}{n}+\sqrt{\left(\frac{\tau}{n}\right)^2+\frac{2\tau}{n}}\right)^{p+1}. \tag{19}$$

Since $\tau = O(n)$ and $\theta = \Theta(1)$, $\Psi_0 = O(n)$.

In the following theorem we provide a lower bound for δ in terms of the proximity function $\Psi(v)$.

Theorem 4.11. *Let δ be the norm-based proximity measure as defined in (9). If $\Psi := \Psi(v) \geq \tau$ for $\tau \geq 1$, then we have*

$$\delta \geq \frac{1}{6}\Psi^{\frac{p}{p+1}}.$$

Proof. By Theorem 4.9 in [1] and $e^{\frac{1}{\varrho(\Psi)}-1} \leq 1$ for $\varrho(\Psi) \geq 1$, we have

$$\begin{aligned}
\delta \geq \frac{1}{2}\psi'(\varrho(\Psi)) &= \frac{1}{2}\left(\varrho(\Psi)^p - \frac{e^{\frac{1}{\varrho(\Psi)}-1}}{\varrho(\Psi)^2}\right)\\
&\geq \frac{1}{2}\left(\varrho(\Psi)^p - \frac{1}{\varrho(\Psi)^2}\right)\\
&\geq \frac{1}{2}\left(\varrho(\Psi)^p - \frac{1}{\varrho(\Psi)}\right).
\end{aligned}$$

Then by Lemma 3.2 (i), $\Psi \geq 1$ and $p \in [0,1]$, we have

$$\begin{aligned}
\delta &\geq \frac{1}{2}\left(((p+1)\Psi+1)^{\frac{p}{p+1}} - \frac{1}{((p+1)\Psi+1)^{\frac{1}{p+1}}}\right) = \frac{1}{2}\left(\frac{((p+1)\Psi+1)^{\frac{p+1}{p+1}}-1}{((p+1)\Psi+1)^{\frac{1}{p+1}}}\right)\\
&= \frac{1}{2}\frac{(p+1)\Psi}{((p+1)\Psi+1)^{\frac{1}{p+1}}} \geq \frac{(p+1)\Psi}{2(2\Psi+1)^{\frac{1}{p+1}}} \geq \frac{(p+1)\Psi}{6\Psi^{\frac{1}{p+1}}} \geq \frac{1}{6}\Psi^{\frac{p}{p+1}}.
\end{aligned}$$

□

In the following we compute the total number of iterations of the algorithm to get an ε-approximate solution. We need the following technical lemma to obtain iteration bounds. For the proof the reader can refer [11].

Lemma 4.12 (Lemma A.2 in [1]). *Let $t_0, t_1, \cdots, t_K$ be a sequence of positive numbers such that $t_{k+1} \le t_k - \beta t_k^{1-\gamma}$, $k = 0, 1, \cdots, K-1$, where $\beta > 0$ and $0 < \gamma \le 1$. Then $K \le \lfloor \frac{t_0^\gamma}{\beta\gamma} \rfloor$.*

We define the value of $\Psi(v)$ after the μ-update as Ψ_0 and the subsequent values in the same outer iteration are denoted as Ψ_k, $k = 1, 2, \cdots$. Let K denote the total number of inner iterations in the outer iteration. Then by the definition of K, we have $\Psi_{K-1} > \tau$, $0 \le \Psi_K \le \tau$.

In the following lemma, we compute the upper bound for the total number of inner iterations which we needed to return to the τ-neighborhood again. For notational convenience we denote $\Psi(v)$ by Ψ and $a = 1 + \frac{1}{\sqrt{1+2\kappa}}$.

Lemma 4.13. *Let K be the total number of inner iterations in an outer iteration. Then we have*

$$K \le 216(1+2\kappa)(p+1)\left(1 + \log\left(\frac{5}{3}\Psi_0^{\frac{p}{p+1}}\right)\right)\Psi_0^{\frac{1}{p+1}},$$

where Ψ_0 denotes the value of $\Psi(v)$ after the μ-update.

Proof. From Theorem 4.8, Theorem 4.11 and Lemma 4.9, we have

$$f(\tilde{\alpha}) \le -\frac{1}{1+2\kappa}\,\frac{\delta^2}{\psi''(\rho(a\delta))} \le -\frac{1}{36(1+2\kappa)}\,\frac{\Psi^{\frac{2p}{p+1}}}{\psi''(\rho(\frac{a}{6}\Psi^{\frac{p}{p+1}}))}.$$

Let $\underline{\rho}\left(1 + \frac{a}{3}\Psi^{\frac{p}{p+1}}\right) = t$. Then by definition of $\underline{\rho}$,

$$1 + \frac{a}{3}\Psi^{\frac{p}{p+1}} = \frac{e^{\frac{1}{t}-1}}{t^2}. \tag{20}$$

By Lemma 3.2 (ii) and (11), we have

$$1 \ge \rho\left(\frac{a}{6}\Psi^{\frac{p}{p+1}}\right) \ge \underline{\rho}\left(1 + \frac{a}{3}\Psi^{\frac{p}{p+1}}\right) = t \ge \frac{1}{1 + \log\left(1 + \frac{a}{3}\Psi^{\frac{p}{p+1}}\right)}. \tag{21}$$

Then by $\psi''' < 0$ and (21), we get

$$f(\tilde{\alpha}) \le -\frac{1}{36(1+2\kappa)}\,\frac{\Psi^{\frac{2p}{p+1}}}{\psi''(\underline{\rho}(1 + \frac{a}{3}\Psi^{\frac{p}{p+1}}))} = -\frac{1}{36(1+2\kappa)}\,\frac{\Psi^{\frac{2p}{p+1}}}{pt^{p-1} + \frac{1+2t}{t^4}e^{\frac{1}{t}-1}}.$$

Using the fact $0 < t \le 1$, (20) and (21), we have

$$pt^{p-1} + \frac{1+2t}{t^4}e^{\frac{1}{t}-1} \le pt^{p-1} + \frac{3}{t^4}e^{\frac{1}{t}-1} \le pt^{p-1} + \frac{3(1 + \frac{a}{3}\Psi^{\frac{p}{p+1}})}{t^2}$$

$$\leq p\left(1+\log\left(1+\frac{a}{3}\Psi^{\frac{p}{p+1}}\right)\right)^{1-p}+3(1+\frac{a}{3}\Psi^{\frac{p}{p+1}})\left(1+\log\left(1+\frac{a}{3}\Psi^{\frac{p}{p+1}}\right)\right)^{2}.$$

Without loss of generality we may assume that $\Psi_0 \geq \Psi \geq \tau \geq 1$. Since $a = 1+\frac{1}{\sqrt{1+2\kappa}} \leq 2$, we have $1+\frac{a}{3}\Psi^{\frac{p}{p+1}} \leq (1+\frac{2}{3})\Psi^{\frac{p}{p+1}} = \frac{5}{3}\Psi^{\frac{p}{p+1}}$. Then we have

$$\begin{aligned} pt^{p-1}+\frac{1+2t}{t^4}e^{\frac{1}{t}-1} &\leq p\left(1+\log\left(\frac{5}{3}\Psi^{\frac{p}{p+1}}\right)\right)^{1-p}+5\Psi^{\frac{p}{p+1}}\left(1+\log\left(\frac{5}{3}\Psi^{\frac{p}{p+1}}\right)\right)^{2} \\ &\leq 6\Psi^{\frac{p}{p+1}}\left(1+\log\left(\frac{5}{3}\Psi_0^{\frac{p}{p+1}}\right)\right)^{2}. \end{aligned}$$

Thus

$$f(\tilde{\alpha}) \leq -\frac{1}{216(1+2\kappa)}\frac{\Psi^{\frac{p}{p+1}}}{\left(1+\log\left(\frac{5}{3}\Psi_0^{\frac{p}{p+1}}\right)\right)^{2}}.$$

This implies that $\Psi_{k+1} \leq \Psi_k - \beta{\Psi_k}^{1-\gamma},\ k = 0,1,2,\cdots,K-1$, where

$$\beta = \frac{1}{216(1+2\kappa)\left(1+\log\left(\frac{5}{3}\Psi_0^{\frac{p}{p+1}}\right)\right)^{2}},\ \gamma = \frac{1}{p+1}.$$

Hence by Lemma 4.12, K is bounded above by

$$K \leq \frac{\Psi_0^{\gamma}}{\beta\gamma} = 216(1+2\kappa)(p+1)\left(1+\log\left(\frac{5}{3}\Psi_0^{\frac{p}{p+1}}\right)\right)^{2}\Psi_0^{\frac{1}{p+1}}. \tag{22}$$

This completes the proof. □

From (19), we have

$$\Psi_0 \;\leq\; \frac{n}{(p+1)(1-\theta)^{\frac{p+1}{2}}}\left(1+\frac{\tau}{n}+\sqrt{\left(\frac{\tau}{n}\right)^2+\frac{2\tau}{n}}\right)^{p+1}.$$

From (22), we have

$$K \leq 216(1+2\kappa)(p+1)^{\frac{p}{p+1}}\frac{n^{\frac{1}{p+1}}}{\sqrt{1-\theta}}\left(1+\frac{\tau}{n}+\sqrt{\left(\frac{\tau}{n}\right)^2+\frac{2\tau}{n}}\right).$$

The upper bound for the total number of iterations is obtained by multiplying the number K by the number of central path parameter updates. If the central path parameter μ has the initial value μ^0 and is updated by multiplying $1-\theta$, with $0 < \theta < 1$, then after at most

$$\left\lceil \frac{1}{\theta}\ \log\frac{n\mu^0}{\epsilon} \right\rceil$$

iterations we have $n\mu \leq \epsilon$. So we obtain the main result.

Theorem 4.14. *Let a $P_*(\kappa)$ linear complementarity problem be given, where $\kappa \geq 0$. Assume that a strictly feasible starting point (x^0, s^0) is given with $\Psi(x^0, s^0, \mu^0) \leq \tau$ for some $\mu^0 > 0$. Then the total number of iterations to get an ε-approximate solution for the algorithm is bounded above by*

$$\lceil\, 216(1+2\kappa)(p+1)^{\frac{p}{p+1}} \frac{n^{\frac{1}{p+1}}}{\sqrt{1-\theta}} \left(1 + \frac{\tau}{n} + \sqrt{\left(\frac{\tau}{n}\right)^2 + \frac{2\tau}{n}}\right) \rceil \lceil\, \frac{1}{\theta} \log \frac{n\mu^0}{\epsilon}\, \rceil.$$

Remark 4.15. Since $\tau = O(n)$ and $\theta = \Theta(1)$, the algorithm has $O((1 + 2\kappa)(\log n)^2 n^{\frac{1}{p+1}} \log \frac{n}{\varepsilon})$ complexity.

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