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SOME CRITERION FOR CERTAIN MULTIVALENTLY ANALYTIC FUNCTIONS AND MULTIVALENTLY MEROMORPHIC FUNCTIONS

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Abstract. The aim of the present investigation is to give some criterion for certain multivalently analytic functions and (or) multivalently meromorphic functions in the corresponding domains.

1. Introduction and Motivation

Let the *non-normalized* function $\mathcal{Q}_a$ be *non-constant*, *analytic* and *univalent* in the *unit open disc* $\mathcal{U} = \{z \in \mathbb{C} \ : \ |z| < 1\}$ such that $\mathcal{Q}_a(0) = a$, where $a \in \mathbb{C}$.

In this paper, we establish a theorem characterizing the inclusion property of functions $\mathcal{Q}_a$, by using the well-known Jack's Lemma and Nunakowa's Lemma. Consequences of our main result are considered for certain multivalently analytic functions and multivalently meromorphic functions in the certain domains. See, for example, some interesting papers concerning multivalently analytic functions, multivalently meromorphic functions and (or) certain differential inequalities [6-10].

First, we mention the following results which are used in the next section.

Lemma 1 (Jack's Lemma, [4]). *Let $\omega(z)$ be non-constant and regular in $\mathcal{U}$ with $\omega(0) = 0$. If $|\omega(z)|$ attains its maximum value on the circle $|z| = r$ $(0 < r < 1)$ at the point z_0, then $z_0\omega'(z_0) = c\omega(z_0)$, where $c \geq 1$.*

Lemma 2 (Nunokawa's Lemma, [5]). *Let $p(z)$ be an analytic function in $\mathcal{U}$ with $p(0) = 1$. If there exists a point $z_0 \in \mathcal{U}$ such that*

$$\Re e\,\{p(z)\} > 0 \ (|z| < |z_0|), \ \Re e\,\{p(z_0)\} = 0 \ \text{ and } \ p(z_0) \neq 0,$$

then

$$p(z_0) = ib \ \text{ and } \ \frac{z_0 p'(z_0)}{p(z_0)} = i\frac{c}{2}\left(b + \frac{1}{b}\right),$$

where $b \neq 0$ and $c \geq 1$.

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2. Main Result and Its Certain Applications

Using Lemmas 1 and 2 we now prove the following result.

Theorem 2.1. *If the function $\mathcal{Q}_a$ satisfies one of the following conditions:*

$$\Re e\left\{\frac{z\mathcal{Q}_a'(z)}{1-a+\mathcal{Q}_a(z)}\right\} < \frac{1}{2} \tag{1}$$

and

$$\Re e\left\{\frac{z\mathcal{Q}_a'(z)}{1+\mathcal{Q}_a(z)}\right\}\left\{\begin{array}{ll} >0 & when \ \ \Re e\,\{a\}\geq 0 \\ <0 & when \ \ \Re e\,\{a\}\leq 0 \end{array}\right\} \tag{2}$$

then

$$\Re e\,\{\mathcal{Q}_a(z)\} > \Re e\,\{a\}-1, \tag{3}$$

where $z\in\mathcal{U}$ and $a\in\mathbb{C}$.

Proof. Define a function w by

$$w(z)=\mathcal{Q}_a(z)-a \quad (a\in\mathbb{C},\ z\in\mathcal{U}). \tag{4}$$

It is obvious that w is analytic in $\mathcal{U}$ and $w(0)=0$. We find from (4) that

$$\frac{z\mathcal{Q}_a'(z)}{1-a+\mathcal{Q}_a(z)}=\frac{zw'(z)}{1+w(z)} \quad (z\in\mathcal{U}). \tag{5}$$

Assume that there exist a point $z_0\in\mathcal{U}$ such that

$$|w(z_0)|=1 \ \ \textit{and} \ \ |w(z)|<1 \ \ \textit{when} \ \ |z|<|z_0| \ \ (z\in\mathcal{U}).$$

Then, applying Lemma 1, we have

$$z_0w'(z_0)=cw(z_0) \quad \left(c\geq 1,\ w(z_0)=e^{i\varphi}\neq -1\right). \tag{6}$$

From (5) and (6) we obtain

$$\Re e\left\{\left.\frac{z\mathcal{Q}_a'(z)}{1-a+\mathcal{Q}_a(z)}\right|_{z=z_0}\right\}$$
$$=\Re e\left\{\left.\frac{zw'(z)}{1+w(z)}\right|_{z=z_0}\right\}=\Re e\left\{\frac{ce^{i\varphi}}{1+e^{i\varphi}}\right\}=\frac{c}{2},$$

which obviously contradicts the condition (1) when $c=1$. It follows that $|w(z)|<1$ for all $z\in\mathcal{U}$, that is

$$|\mathcal{Q}_a(z)-a|=|w(z)|<1,$$

which implies the inequality (3).

Next, we again define a new function q by

$$q(z)=1+\mathcal{Q}_a(z)-a \quad (a\in\mathbb{C},\ z\in\mathcal{U}). \tag{7}$$

Then it is easy to see that q is analytic in $\mathcal{U}$ with $q(0) = 1$. We have from (7) that

$$\frac{z\mathcal{Q}_a'(z)}{1+\mathcal{Q}_a(z)} = \frac{zq'(z)}{a+q(z)} \quad (z \in \mathcal{U}). \tag{8}$$

Suppose now that there exist a point $z_0 \in \mathcal{U}$ such that

$$\Re e\,\{q(z)\} > 0 \ (|z| < |z_0|), \ \Re e\,\{q(z_0)\} = 0, \ \textit{and} \ q(z_0) \neq 0 \ (z \in \mathcal{U}).$$

Then, by using Lemma 2, we have

$$q(z_0) = ib \ \ \textit{and} \ \ \frac{z_0 q'(z_0)}{q(z_0)} = i\frac{c}{2}\left(b + \frac{1}{b}\right) \quad (b \neq 0, \ c \geq 1). \tag{9}$$

Thus, we have from (8) and (9) that

$$\Re e\left\{ \left.\frac{z\mathcal{Q}_a'(z)}{1+\mathcal{Q}_a(z)}\right|_{z=z_0} \right\} = \Re e\left\{ \frac{z_0 q'(z_0)}{q(z_0)} \cdot \frac{q(z_0)}{a+q(z_0)} \right\}$$

$$= \Re e\left\{ i\frac{c}{2}\left(b+\frac{1}{b}\right)\frac{ib(\overline{a+ib})}{|a+ib|^2} \right\} = -\frac{c}{2}\frac{1+b^2}{|a+ib|^2}\Re e\,\{\overline{a}\}\,. \tag{10}$$

Therefore, we easily get from (10):

$$\Re e\left\{\frac{z_0\mathcal{Q}_a'(z_0)}{1+\mathcal{Q}_a(z_0)}\right\}\left\{ \begin{array}{ll} \leq 0 \ \ if & \Re e\,\{a\} \geq 0 \\ \geq 0 \ \ if & \Re e\,\{a\} \leq 0 \end{array} \right\},$$

which it contradicts the condition (2). So, it follows that $\Re e\,\{q(z)\} > 0$ for all $z \in \mathcal{U}$, that is that

$$\Re e\,\{1 - a + \mathcal{Q}_a(z)\} > 0.$$

Hence, the proof of Theorem is completed. □

Denote by $\mathcal{T}(p)$ and $\mathcal{M}(p)$, where $p \in \mathbb{N} = \{0, 1, 2, 3, \dots\}$, the classes of normalized functions f and g of the forms:

$$f(z) = z^p + a_{p+1}z^{p+1} + a_{p+2}z^{p+2} + \cdots$$

and

$$g(z) = z^{-p} + a_{-p+1}z^{-p+1} + a_{-p+2}z^{-p+2} + \cdots,$$

which are analytic and multivalent in the unit disc $\mathcal{U}$ and meromorphic multivalent in the punctured unit disc $\mathcal{D} = \mathcal{U} \setminus \{0\}$, respectively.

In order to obtain several useful results in the Analytic Function Theory and Geometric Function Theory we apply our Theorem to the classes $\mathcal{T}(p)$ and $\mathcal{M}(p)$. First we recall some well-known subclasses of $\mathcal{T}(p)$ and $\mathcal{M}(p)$, which play an important role in the Geometric Function Theory (see, for details, [1-3]):

$$\mathcal{TS}(p,\alpha) = \left\{ f \in \mathcal{T}(p) : \ \Re e\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \ (z \in \mathcal{U}) \right\}, \tag{11}$$

$$\mathcal{TC}(p,\alpha) = \left\{ f \in \mathcal{T}(p):\ \Re e\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \alpha\ (z\in\mathcal{U})\right\}, \tag{12}$$

$$\mathcal{TK}(p,\alpha) = \left\{ f \in \mathcal{T}(p):\ \Re e\left\{\frac{f'(z)}{z^{p-1}}\right\} > \alpha\ (z\in\mathcal{U})\right\}, \tag{13}$$

$$\mathcal{MS}(p,\alpha) = \left\{ f \in \mathcal{M}(p):\ \Re e\left\{-\frac{zf'(z)}{f(z)}\right\} > \alpha\ (z\in\mathcal{D})\right\} \tag{14}$$

and

$$\mathcal{MC}(p,\alpha) = \left\{ f \in \mathcal{M}(p):\ \Re e\left\{-1-\frac{zf'(z)}{f(z)}\right\} > \alpha\ (z\in\mathcal{D})\right\}, \tag{15}$$

where $p\in\mathbb{N}$ and $0\le\alpha<p$.

In the literature, the functions in (11) - (15) are called, respectively, multivalently starlike of order α in $\mathcal{U}$, multivalently convex of order α in $\mathcal{U}$, multivalently close-to-convex of order α in $\mathcal{U}$, multivalently meromorphic starlike of order α in $\mathcal{D}$ and multivalently meromorphic convex of order α in $\mathcal{D}$. See, for their details, [1-3]. Now we can give some interesting and (or) important results conserning the above well-known results:

Corollary 1. *Let $f\in\mathcal{T}(p)$ $(p\in\mathbb{N})$, $\mathcal{Q}_1(z) := z^{-p}f(z)$ and also let the function $\mathcal{Q}_1$ satisfy the assumptions of the Theorem. Then*

$$\Re e\left\{\mathcal{Q}_1(z)\right\} = \Re e\left\{z^{-p}f(z)\right\} > 0.$$

Corollary 2. *Let $f\in\mathcal{T}(p)$ $(p\in\mathbb{N})$, $\mathcal{Q}_p(z) := z^{1-p}f'(z)$ and also let the function $\mathcal{Q}_p$ satisfy the assumptions of the Theorem. Then*

$$\Re e\left\{\mathcal{Q}_p(z)\right\} = \Re e\left\{z^{1-p}f'(z)\right\} > p-1,$$

i.e., $f\in\mathcal{TK}(p,p-1)$.

Corollary 3. *Let $f\in\mathcal{T}(p)$ $(p\in\mathbb{N})$, $\mathcal{Q}_p(z) := \frac{zf'(z)}{f(z)}$ and also let the function $\mathcal{Q}_p$ satisfy the assumptions of the Theorem. Then*

$$\Re e\left\{\mathcal{Q}_p(z)\right\} = \Re e\left\{\frac{zf'(z)}{f(z)}\right\} > p-1,$$

i.e., $f\in\mathcal{TS}(p,p-1)$.

Corollary 4. *Let $f\in\mathcal{T}(p)$ $(p\in\mathbb{N})$, $\mathcal{Q}_p(z) := 1+\frac{zf''(z)}{f'(z)}$ and also let the function $\mathcal{Q}_p$ satisfy the assumptions of the Theorem. Then*

$$\Re e\left\{\mathcal{Q}_p(z)\right\} = \Re e\left\{\frac{zf''(z)}{f'(z)}\right\} > p-2,$$

i.e., $f\in\mathcal{TC}(p,p-1)$.

Corollary 5. *Let $f \in \mathcal{M}(p)$ $(p \in \mathbb{N})$, $\mathcal{Q}_p(z) := -\frac{zf'(z)}{f(z)}$ and also let the function $\mathcal{Q}_p$ satisfy the assumptions of the Theorem. Then*

$$\Re e \{\mathcal{Q}_p(z)\} = \Re e \left\{ -\frac{zf'(z)}{f(z)} \right\} > p - 1,$$

i.e., $f \in \mathcal{MS}(p, p-1)$.

Corollary 6. *Let $f \in \mathcal{M}(p)$, $(p \in \mathbb{N})$, $\mathcal{Q}_p(z) := -1 - \frac{zf''(z)}{f'(z)}$ and let also the function $\mathcal{Q}_p$ satisfy the assumptions of the Theorem. Then*

$$\Re e \{\mathcal{Q}_q(z)\} = \Re e \left\{ -\frac{zf''(z)}{f'(z)} \right\} > q,$$

i.e., $f \in \mathcal{MC}(p, p-1)$.

Corollary 7. *Let $f \in \mathcal{T}(p)$, $g \in \mathcal{M}(q)$ $(p, q \in \mathbb{N})$ and $\mathcal{Q}_1(z) := z^{-p-q}\frac{f(z)}{g(z)}$. If the function $\mathcal{Q}_1$ satisfies the assumptions of the Theorem, then*

$$\Re e \{\mathcal{Q}_1(z)\} = \Re e \left\{ z^{-p-q}\frac{f(z)}{g(z)} \right\} > 0.$$

Corollary 8. *Let $f \in \mathcal{T}(p)$, $g \in \mathcal{M}(q)$ $(p, q \in \mathbb{N})$ and $\mathcal{Q}_{-\frac{q}{p}}(z) := z^{p+q}\frac{g'(z)}{f'(z)}$. If the function $\mathcal{Q}_{-\frac{q}{p}}$ satisfies the assumptions of the Theorem, then*

$$\Re e \left\{ \mathcal{Q}_{-\frac{q}{p}}(z) \right\} = \Re e \left\{ z^{p+q}\frac{g'(z)}{f'(z)} \right\} > -\frac{q}{p} - 1.$$

When we take $p = 1$ in the definitions (11) – (15), then we obtain the other interesting subclasses: $\mathcal{TS}(\alpha) := \mathcal{TS}(1, \alpha)$, $\mathcal{TC}(\alpha) := \mathcal{TC}(1, \alpha)$, $\mathcal{TK}(\alpha) := \mathcal{TK}(1, \alpha)$, $\mathcal{MS}(\alpha) := \mathcal{MS}(1, \alpha)$ and $\mathcal{MC}(\alpha) := \mathcal{MC}(1, \alpha)$ which consist, respectively, starlike functions of order α in $\mathcal{U}$, convex functions of order α in $\mathcal{U}$, close-to-convex functions of order α in $\mathcal{U}$, meromorphically starlike functions of order α in $\mathcal{D}$ and meromorphically convex functions of order α in $\mathcal{D}$, where $0 \leq \alpha < 1$. See, [1-3] and (for example) [8-10]. When we choose the functions f and (or) g of such subclasses of $\mathcal{T} := \mathcal{T}(1)$ and (or) $\mathcal{M} := \mathcal{M}(1)$, we can obtain a lot of results as in the Corollaries 1-8.

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