# SENSITIVITY ANALYSIS FOR COMPLETELY GENERALIZED NONLINEAR VARIATIONAL INCLUSIONS 

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#### Abstract

In this paper, by using the technique of the resolvent operators, we study the behaviour and sensitivity analysis of the solutions set for a class of parametric completely generalized nonlinear variational inclusions with set-valued mappings.


## 1. Introduction

Sensitivity analysis of solutions of variational inequalities with single-valued mappings have been studied by many authors via quite different techniques. By using the projection method, Dafermos [3], Yen [13], Robinson [10], Qiu and Magnanti [9], Mukherjee and Verma [5] and Pan [8] studied the sensitivity analysis of solutions of some variational inequalities with single valued mappings in finite dimensional spaces and Hilbert spaces.

Our inspiration and motivation is devoted to $[4,7,11,12,14]$, we use the technique of resolvent operator, study the behaviour and sensitivity analysis of the solution set for a class of parametric completely generalized nonlinear variational inclusions with set-valued mappings. Our results improves the results of Dafermos [3], Ding and Lou [4], and Park and Jeong [7], etc.

Let $H$ be a real Hilbert space with norm and inner product denoted by $\|\cdot\|$ and $\langle\cdot\rangle$, respectively. Let $\partial \varphi$ denote the subdifferential of a proper convex and lower semicontinuous function $\varphi: H \times H \rightarrow R \cup\{+\infty\}$. Given setvalued mappings $M, S, T: H \rightarrow 2^{H}$, where $2^{H}$ denotes the family of nonempty bounded subsets of $H$ and single valued mappings $g, F, G, P: H \rightarrow H$ with $\operatorname{Im}(g) \cap \operatorname{dom} \partial \varphi(\cdot, v) \neq \emptyset$. It is known that the subdifferential $\partial \varphi(\cdot, v)$ is a maximal monotone operator.

Let $\delta: 2^{H} \rightarrow[0,+\infty)$ be a function defined by

$$
\delta(A, B)=\sup \{\|a-b\|: a \in A, b \in B\} .
$$

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Let $\hat{H}: C(H) \rightarrow[0, \infty)$ be a function defined by

$$
\hat{H}(A, B)=\max \left\{\sup _{u \in A} d(u, B), \sup _{v \in B} d(A, v)\right\},
$$

where

$$
d(u, B)=\sup _{v \in B}\|u-v\|
$$

then $\left(2^{H}, \delta\right)$ and $(C(H), \hat{H})$ are complete metric spaces.
We consider the following completely generalized nonlinear variational inclusions for finding $u \in H, x \in M(u), y \in S(u)$ and $z \in T(u)$ such that $g(u) \cap \operatorname{dom} \partial \varphi(\cdot, v) \neq \emptyset$ and
(1.1) $\langle P(x)-(F y-G z), v-g(u)\rangle \geq \varphi(g(u), u)-\varphi(v, u), \quad$ for all $v \in H$, considered by Ahmad et al [1].
Definition 1.1 ([2]). If $A$ is a maximal monotone operator on $H$, then the resolvent operator associated with $A$ is defined by

$$
\begin{equation*}
J_{A}(u)=(I+\eta A)^{-1}(u), \quad \text { for all } u \in H \tag{1.2}
\end{equation*}
$$

where $\eta>0$ is a constant and $I$ is an identity operator. Since the subdifferential $\partial \varphi(\cdot, \cdot)$ of a proper, convex and lower semicontinuous function $\varphi(\cdot, \cdot): H \times H \rightarrow$ $R \cup\{+\infty\}$ is a maximal monotone operator with respect to first variable, so we denote by

$$
J_{\partial \varphi(\cdot, u)}=(I+\eta \partial \varphi(\cdot, u))^{-1}
$$

the resolvent operator associated with $\varphi(\cdot, u)$.
Lemma 1.1 ([2]). The resolvent operator $J_{A}$ is a single-valued and nonexpansive, i.e.,

$$
\begin{equation*}
\left\|J_{A}(u)-J_{A}(v)\right\| \leq\|u-v\|, \quad \text { for all } u, v \in H \tag{1.3}
\end{equation*}
$$

## 2. Preliminaries

In this section, we present the parametric version of (1.1), and also provide some pertinent definitions which are essential for the study of our problems. Now, we consider the parametric version of problem (1.1). To formulate the problem, let $\Omega$ be a nonempty open subset of $H$ in which the parameter $\lambda$ takes values. Let $M, S, T: \Omega \times H \rightarrow 2^{H}$ be the set-valued mappings and $g, F, G: \Omega \times$ $H \rightarrow H$ be the single-valued mappings. Since the subdifferential $\partial \varphi$ of a proper convex and lower semicontinuous function $\varphi: H \times H \rightarrow R \cup\{+\infty\}$ is a maximal monotone operator with respect to first argument with $\operatorname{Im}(g) \cap \operatorname{dom} \partial \varphi(\cdot, v) \neq$ $\emptyset, v \in H$. For each fixed $\lambda \in \Omega$, the parametric completely generalized nonlinear variational inclusions consists of finding $u \in H, x(u, \lambda) \in M(u, \lambda), y(u, \lambda) \in$ $S(u, \lambda)$ and $z(u, \lambda) \in T(u, \lambda)$ such that $g(u, \lambda) \cap \operatorname{dom} \partial \varphi(\cdot, v) \neq \emptyset$ and

$$
\begin{align*}
\langle P(x(u, \lambda), \lambda) & -(F(y(u, \lambda), \lambda)-G(z(u, \lambda), \lambda)), v-g(u, \lambda)\rangle \\
& \geq \varphi(g(u, \lambda), u)-\varphi(v, u), \text { for all } v \in H . \tag{2.1}
\end{align*}
$$

Remark 2.1. By the suitable choice of the mappings $M, S, T, G, F, g$ problem (2.1) includes many known results as special cases such as $[3,4,7]$ and the references therein.
Definition 2.1. For all $u, v \in H, \lambda \in \Omega$, the parametric operator $g: H \times \Omega \rightarrow$ $H$ is said to be
(i) strongly monotone if there exists a constant $\alpha>0$ such that

$$
\langle g(u, \lambda)-g(v, \lambda), u-v\rangle \geq \alpha\|u-v\|^{2}
$$

(ii) Lipschitz continuous if there exists a constant $\beta>0$ such that

$$
\|g(u, \lambda)-g(v, \lambda)\| \leq \beta\|u-v\|
$$

Definition 2.2. For all $u, v \in H, \lambda \in \Omega$, the set-valued mapping $S: H \times \Omega \rightarrow$ $2^{H}[S: H \times \Omega \rightarrow C(H)]$ is said to be
(i) $\rho$ - $\delta$-Lipschitz [ $\rho$ - $\hat{H}$-Lipschitz] continuous if there exists a constant $\rho>0$ such that

$$
\begin{aligned}
\delta(S(u, \lambda), S(v, \lambda)) & \leq \rho\|u-v\| \\
{[\hat{H}(S(u, \lambda), S(v, \lambda))} & \leq \rho\|u-v\|]
\end{aligned}
$$

where $\hat{H}(\cdot, \cdot)$ is a Hausdorff metric.
(ii) relaxed Lipschitz continuous with respect to a mapping $F: H \times \Omega \rightarrow H$ if there exists a constant $s \geq 0$ such that

$$
\langle F(y(u, \lambda), \lambda)-F(y(v, \lambda), \lambda), u-v\rangle \leq-s\|u-v\|^{2}
$$

for all $y(u, \lambda) \in S(u, \lambda)$ and $y(v, \lambda) \in S(v, \lambda)$,
(iii) relaxed monotone with respect to a mapping $G: H \times \Omega \rightarrow H$ if there exists a constant $c>0$ such that

$$
\langle G(y(u, \lambda), \lambda)-G(y(v, \lambda), \lambda), u-v\rangle \geq-c\|u-v\|^{2} .
$$

for all $y(u, \lambda) \in S(u, \lambda)$ and $y(v, \lambda) \in S(v, \lambda)$.
Assumption 2.1. For all $u, v, w \in H$, the operator $J_{\partial \varphi(\cdot, \cdot)}$ satisfies the condition

$$
\left\|J_{\partial \varphi(\cdot, u)}(w)-J_{\partial \varphi(\cdot, v)}(w)\right\| \leq \mu\|u-v\|
$$

where $\mu>0$ is a constant.

## 3. Main Results

First of all, we prove the following lemma.
Lemma 3.1. Fixed $\bar{\lambda} \in \Omega, \bar{u}=u(\bar{\lambda}) \in H, x(u(\bar{\lambda}), \bar{\lambda}) \in M(u(\bar{\lambda}), \bar{\lambda}), y(u(\bar{\lambda}), \bar{\lambda}) \in$ $S(u(\bar{\lambda}), \bar{\lambda}), z(u(\bar{\lambda}), \bar{\lambda}) \in T(u(\bar{\lambda}), \bar{\lambda})$ is a solution of problem (2.1) if and only if for some given $\eta>0$, the set valued mapping $Q: H \times \Omega \rightarrow 2^{H}$ defined by (3.1)
$Q(u, \lambda)=\bigcup_{x \in M(u, \lambda), y \in S(u, \lambda), z \in T(u, \lambda)}\left[u-g(u, \lambda)+J_{\partial \varphi(\cdot u)}(g(u, \lambda)-\eta(P(x, \lambda)-(F(y, \lambda)-G(z, \lambda))))\right]$
has a fixed point $u(\bar{\lambda})$.

Proof. For any fixed $\bar{\lambda} \in \Omega$, let $\bar{u}=u(\bar{\lambda}), x(\bar{u}, \bar{\lambda}) \in M(\bar{u}, \bar{\lambda}), y(\bar{u}, \bar{\lambda}) \in S(\bar{u}, \bar{\lambda})$, $z(\bar{u}, \bar{\lambda}) \in T(\bar{u}, \bar{\lambda})$ be a solution of $(2.1)$. Then $\bar{u} \in H, x(\bar{u}, \bar{\lambda}) \in M(\bar{u}, \bar{\lambda})$, $y(\bar{u}, \bar{\lambda}) \in S(\bar{u}, \bar{\lambda})$ and $z(\bar{u}, \bar{\lambda}) \in T(\bar{u}, \bar{\lambda})$ such that

$$
\begin{aligned}
\langle P(x(\bar{u}, \bar{\lambda}), \bar{\lambda}) & -(F(y(\bar{u}, \bar{\lambda}), \bar{\lambda})-G(z(\bar{u}, \bar{\lambda}), \bar{\lambda})), v-g(\bar{u}, \bar{\lambda})\rangle \\
& \geq \varphi(g(\bar{u}, \bar{\lambda}), u)-\varphi(v, u), \text { for all } v \in H .
\end{aligned}
$$

By definition of $\partial \varphi$, we have

$$
g(\bar{u}, \bar{\lambda})-\eta(P(x(\bar{u}, \bar{\lambda}), \bar{\lambda})-(F(y(\bar{u}, \bar{\lambda}), \bar{\lambda})-G(z(\bar{u}, \bar{\lambda}), \bar{\lambda}))) \in g(\bar{u}, \bar{\lambda})
$$

Thus we obtain

$$
\begin{aligned}
& (I+\eta \partial \varphi(\cdot, u))^{-1}[g(\bar{u}, \bar{\lambda})-\eta(P(x(\bar{u}, \bar{\lambda}), \bar{\lambda})-(F(y(\bar{u}, \bar{\lambda}), \bar{\lambda})-G(z(\bar{u}, \bar{\lambda}), \bar{\lambda})))] \\
& =g(\bar{u}, \bar{\lambda})
\end{aligned}
$$

i.e.

$$
J_{\partial \varphi(\cdot, u)}[g(\bar{u}, \bar{\lambda})-\eta(P(x(\bar{u}, \bar{\lambda}), \bar{\lambda})-(F(y(\bar{u}, \bar{\lambda}), \bar{\lambda})-G(z(\bar{u}, \bar{\lambda}), \bar{\lambda})))]=g(\bar{u}, \bar{\lambda})
$$

Hence, we have

$$
\begin{aligned}
\bar{u}= & \bar{u}-g(\bar{u}, \bar{\lambda}) \\
& +J_{\partial \varphi(\cdot, u)}[g(\bar{u}, \bar{\lambda})-\eta(P(x(\bar{u}, \bar{\lambda}), \bar{\lambda})-(F(y(\bar{u}, \bar{\lambda}), \bar{\lambda})-G(z(\bar{u}, \bar{\lambda}), \bar{\lambda})))] \\
\in & \left.\bigcup_{x(\bar{u}, \bar{\lambda}) \in M(\bar{u}, \bar{\lambda}), y(\bar{u}, \bar{\lambda}) \in S(\bar{u}, \bar{\lambda}), z(\bar{u}, \bar{\lambda}) \in T(\bar{u}, \bar{\lambda})} J_{\partial \varphi(\cdot u}\{g(\bar{u}, \bar{\lambda})-\eta(P(x(\bar{u}, \bar{\lambda}), \bar{\lambda})-(F(y(\bar{u}, \bar{\lambda}), \bar{\lambda})-G(z \bar{u}, \bar{\lambda}), \bar{\lambda})))\right\} \\
= & Q(\bar{u}, \bar{\lambda}) .
\end{aligned}
$$

This means that $\bar{u}=u(\bar{\lambda})$ is a fixed point of $Q(\bar{u}, \bar{\lambda})$. Now, for any fixed $\bar{\lambda} \in \Omega$, let $\bar{u}=u(\bar{\lambda})$ be a fixed point of $Q(\bar{u}, \bar{\lambda})$. By the definition of $Q$, there exists $x(\bar{u}, \bar{\lambda})=M(\bar{u}, \bar{\lambda}), y(\bar{u}, \bar{\lambda})=S(\bar{u}, \bar{\lambda})$ and $z(\bar{u}, \bar{\lambda})=T(\bar{u}, \bar{\lambda})$ such that

$$
\begin{aligned}
\bar{u}= & \bar{u}-g(\bar{u}, \bar{\lambda}) \\
& +J_{\partial \varphi(\cdot, u)}[g(\bar{u}, \bar{\lambda})-\eta(P(x(\bar{u}, \bar{\lambda}), \bar{\lambda})-(F(y(\bar{u}, \bar{\lambda}), \bar{\lambda})-G(z(\bar{u}, \bar{\lambda}), \bar{\lambda})))]
\end{aligned}
$$

i.e.,
$g(\bar{u}, \bar{\lambda})=J_{\partial \varphi(\cdot, u)}[g(\bar{u}, \bar{\lambda})-\eta(P(x(\bar{u}, \bar{\lambda}), \bar{\lambda})-(F(y(\bar{u}, \bar{\lambda}), \bar{\lambda})-G(z(\bar{u}, \bar{\lambda}), \bar{\lambda})))]$.
By definition of $J_{\partial \varphi}$ we get

$$
\begin{aligned}
& g(\bar{u}, \bar{\lambda})-\eta(P(x(\bar{u}, \bar{\lambda}), \bar{\lambda})-(F(y(\bar{u}, \bar{\lambda}), \bar{\lambda})-G(z(\bar{u}, \bar{\lambda}), \bar{\lambda}))) \\
& \quad \in g(\bar{u}, \bar{\lambda})+\eta \partial \varphi(\cdot, u)(g(\bar{u}, \bar{\lambda}))
\end{aligned}
$$

i.e.,

$$
(F(y(\bar{u}, \bar{\lambda}), \bar{\lambda})-G(z(\bar{u}, \bar{\lambda}), \bar{\lambda}))-P(x(\bar{u}, \bar{\lambda}), \bar{\lambda}) \in \partial \varphi(\cdot, u)(g(\bar{u}, \bar{\lambda}))
$$

Hence by definition of $\partial \varphi$

$$
\begin{aligned}
\langle P(x(\bar{u}, \bar{\lambda}), \bar{\lambda}) & -(F(y(\bar{u}, \bar{\lambda}), \bar{\lambda})-G(z(\bar{u}, \bar{\lambda}), \bar{\lambda})), v-g(\bar{u}, \bar{\lambda})\rangle \\
& \geq \varphi(g(\bar{u}, \bar{\lambda}), u)-\varphi(v, u), \text { for all } v \in H
\end{aligned}
$$

This completes the proof.
Theorem 3.1. Let $g: \Omega \times H \rightarrow H$ be a strongly monotone and locally Lipschitz continuous with corresponding constants $\alpha>0$ and $\beta>0$ respectively. Let $F, G, P: \Omega \times H \rightarrow H$ be the locally Lipschitz continuous with corresponding constants $\xi>0, \rho>0$ and $\sigma>0$ respectively. Let $M, S, T: H \times \Omega \rightarrow 2^{H}$ be the locally $\delta$-Lipschitz continuous [ $\hat{H}$-Lipschitz continuous] with constants $\nu>0$, $\gamma>0$ and $\epsilon>0$ respectively and operator $S$ be the relaxed Lipschitz continuous with respect to $F$ with constant $s \geq 0$ and operator $T$ is relaxed monotone with respect to $G$ with constant $\nu$. Let $\varphi: H \times H \rightarrow R \cup\{+\infty\}$ be such that for each fixed $\nu \in H, \varphi(\cdot, v)$ is a proper, convex and lower semicontinuous function on $H$, such that $g(u, \lambda) \cap \operatorname{dom} \partial \varphi(\cdot, v) \neq \emptyset$. Assume that Assumption 2.1 holds. If there exists a constant $\eta>0$ such that

$$
\begin{aligned}
& \left|\eta-\frac{(s-c)(1-\kappa) \sigma \nu}{(\xi \gamma+\rho \epsilon)^{2}-\sigma^{2} \nu^{2}}\right|<\frac{\sqrt{((s-c)(1-\kappa) \sigma \nu)^{2}-\kappa(2-\kappa)\left((\xi \gamma+\rho \epsilon)^{2}-\sigma^{2} \nu^{2}\right)}}{(\xi \gamma+\rho \epsilon)^{2}-\sigma^{2} \nu^{2}} \\
& (s-c)>\sigma \nu(1-k)+\sqrt{\kappa(2-\kappa)\left((\xi \gamma+\rho \epsilon)^{2}-\sigma^{2} \nu^{2}\right)} \\
& s .2) \quad s-c>\sigma \nu(1-\kappa) \\
& \kappa<1 \\
& \quad \sigma \nu<\xi \gamma+\rho \epsilon
\end{aligned}
$$

Then the set-valued mapping $Q: H \times \Omega \rightarrow 2^{H}$ defined by (3.1) is a uniform $\theta-\delta$ set-valued $[\theta-\hat{H}$-set valued $]$ contraction mapping with respect to $\lambda \in \Omega$, where

$$
\begin{aligned}
\theta & =\kappa+\eta \sigma \nu+t(\eta)<1 \\
\kappa & =2 \sqrt{1-2 \alpha+\beta^{2}}+\mu \\
t(\eta) & =\sqrt{1-2(s-c) \eta+\eta^{2}(\xi \gamma+\rho \epsilon)^{2}}
\end{aligned}
$$

Proof. By the definition of $Q$, for any $u, v \in H, a(u, \lambda) \in Q(u, \lambda)$ and $b(v, \lambda) \in$ $Q(v, \lambda)$, there exists $x_{1}(u, \lambda) \in M(u, \lambda), y_{1}(u, \lambda) \in S(u, \lambda), z_{1}(u, \lambda) \in T(u, \lambda)$, $x_{2}(v, \lambda) \in M(v, \lambda), y_{2}(v, \lambda) \in S(v, \lambda)$ and $z_{2}(v, \lambda) \in T(v, \lambda)$ such that

$$
\begin{aligned}
a(u, \lambda)= & (u-g(u, \lambda) \\
& +J_{\partial \varphi(\cdot, u)}\left[g(u, \lambda)-\eta\left(P\left(x_{1}(u, \lambda), \lambda\right)-\left(F\left(y_{1}(u, \lambda), \lambda\right)-G\left(z_{1}(u, \lambda), \lambda\right)\right)\right)\right] \\
b(v, \lambda)= & (v-g(v, \lambda) \\
& +J_{\partial \varphi(\cdot, v)}\left[g(v, \lambda)-\eta\left(P\left(x_{2}(v, \lambda), \lambda\right)-\left(F\left(y_{2}(v, \lambda), \lambda\right)-G\left(z_{2}(v, \lambda), \lambda\right)\right)\right)\right]
\end{aligned}
$$

Then by Lemma 1.1 and Assumption 2.1,

$$
\begin{aligned}
\| a(u, \lambda)- & b(v, \lambda) \| \\
\leq & \|u-v-(g(u, \lambda)-g(v, \lambda))\| \\
& +\| J_{\partial \varphi(\cdot, u)}\left[g(u, \lambda)-\eta\left(P\left(x_{1}(u, \lambda), \lambda\right)-\left(F\left(y_{1}(u, \lambda), \lambda\right)\right.\right.\right. \\
& \left.-G\left(z_{1}(u, \lambda), \lambda\right)\right)-J_{\partial \varphi(\cdot, u)}\left[g(v, \lambda)-\eta\left(P\left(x_{2}(v, \lambda), \lambda\right)\right.\right. \\
& \left.-\left(F\left(y_{2}(v, \lambda), \lambda\right)-G\left(z_{2}(v, \lambda), \lambda\right)\right)\right)\|+\| J_{\partial \varphi(\cdot, u)}[g(v, \lambda) \\
& -\eta\left(P\left(x_{2}(v, \lambda), \lambda\right)-\left(F\left(y_{2}(v, \lambda), \lambda\right)-G\left(z_{2}(v, \lambda), \lambda\right)\right)\right) \\
& -J_{\partial \varphi}(\cdot, v)\left[g(v, \lambda)-\eta\left(P\left(x_{2}(v, \lambda), \lambda\right)\right.\right. \\
& \left.-\left(F\left(y_{2}(v, \lambda), \lambda\right)-G\left(z_{2}(v, \lambda), \lambda\right)\right)\right) \| \\
\leq & \|u-v-(g(u, \lambda)-g(v, \lambda))\| \\
& +\| g(u, \lambda)-g(v, \lambda)-\eta\left(P\left(x_{1}(u, \lambda), \lambda\right)-P\left(x_{2}(v, \lambda), \lambda\right)\right. \\
& -\left(F\left(y_{1}(u, \lambda), \lambda\right)-G\left(z_{1}(u, \lambda), \lambda\right)\right)+\left(F\left(y_{2}(v, \lambda), \lambda\right)\right. \\
& \left.-G\left(z_{2}(v, \lambda), \lambda\right)\right)\|+\mu\| u-v \| \\
\leq & 2 \| u-v-g(u, \lambda)-g(v, \lambda)) \| \\
& +\eta\left\|P\left(x_{1}(u, \lambda), \lambda\right)-P\left(x_{2}(v, \lambda), \lambda\right)\right\|+\| u-v \\
& +\eta\left(F\left(y_{1}(u, \lambda), \lambda\right)-F\left(y_{2}(v, \lambda), \lambda\right)\right)-\eta\left(G\left(z_{1}(u, \lambda), \lambda\right)\right. \\
& \left.-G\left(z_{2}(v, \lambda), \lambda\right)\right)\|+\mu\| u-v \| .
\end{aligned}
$$

Since $g$ is strongly monotone and Lipschitz continuous, we have

$$
\begin{equation*}
\|u-v-(g(u, \lambda)-g(v, \lambda))\| \leq \sqrt{1-2 \alpha+\beta^{2}}\|u-v\| \tag{3.4}
\end{equation*}
$$

Since $M, S, T$ are locally $\delta$-Lipschitz continuous, we have and $P, F, G$ are locally Lipschitz continuous

$$
\begin{align*}
\left\|P\left(x_{1}(u, \lambda), \lambda\right)-P\left(x_{2}(v, \lambda), \lambda\right)\right\| & \leq \sigma\left\|x_{1}(u, \lambda)-x_{2}(v, \lambda)\right\| \\
& \leq \sigma \delta(M(u, \lambda), M(v, \lambda))  \tag{3.5}\\
& \leq \sigma \nu\|u-v\|,
\end{align*}
$$

$$
\begin{align*}
\left\|F\left(y_{1}(u, \lambda), \lambda\right)-F\left(y_{2}(v, \lambda), \lambda\right)\right\| & \leq \xi\left\|y_{1}(u, \lambda)-y_{2}(v, \lambda)\right\| \\
& \leq \xi \delta(S(u, \lambda), S(v, \lambda))  \tag{3.6}\\
& \leq \xi \gamma\|u-v\|,
\end{align*}
$$

$$
\begin{aligned}
\left\|G\left(z_{1}(u, \lambda), \lambda\right)-G\left(z_{2}(v, \lambda), \lambda\right)\right\| & \leq \rho\left\|z_{1}(u, \lambda)-z_{2}(v, \lambda)\right\| \\
& \leq \rho \delta(T(u, \lambda), T(v, \lambda)) \\
& \leq \rho \epsilon\|u-v\| .
\end{aligned}
$$

Further since $S$ is locally relaxed Lipschitz continuous and $T$ is locally relaxed monotone, we have

$$
\begin{align*}
& \left\|u-v+\eta\left(F\left(y_{1}(u, \lambda), \lambda\right)-F\left(y_{2}(v, \lambda), \lambda\right)\right)-\eta\left(G\left(z_{1}(u, \lambda), \lambda\right)-G\left(z_{2}(v, \lambda), \lambda\right)\right)\right\|^{2}  \tag{3.8}\\
& \leq\|u-v\|^{2}+2 \eta\left\langle F\left(y_{1}(u, \lambda), \lambda\right)-F\left(y_{2}(v, \lambda), \lambda\right), u-v\right\rangle \\
& \quad-2 \eta\left\langle G\left(z_{1}(u, \lambda), \lambda\right)-G\left(z_{2}(\nu, \lambda), \lambda\right), u-v\right\rangle \\
& \quad+\eta^{2}\left\|F\left(y_{1}(u, \lambda), \lambda\right)-F\left(y_{2}(v, \lambda), \lambda\right)-\left(G\left(z_{1}(u, \lambda), \lambda\right)-G\left(z_{2}(\nu, \lambda), \lambda\right)\right)\right\|^{2} \\
& \leq\left[1-2 \eta(s-c)+\eta^{2}(\xi \gamma+\rho \epsilon)^{2}\right]\|u-v\| .
\end{align*}
$$

From (3.3) and (3.8), we get
(3.9)

$$
\begin{aligned}
& \|a(u, \lambda)-b(v, \lambda)\| \\
& \leq\left[2 \sqrt{1-2 \alpha+\beta^{2}}+\sigma \nu \eta+\sqrt{1-2 \eta(s-c)+\eta^{2}(\xi \gamma+\rho \epsilon)^{2}}+\mu\right]\|u-v\| \\
& \leq(\kappa+\eta \sigma \nu+t(\eta))\|u-v\|
\end{aligned}
$$

where $\kappa=2 \sqrt{1-2 \alpha+\beta^{2}}+\mu$ and $t(\eta)=\sqrt{1-2 \eta(s-c)+\eta^{2}(\xi \gamma+\rho \epsilon)^{2}}$.
By the arbitrariness of $a(u, \lambda)$ and $b(v, \lambda)$, we have

$$
\delta(Q(u, \lambda), Q(v, \lambda)) \leq \theta d(u, v)
$$

where $\theta=\kappa+\eta \sigma \nu+t(\eta)$. By the condition (3.2) we have $\theta<1$. This proves that $Q$ is a uniform $\theta-\delta$-set-valued contraction mapping with respect to $\lambda \in \Omega$.

Theorem 3.2. Assume that $g, F, G, P, M, S, T$ are follows all hypothesis in Theorem 3.1. Suppose there exists a constant $\eta>0$ such that (3.2) in Theorem 3.1 and Assumption 2.1 hold. Then
(i) the set-valued mapping $Q: H \times \Omega \rightarrow 2^{H}$ defined by (3.1) in Lemma 3.1 is a compact uniform $\theta$ - $\hat{H}$-contraction mapping with respect to $\lambda \in \Omega$.
(ii) for each $\lambda \in \Omega$, the (2.1) has nonempty solution set $N(\lambda)$ and $M(\lambda)$ is closed in $H$.

Proof. For each given $(u, \lambda) \in H \times \Omega$. Since $M(u, \lambda), S(u, \lambda), T(u, \lambda) \in$ $C(H)$ and $J_{\partial \varphi(\cdot, u)}$ is continuous, it follows from the definition of $Q(u, \lambda)$ that $Q(u, \lambda) \in C(H)$.

Now, we show that $Q(u, \lambda)$ is a uniform $\theta-\hat{H}$-contraction mapping with respect to $\lambda \in \Omega$. For any given $(u, \lambda),(v, \lambda) \in H \times \Omega$, and for any $a \in Q(u, \lambda)$ there exist $x \in M(u, \lambda), y \in S(u, \lambda)$ and $z \in T(u, \lambda) \in C(H)$ such that

$$
a=u-g(u, \lambda)+J_{\partial \phi(\cdot, u)}[g(u, \lambda)-\eta(P(x, \lambda)-(F(y, \lambda)-G(z, \lambda)))] .
$$

Note that $M(u, \lambda), S(u, \lambda), T(u, \lambda) \in C(H)$, there exist $x_{1} \in M(v, \lambda), y_{1} \in$ $S(v, \lambda)$ and $z \in T(v, \lambda)$ such that

$$
\begin{align*}
\left\|x-x_{1}\right\| & \leq \hat{H}(M(u, \lambda), M(v, \lambda)) \\
\left\|y-y_{1}\right\| & \leq \hat{H}(S(u, \lambda), S(v, \lambda))  \tag{3.10}\\
\left\|z-z_{1}\right\| & \leq \hat{H}(T(u, \lambda), T(v, \lambda))
\end{align*}
$$

Let

$$
b=v-g(v, \lambda)+J_{\partial \phi(\cdot, v)}\left[g(v, \lambda)-\eta\left(P\left(x_{1}, \lambda\right)-\left(F\left(y_{1}, \lambda\right)-G\left(z_{1}, \lambda\right)\right)\right)\right]
$$

Then $b \in Q(v, \lambda)$. Note inequalities (3.10), by using a similar argument as in the proof of Theorem 3.1, we can obtain

$$
\hat{H}(Q(u, \lambda), Q(v, \lambda)) \leq \theta\|u-v\|
$$

This proves that $Q(u, \lambda)$ is a uniform $\theta-\hat{H}$-contraction mapping with respect to $\lambda \in \Omega$.
(ii) Since $Q(u, \lambda)$ is a uniform $\theta-\hat{H}$-contraction mapping with respect to $\lambda \in \Omega$, by the Nadler fixed point theorem [6], $Q(u, \lambda)$ has a fixed point $u(\lambda)$ for each $\lambda \in \Omega$. By Lemma 3.1. $N(\lambda) \neq \emptyset$. For each $\lambda \in \Omega$, let $\left\{u_{n}\right\} \subset N(\lambda)$ and let $u_{n} \rightarrow u_{0}$, as $n \rightarrow \infty$. Then we have

$$
u_{n} \in Q\left(u_{n}, \lambda\right), n=1,2, \cdots
$$

By (i), we have

$$
\hat{H}\left(Q\left(u_{n}, \lambda\right), Q\left(u_{0}, \lambda\right)\right) \leq \theta\left\|u_{n}-u_{0}\right\|
$$

It follows that

$$
\begin{aligned}
d\left(u_{0}, Q\left(u_{0}, \lambda\right)\right) & \leq\left\|u_{0}-u_{n}\right\|+d\left(u_{n}, Q\left(u_{n}, \lambda\right)\right)+\hat{H}\left(Q\left(u_{n}, \lambda\right), Q\left(u_{0}, \lambda\right)\right) \\
& \leq(1+\theta)\left\|u_{n}-u_{0}\right\| \rightarrow 0 \text { as } n \rightarrow \infty
\end{aligned}
$$

and hence $u_{0} \in Q\left(u_{0}, \lambda\right)$ and $x_{0} \in N(\lambda)$. These $N(\lambda)$ is closed in $H$.

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