

## CONVERGENCE THEOREMS OF THE MODIFIED ISHIKAWA ITERATIVE PROCESS FOR NONEXPANSIVE MAPPINGS

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ABSTRACT. In this paper, we introduce an iterative method for a pair of nonexpansive mappings. Strong convergence theorems are established in a real uniformly smooth Banach space.

### 1. Introduction and preliminaries

Let  $E$  be a real Banach space and let  $J$  denote the normalized duality mapping from  $E$  into  $2^{E^*}$  given by

$$J(x) = \{f \in E^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2\}, \quad \forall x \in E,$$

where  $E^*$  denotes the dual space of  $E$  and  $\langle \cdot, \cdot \rangle$  denotes the generalized duality pairing. The norm of  $E$  is said to be *Gâteaux differentiable* (and  $E$  is said to be *smooth*) if

$$\lim_{t \rightarrow 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each  $x, y$  in its unit sphere  $U = \{x \in E : \|x\| = 1\}$ . A Banach space  $E$  whose norm is uniformly Gâteaux differentiable, then the duality map  $J$  is single-valued and norm-to-weak\* uniformly continuous on bounded sets of  $E$ . It is said to be *uniformly Fréchet differentiable* (and  $E$  is said to be *uniformly smooth*) if the limit is attained uniformly for  $(x, y) \in U \times U$ .

Let  $C$  a nonempty closed convex subset of a real Banach space  $E$  and  $T : C \rightarrow C$  a nonlinear mapping. A point  $x \in C$  is a *fixed point* of  $T$  provided that  $Tx = x$ . Denote by  $F(T)$  the set of fixed points of  $T$ , that is,  $F(T) = \{x \in C : Tx = x\}$ .

Recall that the mapping  $T$  is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

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The mapping  $T$  is said to be *contractive* if there exists a constant  $\alpha \in (0, 1)$  such that

$$\|Tx - Ty\| \leq \alpha\|x - y\|, \quad \forall x, y \in C.$$

Some iteration processes are often used to approximate a fixed point of a nonexpansive mapping  $T$ . The first iteration process is now known as normal Mann's iteration process [9] which generates the sequence  $\{x_n\}$  by the following manner:

$$(1.1) \quad x_0 \in C, \quad x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad \forall n \geq 0,$$

where the initial guess  $x_0$  is taken in  $C$  arbitrarily and the sequence  $\{\alpha_n\}_{n=0}^{\infty}$  is in the interval  $[0, 1]$ .

The second iteration process is referred to as Ishikawa's [7] iteration process which is defined recursively by

$$(1.2) \quad \begin{cases} x_0 \in C, \\ y_n = \beta_n x_n + (1 - \beta_n)Tx_n, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Ty_n, \end{cases} \quad \forall n \geq 0,$$

where the initial guess  $x_0$  is taken in  $C$  arbitrarily,  $\{\alpha_n\}$  and  $\{\beta_n\}$  are sequences in the interval  $[0, 1]$ .

But both (1.1) and (1.2) have only weak convergence, in general (see [6] for an example). For example, Reich [18] showed that if  $E$  is a uniformly convex and has a Fréchet differentiable norm and if the sequence  $\{\alpha_n\}$  is such that  $\sum_{n=0}^{\infty} \alpha_n(1 - \alpha_n) = \infty$ , then the sequence  $\{x_n\}$  generated by the process (1.1) converges weakly to a point in  $F(T)$  (An extension of this result to the process (1.2) can be found in [21]).

Attempts to modify the iterative processes (1.1) and (1.2) so that strong convergence is guaranteed have recently been made, see, for example, [2-4,8,10-16,19]. In 2005, Kim and Xu [8] modified the Mann's iterative process and obtained a strong convergence theorem in uniformly smooth Banach spaces. To be more precise, they proved the following results.

**Theorem KX.** *Let  $C$  be a closed convex subset of a uniformly smooth Banach space  $E$  and let  $T : C \rightarrow C$  be a nonexpansive mapping such that  $F(T) \neq \emptyset$ . Given a point  $u \in C$  and given sequences  $\{\alpha_n\}$  and  $\{\beta_n\}$  in  $(0, 1)$ , the following conditions are satisfied*

- (a)  $\alpha_n \rightarrow 0$  and  $\beta_n \rightarrow 0$  as  $n \rightarrow \infty$ ,
- (b)  $\sum_{n=0}^{\infty} \alpha_n = \infty$  and  $\sum_{n=0}^{\infty} \beta_n = \infty$ ,
- (c)  $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$  and  $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$ .

Define a sequence  $\{x_n\}$  in  $C$  by

$$\begin{cases} x_0 = x \in C \text{ chosen arbitrarily,} \\ y_n = \beta_n x_n + (1 - \beta_n)Tx_n, \\ x_{n+1} = \alpha_n u + (1 - \alpha_n)y_n, \end{cases} \quad \forall n \geq 0.$$

Then  $\{x_n\}$  strongly converges to a fixed point of  $T$ .

Very recently, Qin, Su and Shang [14] further improved Kim and Xu [8]'s results by modifying Ishikawa iterative process. More precisely, they obtained the following result.

**Theorem QSS.** *Let  $C$  be a closed convex subset of a uniformly smooth Banach space  $E$  and let  $T : C \rightarrow C$  be a nonexpansive mapping such that  $F(T) \neq \emptyset$ . Given a point  $u \in C$ , the initial guess  $x_0 \in C$  is chosen arbitrarily and given sequences  $\{\alpha_n\}_{n=0}^{\infty}$ ,  $\{\beta_n\}_{n=0}^{\infty}$  and  $\{\gamma_n\}_{n=0}^{\infty}$  in  $[0, 1]$ , the following conditions are satisfied*

- (a)  $\sum_{n=0}^{\infty} \alpha_n = \infty$  as  $n \rightarrow \infty$ ,
- (b)  $\alpha_n \rightarrow 0$ ,  $\beta_n \rightarrow 0$  and  $0 < a \leq \gamma_n$  for some  $a \in (0, 1)$ ,
- (c)  $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ ,  $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$  and  $\sum_{n=0}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$ .

Let  $\{x_n\}_{n=1}^{\infty}$  be a sequence defined by

$$\begin{cases} x_0 = x \in C & \text{chosen arbitrarily,} \\ z_n = \gamma_n x_n + (1 - \gamma_n)Tx_n, \\ y_n = \beta_n x_n + (1 - \beta_n)Tz_n, \\ x_{n+1} = \alpha_n u + (1 - \alpha_n)y_n, & \forall n \geq 0. \end{cases}$$

Then  $\{x_n\}_{n=1}^{\infty}$  converges strongly to a fixed point of  $T$ .

In this paper, motivated by Kim and Xu [8] and Qin, Su and Shang [14], we modify the Ishikawa iterative process for a pair of nonexpansive mappings to have strong convergence by viscosity approximation methods. Strong convergence theorems are established in a real uniformly smooth Banach space under some appropriate restrictions imposed on the control sequences.

In order to prove our main results, we need the following definitions and lemmas.

Recall that if  $C$  and  $D$  are nonempty subsets of a Banach space  $E$  such that  $C$  is nonempty, closed, convex and  $D \subset C$ , then a map  $Q : C \rightarrow D$  is sunny([1], [17]) provided  $Q(x + t(x - Q(x))) = Q(x)$  for all  $x \in C$  and  $t \geq 0$  whenever  $x + t(x - Q(x)) \in C$ . A sunny nonexpansive retraction is a sunny retraction, which is also nonexpansive. Sunny nonexpansive retractions play an important role in our argument. They are characterized as follows [1,17]: if  $E$  is a smooth Banach space, then  $Q : C \rightarrow D$  is a sunny nonexpansive retraction if and only if there holds the inequality

$$\langle x - Qx, J(y - Qx) \rangle \leq 0, \quad \forall x \in C, y \in D.$$

Reich [17] showed that if  $E$  is uniformly smooth and if  $D$  is the fixed point set of a nonexpansive mapping from  $C$  into itself, then there is a sunny nonexpansive retraction from  $C$  onto  $D$  and it can be constructed as follows.

**Lemma 1.1.** ([17]) *Let  $E$  be a uniformly smooth Banach space and let  $T : C \rightarrow C$  be a nonexpansive mapping with a fixed point  $x_t \in C$  of the contraction*

$C \ni x \mapsto tu + (1-t)x$ . Then  $\{x_t\}$  converges strongly as  $t \rightarrow 0$  to a fixed point of  $T$ . Define  $Q : C \rightarrow F(T)$  by  $Qu = \lim_{t \rightarrow 0} x_t$ . Then  $Q$  is the unique sunny nonexpansive retract from  $C$  onto  $F(T)$ , that is,  $Q$  satisfies the property

$$\langle u - Qu, J(z - Qu) \rangle \leq 0, \quad \forall u \in C, z \in F(T).$$

**Lemma 1.2.** *In a Banach space  $E$ , there holds the inequality*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y) \rangle, \quad \forall x, y \in E,$$

where  $j(x + y) \in J(x + y)$ .

**Lemma 1.3.** ([20]) *Let  $\{x_n\}$  and  $\{y_n\}$  be bounded sequences in a Banach space  $E$  and let  $\beta_n$  be a sequence in  $[0, 1]$  with  $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$ . Suppose that  $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$  for all  $n \geq 0$  and*

$$\limsup_{n \rightarrow \infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Then  $\lim_{n \rightarrow \infty} \|y_n - x_n\| = 0$ .

**Lemma 1.4.** ([22]) *Let  $\{\alpha_n\}_{n=0}^{\infty}$  be a sequence of nonnegative real numbers satisfying the property*

$$\alpha_{n+1} \leq (1 - \gamma_n)\alpha_n + \gamma_n \sigma_n, \quad \forall n \geq 0,$$

where  $\{\gamma_n\}_{n=0}^{\infty} \subset (0, 1)$  and  $\{\sigma_n\}_{n=0}^{\infty}$  are such that

- (a)  $\lim_{n \rightarrow \infty} \gamma_n = 0$  and  $\sum_{n=0}^{\infty} \gamma_n = \infty$ ,
- (b) either  $\limsup_{n \rightarrow \infty} \sigma_n \leq 0$  or  $\sum_{n=0}^{\infty} |\gamma_n \sigma_n| < \infty$ .

Then  $\{\alpha_n\}_{n=0}^{\infty}$  converges to zero.

## 2. Main results

**Theorem 2.1.** *Let  $C$  be a closed convex subset of a uniformly smooth Banach space  $E$ ,  $T_1, T_2 : C \rightarrow C$  nonexpansive mappings such that  $F(T_1) \cap F(T_2) = F(T_1 T_2) \neq \emptyset$  and  $f : C \rightarrow C$  a contraction with the coefficient  $\alpha \in (0, 1)$ . Let  $\{x_n\}$  be a sequence defined by the following iterative process*

$$\begin{cases} x_0 = x \in C & \text{chosen arbitrarily,} \\ z_n = \gamma_n x_n + (1 - \gamma_n) T_2 x_n, \\ y_n = \beta_n x_n + (1 - \beta_n) T_1 z_n, \\ x_{n+1} = \alpha_n f(x_n) + \lambda_n x_n + \delta_n y_n, \end{cases} \quad \forall n \geq 0,$$

where  $\{\alpha_n\}$ ,  $\{\beta_n\}$ ,  $\{\gamma_n\}$ ,  $\{\lambda_n\}$  and  $\{\delta_n\}$  are sequences in  $[0, 1]$ . If the following conditions are satisfied

- (a)  $\alpha_n + \lambda_n + \delta_n = 1$  for all  $n \geq 0$ ,
- (b)  $\sum_{n=0}^{\infty} \alpha_n = \infty$  and  $\alpha_n \rightarrow 0$  as  $n \rightarrow \infty$ ,
- (c)  $\beta_n \rightarrow 0$  and  $\gamma_n \rightarrow 0$  as  $n \rightarrow \infty$ ,
- (d)  $0 < \liminf_{n \rightarrow \infty} \lambda_n \leq \limsup_{n \rightarrow \infty} \lambda_n < 1$ ,

then  $\{x_n\}$  converges strongly to a common fixed point of  $T_1$  and  $T_2$ .

*Proof.* First, we show that  $\{x_n\}$  is bounded. Indeed, taking  $p \in F(T_1) \cap F(T_2)$ , we have

$$\|z_n - p\| \leq \gamma_n \|x_n - p\| + (1 - \gamma_n) \|T_2 x_n - p\| \leq \|x_n - p\|.$$

It follows that

$$\begin{aligned} \|y_n - p\| &\leq \beta_n \|x_n - p\| + (1 - \beta_n) \|T_1 z_n - p\| \\ &\leq \beta_n \|x_n - p\| + (1 - \beta_n) \|z_n - p\| \\ &\leq \|x_n - p\|. \end{aligned}$$

This implies that

$$\begin{aligned} \|x_{n+1} - p\| &\leq \alpha_n \|f(x_n) - p\| + \lambda_n \|x_n - p\| + \delta_n \|y_n - p\| \\ &\leq \alpha_n \|f(x_n) - f(p)\| + \alpha_n \|f(p) - p\| + \lambda_n \|x_n - p\| + \delta_n \|y_n - p\| \\ &\leq \alpha_n \|f(x_n) - f(p)\| + \alpha_n \|f(p) - p\| + (1 - \alpha_n) \|x_n - p\| \\ &\leq \max \left\{ \frac{1}{1 - \alpha} \|f(p) - p\|, \|x_n - p\| \right\}. \end{aligned}$$

Now, an induction yields

$$\|x_n - p\| \leq \max \left\{ \frac{1}{1 - \alpha} \|f(p) - p\|, \|x_0 - p\| \right\}, \quad \forall n \geq 0,$$

which implies that  $\{x_n\}$  is bounded, so are  $\{y_n\}$  and  $\{z_n\}$ .

Next, we claim that

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0.$$

Putting  $l_n = \frac{x_{n+1} - \lambda_n x_n}{1 - \lambda_n}$ , we have

$$(2.1) \quad x_{n+1} = (1 - \lambda_n) l_n + \lambda_n x_n, \quad \forall n \geq 0.$$

Now, we compute  $\|l_{n+1} - l_n\|$ . From

$$\begin{aligned} l_{n+1} - l_n &= \frac{\alpha_{n+1} f(x_{n+1}) + \delta_{n+1} y_{n+1}}{1 - \lambda_{n+1}} - \frac{\alpha_n f(x_n) + \delta_n y_n}{1 - \lambda_n} \\ &= \frac{\alpha_{n+1} f(x_{n+1}) + (1 - \lambda_{n+1} - \alpha_{n+1}) y_{n+1}}{1 - \lambda_{n+1}} \\ &\quad - \frac{\alpha_n f(x_n) + (1 - \lambda_n - \alpha_n) y_n}{1 - \lambda_n} \\ &= \frac{\alpha_{n+1}}{1 - \lambda_{n+1}} (f(x_{n+1}) - y_{n+1}) - \frac{\alpha_n}{1 - \lambda_n} (f(x_n) - y_n) + y_{n+1} - y_n, \end{aligned}$$

we have

$$(2.2) \quad \begin{aligned} \|l_{n+1} - l_n\| &\leq \frac{\alpha_{n+1}}{1 - \lambda_{n+1}} \|f(x_{n+1}) - y_{n+1}\| \\ &\quad + \frac{\alpha_n}{1 - \lambda_n} \|y_n - f(x_n)\| + \|y_{n+1} - y_n\|. \end{aligned}$$

On the other hand, we have

$$\begin{cases} z_n = \gamma_n x_n + (1 - \gamma_n)T_2 x_n, \\ z_{n-1} = \gamma_{n-1} x_{n-1} + (1 - \gamma_{n-1})T_2 z_{n-1}, \end{cases}$$

from which it follows that

$$\begin{aligned} z_n - z_{n-1} &= (1 - \gamma_n)(T_2 x_n - T_2 x_{n-1}) + \gamma_n(x_n - x_{n-1}) \\ &\quad + (\gamma_{n-1} - \gamma_n)(T_2 x_{n-1} - x_{n-1}). \end{aligned}$$

This implies that

$$(2.3) \quad \|z_n - z_{n-1}\| \leq \|x_n - x_{n-1}\| + (\gamma_{n-1} + \gamma_n)M_1,$$

where  $M_1$  is an appropriate constant such that  $M_1 \geq \sup_{n \geq 1} \{\|T_2 x_{n-1} - x_{n-1}\|\}$ . In similar way, we have

$$\begin{cases} y_n = \beta_n x_n + (1 - \beta_n)T_1 z_n, \\ y_{n-1} = \beta_{n-1} x_{n-1} + (1 - \beta_{n-1})T_1 z_{n-1}, \end{cases}$$

from which it follows that

$$\begin{aligned} y_n - y_{n-1} &= (1 - \beta_n)(T_1 z_n - T_1 z_{n-1}) + \beta_n(x_n - x_{n-1}) \\ &\quad + (T_1 z_{n-1} - x_{n-1})(\beta_{n-1} - \beta_n). \end{aligned}$$

This implies that

$$(2.4) \quad \begin{aligned} &\|y_n - y_{n-1}\| \\ &\leq (1 - \beta_n)\|T_1 z_n - T_1 z_{n-1}\| + \beta_n\|x_n - x_{n-1}\| \\ &\quad + \|T_1 z_{n-1} - x_{n-1}\|\|\beta_{n-1} - \beta_n\| \\ &\leq (1 - \beta_n)\|z_n - z_{n-1}\| + \beta_n\|x_n - x_{n-1}\| + (\beta_{n-1} + \beta_n)M_2, \end{aligned}$$

where  $M_2$  is an appropriate constant such that  $M_2 \geq \sup_{n \geq 1} \{\|T_1 z_{n-1} - x_{n-1}\|\}$ . Substituting (2.3) into (2.4), we arrive at

$$(2.5) \quad \|y_n - y_{n-1}\| \leq \|x_n - x_{n-1}\| + (\gamma_{n-1} + \gamma_n + \beta_{n-1} + \beta_n)M_3,$$

where  $M_3$  is an appropriate constant such that  $M_3 = \max\{M_1, M_2\}$ . Substituting (2.5) into (2.2), we obtain

$$\begin{aligned} &\|l_{n+1} - l_n\| - \|x_n - x_{n-1}\| \\ &\leq \frac{\alpha_{n+1}}{1 - \lambda_{n+1}}\|f(x_{n+1}) - y_{n+1}\| + \frac{\alpha_n}{1 - \lambda_n}\|y_n - f(x_n)\| \\ &\quad + (\gamma_{n-1} + \gamma_n + \beta_{n-1} + \beta_n)M_3. \end{aligned}$$

From the conditions (b) and (c), we have that

$$\limsup_{n \rightarrow \infty} (\|l_{n+1} - l_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

It follows from Lemma 1.3 that

$$\lim_{n \rightarrow \infty} \|l_n - x_n\| = 0.$$

Thanks to (2.1), we see that

$$x_{n+1} - x_n = (1 - \beta_n)(l_n - x_n).$$

This implies that

$$(2.6) \quad \lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0.$$

Noting that

$$\begin{aligned} \|x_n - y_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| \\ &\leq \|x_n - x_{n+1}\| + \alpha_n \|f(x_n) - y_n\| + \lambda_n \|x_n - y_n\|. \end{aligned}$$

That is,

$$(1 - \lambda_n) \|x_n - y_n\| \leq \|x_n - x_{n+1}\| + \alpha_n \|f(x_n) - y_n\|.$$

From the conditions (b) and (d), we obtain

$$(2.7) \quad \lim_{n \rightarrow \infty} \|y_n - x_n\| = 0.$$

On the other hand, we have

$$\begin{aligned} \|T_1 T_2 x_n - x_n\| &\leq \|x_n - y_n\| + \|y_n - T_1 z_n\| + \|T_1 z_n - T_1 T_2 x_n\| \\ &\leq \|x_n - y_n\| + \beta_n \|x_n - T_1 z_n\| + \|z_n - T_2 x_n\| \\ &\leq \|x_n - y_n\| + \beta_n \|x_n - T_1 z_n\| + \gamma_n \|x_n - T_2 x_n\|. \end{aligned}$$

From the assumption  $\lim_{n \rightarrow \infty} \beta_n = \lim_{n \rightarrow \infty} \gamma_n = 0$  and (2.7), we arrive at

$$(2.8) \quad \lim_{n \rightarrow \infty} \|T_1 T_2 x_n - x_n\| = 0.$$

Put  $T = T_1 T_2$ . Since  $T_1$  and  $T_2$  are nonexpansive, we have that  $T$  is also nonexpansive.

Next, we claim that

$$(2.9) \quad \limsup_{n \rightarrow \infty} \langle f(q) - q, J(x_n - q) \rangle \leq 0,$$

where  $q = Qf = \lim_{t \rightarrow 0} x_t$  with  $x_t$  being the fixed point of the contraction  $x \mapsto tf(x) + (1-t)Tx$ . From  $x_t$  solves the fixed point equation

$$x_t = tf(x_t) + (1-t)Tx_t.$$

Thus we have

$$\|x_t - x_n\| = \|(1-t)(Tx_t - x_n) + t(f(x_t) - x_n)\|.$$

It follows from Lemma 1.2 that

$$\begin{aligned} \|x_t - x_n\|^2 &\leq (1-t)^2 \|Tx_t - x_n\|^2 + 2t \langle f(x_t) - x_n, J(x_t - x_n) \rangle \\ (2.10) \quad &\leq (1-2t+t^2) \|x_t - x_n\|^2 + f_n(t) \\ &\quad + 2t \langle f(x_t) - x_t, J(x_t - x_n) \rangle + 2t \|x_t - x_n\|^2, \end{aligned}$$

where

$$(2.11) \quad f_n(t) = (2\|x_t - x_n\| + \|x_n - Tx_n\|) \|x_n - Tx_n\| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

It follows that

$$(2.12) \quad \langle x_t - f(x_t), J(x_t - x_n) \rangle \leq \frac{t}{2} \|x_t - x_n\|^2 + \frac{1}{2t} f_n(t).$$

Let  $n \rightarrow \infty$  in (2.12) and note (2.11) yields

$$(2.13) \quad \limsup_{n \rightarrow \infty} \langle x_t - f(x_t), J(x_t - x_n) \rangle \leq \frac{t}{2} M_4,$$

where  $M_4$  is an appropriate constant such that  $M_4 \geq \|x_t - x_n\|^2$  for all  $t \in (0, 1)$  and  $n \geq 0$ . Taking  $t \rightarrow 0$  from (2.13), we have

$$\limsup_{t \rightarrow 0} \limsup_{n \rightarrow \infty} \langle x_t - f(x_t), J(x_t - x_n) \rangle \leq 0.$$

So, for any  $\epsilon > 0$ , there exists a positive number  $\delta_1$  such that, for  $t \in (0, \delta_1)$ , we get

$$(2.14) \quad \limsup_{n \rightarrow \infty} \langle x_t - f(x_t), J(x_t - x_n) \rangle \leq \frac{\epsilon}{2}.$$

On the other hand, since  $x_t \rightarrow q$  as  $t \rightarrow 0$ , we have that there exists  $\delta_2 > 0$  such that, for  $t \in (0, \delta_2)$  we have

$$\begin{aligned} & |\langle f(q) - q, J(x_n - q) \rangle - \langle x_t - f(x_t), J(x_t - x_n) \rangle| \\ & \leq |\langle f(q) - q, J(x_n - q) \rangle - \langle f(q) - q, J(x_n - x_t) \rangle| \\ & \quad + |\langle f(q) - q, J(x_n - x_t) \rangle - \langle f(x_t) - x_t, J(x_n - x_t) \rangle| \\ & \leq |\langle f(q) - q, J(x_n - q) - J(x_n - x_t) \rangle| + |\langle f(q) - f(x_t) - q + x_t, J(x_n - q) \rangle| \\ & \leq \|f(q) - q\| \|J(x_n - q) - J(x_n - x_t)\| + \|f(q) - f(x_t) - q + x_t\| \|x_n - q\| \\ & < \frac{\epsilon}{2}. \end{aligned}$$

Picking  $\delta = \min\{\delta_1, \delta_2\}$  for all  $t \in (0, \delta)$ , we have

$$\langle f(q) - q, J(x_n - q) \rangle \leq \langle x_t - f(x_t), J(x_t - x_n) \rangle + \frac{\epsilon}{2}.$$

That is,

$$\limsup_{n \rightarrow \infty} \langle f(q) - q, J(x_n - q) \rangle \leq \limsup_{n \rightarrow \infty} \langle x_t - f(x_t), J(x_t - x_n) \rangle + \frac{\epsilon}{2}.$$

It follows from (2.14) that

$$\limsup_{n \rightarrow \infty} \langle f(q) - q, J(x_n - q) \rangle \leq \epsilon.$$

Since  $\epsilon$  is chosen arbitrarily, we have

$$(2.15). \quad \limsup_{n \rightarrow \infty} \langle f(q) - q, J(x_n - q) \rangle \leq 0.$$



Finally, we show that  $x_n \rightarrow q$  strongly and this concludes the proof. It follows from Lemma 1.2 that

$$\begin{aligned}
& \|x_{n+1} - q\|^2 \\
&= \|\lambda_n(x_n - q) + \delta_n(y_n - q) + \alpha_n(f(x_n) - q)\|^2 \\
&\leq \|\lambda_n(x_n - q) + \delta_n(y_n - q)\|^2 + 2\alpha_n \langle f(x_n) - q, J(x_{n+1} - q) \rangle \\
&\leq (\lambda_n \|x_n - q\| + \delta_n \|y_n - q\|)^2 + 2\alpha_n \langle f(x_n) - f(q), J(x_{n+1} - q) \rangle \\
&\quad + 2\alpha_n \langle f(q) - q, J(x_{n+1} - q) \rangle \\
&\leq (1 - \alpha_n)^2 \|x_n - q\|^2 + 2\alpha_n \langle f(x_n) - f(q), J(x_{n+1} - q) \rangle \\
&\quad + 2\alpha_n \langle f(q) - q, J(x_{n+1} - q) \rangle \\
&\leq (1 - \alpha_n)^2 \|x_n - q\|^2 + \alpha_n \alpha (\|x_n - q\|^2 + \|x_{n+1} - q\|^2) \\
&\quad + 2\alpha_n \langle f(q) - q, J(x_{n+1} - q) \rangle.
\end{aligned}$$

This implies that

$$\begin{aligned}
& \|x_{n+1} - q\|^2 \\
&\leq \frac{1 - (2 - \alpha)\alpha_n + \alpha_n^2}{1 - \alpha\alpha_n} \|x_n - q\|^2 - \frac{2\alpha_n}{1 - \alpha\alpha_n} \langle f(q) - q, J(x_{n+1} - q) \rangle \\
&\leq \frac{1 - (2 - \alpha)\alpha_n}{1 - \alpha\alpha_n} \|x_n - q\|^2 - \frac{2\alpha_n}{1 - \alpha\alpha_n} \langle f(q) - q, J(x_{n+1} - q) \rangle + M_4 \alpha_n^2 \\
&= \left(1 - \frac{2(1 - \alpha)\alpha_n}{1 - \alpha\alpha_n}\right) \|x_n - q\|^2 \\
&\quad + \frac{2(1 - \alpha)\alpha_n}{1 - \alpha\alpha_n} \left(\frac{M_4(1 - \alpha\alpha_n)\alpha_n}{2(1 - \alpha)} + \frac{1}{1 - \alpha} \langle f(q) - q, J(x_{n+1} - q) \rangle\right),
\end{aligned}$$

where  $M_4$  is an appropriate constant such that  $M_4 \geq \sup_{n \geq 0} \left\{ \frac{\|x_n - q\|^2}{1 - \alpha\alpha_n} \right\}$ . From Lemma 1.4, we see that  $\|x_n - q\| \rightarrow 0$  as  $n \rightarrow \infty$ . This completes the proof.  $\square$

From Theorem 2.1, we have the following results immediately.

**Corollary 2.1.** *Let  $C$  be a closed convex subset of a uniformly smooth Banach space  $E$ ,  $T_1 : C \rightarrow C$  nonexpansive mappings such that  $F(T_1) \neq \emptyset$  and  $f : C \rightarrow C$  a contraction with the coefficient  $\alpha \in (0, 1)$ . Let  $\{x_n\}$  be a sequence defined by the following iterative process*

$$\begin{cases} x_0 = x \in C & \text{chosen arbitrarily,} \\ y_n = \beta_n x_n + (1 - \beta_n) T_1 x_n, \\ x_{n+1} = \alpha_n f(x_n) + \lambda_n x_n + \delta_n y_n, & \forall n \geq 0, \end{cases}$$

where  $\{\alpha_n\}$ ,  $\{\beta_n\}$ ,  $\{\lambda_n\}$  and  $\{\delta_n\}$  are sequences in  $[0, 1]$ . If the following conditions are satisfied

- (a)  $\alpha_n + \lambda_n + \delta_n = 1$  for all  $n \geq 0$ ,
- (b)  $\sum_{n=0}^{\infty} \alpha_n = \infty$  and  $\alpha_n \rightarrow 0$  as  $n \rightarrow \infty$ ,
- (c)  $\beta_n \rightarrow 0$  as  $n \rightarrow \infty$ ,

(d)  $0 < \liminf_{n \rightarrow \infty} \lambda_n \leq \limsup_{n \rightarrow \infty} \lambda_n < 1$ ,  
 then  $\{x_n\}$  converges strongly to a fixed point of  $T_1$ .

**Corollary 2.2.** *Let  $C$  be a closed convex subset of a uniformly smooth Banach space  $E$ ,  $T_1 : C \rightarrow C$  nonexpansive mappings such that  $F(T_1) \neq \emptyset$ . Let  $\{x_n\}$  be a sequence defined by the following iterative process*

$$\begin{cases} x_0 = x \in C & \text{chosen arbitrarily,} \\ x_{n+1} = \alpha_n u + \lambda_n x_n + \delta_n T_1 x_n, & \forall n \geq 0, \end{cases}$$

where  $u \in C$  is given point,  $\{\alpha_n\}$ ,  $\{\lambda_n\}$  and  $\{\delta_n\}$  are sequences in  $[0, 1]$ . If the following conditions are satisfied

- (a)  $\alpha_n + \lambda_n + \delta_n = 1$  for all  $n \geq 0$ ,
  - (b)  $\sum_{n=0}^{\infty} \alpha_n = \infty$  and  $\alpha_n \rightarrow 0$  as  $n \rightarrow \infty$ ,
  - (c)  $0 < \liminf_{n \rightarrow \infty} \lambda_n \leq \limsup_{n \rightarrow \infty} \lambda_n < 1$ ,
- then  $\{x_n\}$  converges strongly to a fixed point of  $T_1$ .

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