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# COMMON FIXED POINT THEOREMS FOR COMPATIBLE MAPPINGS OF TYPE $(A)$ AND $(P)$ WITH APPLICATIONS IN DYNAMIC PROGRAMMING

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Abstract. In this paper, the concepts of compatible mappings of types $(A)$ and $(P)$ are introduced in an induced metric space, two common fixed point theorems for two pairs of compatible mappings of types $(A)$ and $(P)$ in an induced complete metric space are established. As their applications, the existence and uniqueness results of common solution for a system of functional equations arising in dynamic programming are discussed.

## 1. Introduction and preliminaries

Ray [18] established two fixed point theorems for the following contractive type mappings

$$d(fx, gy) \le d(hx, hy) - \psi(d(hx, hy)), \quad \forall x, y \in X. \tag{1.1}$$

Liu [7] produced a few common fixed point results for two pairs of compatible mappings $A, S$ and $B, T$ in a complete metric space $(X, d)$ such that for all $x, y \in X$,

$$\begin{aligned} &d(Ax, By) \\ &\le \max\big\{d(Sx, Ty), d(Sx, Ax), d(Ty, By), \frac{1}{2}\big[d(Sx, By) + d(Ty, Ax)\big]\big\} \\ &\quad - \varphi\big(\max\big\{d(Sx, Ty), d(Sx, Ax), d(Ty, By), \\ &\qquad \frac{1}{2}\big[d(Sx, By) + d(Ty, Ax)\big]\big\}\big). \end{aligned} \tag{1.2}$$

In particular, Huang, Lee and Kang [4] got a few common fixed point theorems for two pairs of compatible mappings $A, S$ and $B, T$ in a complete metric space

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$(X, d)$ which satisfy the following condition for all $x, y \in X$,

$$\begin{aligned} &d(Ax, By) \\ (1.3) \quad &\leq \phi\big(\max\big\{d(Sx, Ty), d(Sx, Ax), d(Ty, By), \\ &\qquad \frac{1}{2}\big[d(Sx, By) + d(Ty, Ax)\big]\big\}\big), \end{aligned}$$

where $\phi : [0, +\infty) \to [0, +\infty)$ is nondecreasing, upper semi-continuous and $\phi(t) < t$ for all $t > 0$. Moreover, Pathak, Cho, Kang and Lee [17] obtained several common fixed point theorems for two pairs of compatible mappings of type $(P)$ satisfying (1.3).

As proposed in Bellman and Lee [1], the essential form of the functional equation in dynamic programming is

$$(1.4) \qquad f(x) = \operatorname*{opt}_{y \in D} H(x, y, f(T(x, y))), \quad \forall x \in S,$$

where $x$ and $y$ denote the state and decision vectors, respectively. $T$ denotes the transformation of the process, $f(x)$ denotes the optimal return function with the initial state $x$, and the opt represents sup or inf.

Bhakta and Mitra [3], Huang, Lee and Kang [4], Liu [5-7], Liu, Agarwal and Kang [9], Liu and Kang [10-11], Liu and Kim [12], Liu and Ume [13], Liu, Ume and Kang [14], Liu, Xu, Ume and Kang [15], Pathak and Fisher [16], Pathak, Cho, Kang and Lee [17] and others established the existence and uniqueness of solutions or common solutions for several classes of functional equations or systems of functional equations arising in dynamic programming by means of various fixed and common fixed point theorems. Bhakta and Choudhury [2] obtained two fixed point theorems by using countable family of pseudometrics and discussed also the existence of solutions for the following functional equation:

$$(1.5) \qquad f(x) = \inf_{y \in D} G(x, y, f), \quad \forall x \in S.$$

Aroused and motivated by the above achievements in [1-18], we introduce the following contractive type mappings and system of functional equations arising in dynamic programming, respectively,

$$\begin{aligned} &d_k^{s+t}(fx, gy) \\ &\leq \max\Big\{\big\{p^s q^t : p, q \in \{d_k(lx, hy), d_k(fx, lx), d_k(gy, hy), d_k(fx, gy)\}\big\} \\ (1.6) \quad &\qquad \bigcup\Big\{p^s\Big(\frac{q}{2}\Big)^t, p^t\Big(\frac{q}{2}\Big)^s : p \in \{d_k(lx, hy), d_k(fx, lx), d_k(gy, hy), \\ &\qquad\qquad d_k(fx, gy)\}, q \in \{d_k(fx, hy), d_k(gy, lx)\}\Big\}\Big\} \end{aligned}$$

$$
\begin{aligned}
- w\Big( \max \Big\{ & \{ p^s q^t : p, q \in \{ d_k(lx, hy), d_k(fx, lx), d_k(gy, hy), \\
& d_k(fx, gy) \} \} \bigcup \Big\{ p^s \Big(\frac{q}{2}\Big)^t, p^t \Big(\frac{q}{2}\Big)^s : p \in \{ d_k(lx, hy), \\
& d_k(fx, lx), d_k(gy, hy), d_k(fx, gy) \}, \\
& q \in \{ d_k(fx, hy), d_k(gy, lx) \} \Big\} \Big\} \Big),
\end{aligned}
$$

where $s$ and $t$ are some nonnegative numbers with $s+t>0$ and $w \in W$, where

$$
\begin{aligned}
W = \{ w \mid w : [0, +\infty) \to [0, +\infty) & \text{ is a continuous function satisfying} \\
0 < w(t) < t, \forall t > 0 \}, &
\end{aligned}
$$

and

$$
\begin{aligned}
f_1(x) &= \operatorname*{opt}_{y \in D} \{ u(x, y) + H_1(x, y, f_1(T(x, y))) \}, \quad \forall x \in S, \\
f_2(x) &= \operatorname*{opt}_{y \in D} \{ u(x, y) + H_2(x, y, f_2(T(x, y))) \}, \quad \forall x \in S, \\
g_1(x) &= \operatorname*{opt}_{y \in D} \{ u(x, y) + G_1(x, y, g_1(T(x, y))) \}, \quad \forall x \in S, \\
g_2(x) &= \operatorname*{opt}_{y \in D} \{ u(x, y) + G_2(x, y, g_2(T(x, y))) \}, \quad \forall x \in S.
\end{aligned}
\tag{1.7}
$$

The purpose of this paper is to study the existence and uniqueness of common fixed points for the contractive type mappings (1.6) in an induced complete metric space, which is generated by a countable family of pseudometrics $\{d_k\}_{k\geq 1}$. Under certain conditions, we prove two common fixed point theorems for the contractive type mappings (1.6). As applications, we use the common fixed point theorems presented in this paper to establish the existence and uniqueness results of common solution for the system of functional equations (1.7).

Throughout this paper, let $\mathbb{R}^+ = [0, +\infty)$ and $\{d_k\}_{k\geq 1}$ be a countable family of pseudometrics on a nonempty set $X$ such that for any distinct $x, y \in X, d_k(x, y) \neq 0$ for some $k \geq 1$. Define

$$
d(x, y) = \sum_{k=1}^{\infty} 2^{-k} \frac{d_k(x, y)}{1 + d_k(x, y)}, \quad \forall x, y \in X.
$$

It is clear that $d$ is a metric on $X$ and the metric $d$ and the metric space $(X, d)$ are called, respectively, *an induced metric and induced metric space by the countable family of pseudometircs* $\{d_k\}_{k\geq 1}$, respectively. A sequence $\{x_n\}_{n\geq 1} \subseteq X$ converges to a point $x \in X$ if and only if $d_k(x_n, x) \to 0$ as $n \to \infty$ for each $k \geq 1$, and $\{x_n\}_{n\geq 1} \subseteq X$ is a Cauchy sequence if and only if $d_k(x_n, x_m) \to 0$ as $n, m \to \infty$ for each $k \geq 1$. The induced metric space $(X, d)$ is called *complete* if each Cauchy sequence in $X$ converges to some point in $X$. A self mapping $f$ on $(X, d)$ is said to be *continuous* in $X$ if $\lim_{n\to\infty} fx_n = fx$ whenever $\{x_n\}_{n\geq 1} \subseteq X$ such that $\{x_n\}_{n\geq 1}$ converges to $x \in X$.

## 2. Common fixed point theorems

In this section, let $(X,d)$ be the induced metric space by the countable family of pseudometrics $\{d_k\}_{k\geq 1}$ such that for any distinct $x,y\in X, d_k(x,y)\neq 0$ for some $k\geq 1$.

Now we introduce the concepts of compatible mappings of types $(A)$ and $(P)$ in the induced metric space $(X,d)$ and give two useful lemmas.

**Definition 2.1.** The mappings $g,h:(X,d)\to(X,d)$ are said to be *compatible of type* $(A)$ if

$$\lim_{n\to\infty} d_k(ghx_n,hhx_n)=0 \quad\text{and}\quad \lim_{n\to\infty} d_k(hgx_n,ggx_n)=0,\quad \forall k\geq 1,$$

whenever $\{x_n\}_{n\geq 1}$ is a sequence in $X$ such that $\lim_{n\to\infty} gx_n=\lim_{n\to\infty} hx_n=z$ for some $z\in X$.

**Definition 2.2.** The mappings $g,h:(X,d)\to(X,d)$ are said to be *compatible of type* $(P)$ if

$$\lim_{n\to\infty} d_k(ggx_n,hhx_n)=0,\quad \forall k\geq 1,$$

whenever $\{x_n\}_{n\geq 1}$ is a sequence in $X$ such that $\lim_{n\to\infty} gx_n=\lim_{n\to\infty} hx_n=z$ for some $z\in X$.

It is easy to verify that the following results hold.

**Lemma 2.1.** *Let $f$ and $l$ be compatible mappings of type (A) from the induced metric space $(X,d)$ into itself. If $ft=lt$ for some $t\in X$, then $flt=llt=lft=fft$.*

**Lemma 2.2.** *Let $f$ and $l$ be compatible mappings of type (P) from the induced metric space $(X,d)$ into itself. If $ft=lt$ for some $t\in X$, then $flt=fft=llt=lft$.*

Next we show two common fixed point theorems for the contractive type mappings (1.6) in the induced complete metric spaces $(X,d)$.

**Theorem 2.1.** *Let the induced metric space $(X,d)$ be complete and $f$, $g$, $h$ and $l$ be four self mappings of $(X,d)$ such that*
$(B1)$ *one of $f$, $g$, $h$ and $l$ is continuous;*
$(B2)$ *the pairs $f,l$ and $g,h$ are compatible of type $(A)$;*
$(B3)$ *$f(X)\subseteq h(X)$, $g(X)\subseteq l(X)$;*
$(B4)$ *there exist some $w\in W$, $s,t\in\mathbb{R}^+$ with $s+t>0$ satisfying* (1.6).
*Then $f$, $g$, $h$ and $l$ have a unique common fixed point in $X$.*

*Proof.* Let $x_0$ be an arbitrary point in $X$. In terms of $(B3)$, there exist two sequences $\{x_n\}_{n\geq 0}\subseteq X$ and $\{y_n\}_{n\geq 1}\subseteq X$ such that $fx_{2n}=hx_{2n+1}=y_{2n+1}$ and $gx_{2n+1}=lx_{2n+2}=y_{2n+2}$ for $n\geq 0$. Define $d_{kn}=d_k(y_n,y_{n+1})$ for $n\geq 1$ and $k\geq 1$. We firstly prove that

$$d_{kn+1}^{s+t}\leq d_{kn}^{s+t}-w(d_{kn}^{s+t}),\quad \forall n\geq 1,\ k\geq 1. \tag{2.1}$$

In view of (1.6) we acquire that

$$
\begin{aligned}
d_{k2n+1}^{s+t} &= d_k^{s+t}(fx_{2n}, gx_{2n+1}) \\
&\le \max\Big\{\{p^s q^t : p, q \in \{d_k(lx_{2n}, hx_{2n+1}), d_k(fx_{2n}, lx_{2n}), \\
&\qquad d_k(gx_{2n+1}, hx_{2n+1}), d_k(fx_{2n}, gx_{2n+1})\}\} \\
&\qquad \bigcup\Big\{p^s\Big(\frac{q}{2}\Big)^t, p^t\Big(\frac{q}{2}\Big)^s : p \in \{d_k(lx_{2n}, hx_{2n+1}), \\
&\qquad\quad d_k(fx_{2n}, lx_{2n}), d_k(gx_{2n+1}, hx_{2n+1}), d_k(fx_{2n}, gx_{2n+1})\}, \\
&\qquad\quad q \in \{d_k(fx_{2n}, hx_{2n+1}), d_k(gx_{2n+1}, lx_{2n})\}\Big\}\Big\} \\
&\quad - w\Big(\max\Big\{\{p^s q^t : p, q \in \{d_k(lx_{2n}, hx_{2n+1}), d_k(fx_{2n}, lx_{2n}), \\
&\qquad\quad d_k(gx_{2n+1}, hx_{2n+1}), d_k(fx_{2n}, gx_{2n+1})\}\} \\
&\qquad\quad \bigcup\Big\{p^s\Big(\frac{q}{2}\Big)^t, p^t\Big(\frac{q}{2}\Big)^s : p \in \{d_k(lx_{2n}, hx_{2n+1}), \\
&\qquad\qquad d_k(fx_{2n}, lx_{2n}), d_k(gx_{2n+1}, hx_{2n+1}), d_k(fx_{2n}, gx_{2n+1})\}, \\
&\qquad\qquad q \in \{d_k(fx_{2n}, hx_{2n+1}), d_k(gx_{2n+1}, lx_{2n})\}\Big\}\Big\}\Big) \\
&= \max\Big\{\{p^s q^t : p, q \in \{d_{k2n}, d_{k2n+1}\}\}\bigcup\Big\{p^s\Big(\frac{q}{2}\Big)^t, p^t\Big(\frac{q}{2}\Big)^s : \\
&\qquad p \in \{d_{k2n}, d_{k2n+1}\}, q \in \{0, d_k(y_{2n}, y_{2n+2})\}\Big\}\Big\} \\
&\quad - w\Big(\max\Big\{\{p^s q^t : p, q \in \{d_{k2n}, d_{k2n+1}\}\}\bigcup\Big\{p^s\Big(\frac{q}{2}\Big)^t, p^t\Big(\frac{q}{2}\Big)^s : \\
&\qquad\quad p \in \{d_{k2n}, d_{k2n+1}\}, q \in \{0, d_k(y_{2n}, y_{2n+2})\}\Big\}\Big\}\Big),
\end{aligned}
$$

which infers that

$$
\begin{aligned}
&d_{k2n+1}^{s+t} \\
&\le \max\Big\{d_{k2n}^{s+t}, d_{k2n+1}^{s+t}, d_{k2n}^s d_{k2n+1}^t, d_{k2n}^t d_{k2n+1}^s, \\
&\qquad d_{k2n}^s\Big(\frac{d_k(y_{2n}, y_{2n+2})}{2}\Big)^t, d_{k2n}^t\Big(\frac{d_k(y_{2n}, y_{2n+2})}{2}\Big)^s, \\
&\qquad d_{k2n+1}^s\Big(\frac{d_k(y_{2n}, y_{2n+2})}{2}\Big)^t, d_{k2n+1}^t\Big(\frac{d_k(y_{2n}, y_{2n+2})}{2}\Big)^s\Big\} \\
&\quad - w\Big(\max\Big\{d_{k2n}^{s+t}, d_{k2n+1}^{s+t}, d_{k2n}^s d_{k2n+1}^t, d_{k2n}^t d_{k2n+1}^s, \\
&\qquad\quad d_{k2n}^s\Big(\frac{d_k(y_{2n}, y_{2n+2})}{2}\Big)^t, d_{k2n}^t\Big(\frac{d_k(y_{2n}, y_{2n+2})}{2}\Big)^s, \\
&\qquad\quad d_{k2n+1}^s\Big(\frac{d_k(y_{2n}, y_{2n+2})}{2}\Big)^t, d_{k2n+1}^t\Big(\frac{d_k(y_{2n}, y_{2n+2})}{2}\Big)^s\Big\}\Big)
\end{aligned}
\tag{2.2}
$$

$$= \max\{d_{k2n}^{s+t}, d_{k2n+1}^{s+t}, d_{k2n}^{s} d_{k2n+1}^{t}, d_{k2n}^{t} d_{k2n+1}^{s}\}$$
$$- w(\max\{d_{k2n}^{s+t}, d_{k2n+1}^{s+t}, d_{k2n}^{s} d_{k2n+1}^{t}, d_{k2n}^{t} d_{k2n+1}^{s}\}), \quad \forall k \geq 1.$$

Suppose that $d_{k2n+1} > d_{k2n}$ for some $n \geq 1$ and $k \geq 1$. It follows from (2.2) that $d_{k2n+1}^{s+t} \leq d_{k2n+1}^{s+t} - w(d_{k2n+1}^{s+t}) < d_{k2n+1}^{s+t}$, which is a contradiction. Hence $d_{k2n+1} \leq d_{k2n}$ for any $n \geq 1$ and $k \geq 1$. Thus (2.2) means that $d_{k2n+1}^{s+t} \leq d_{k2n}^{s+t} - w(d_{k2n}^{s+t})$ for any $n \geq 1$ and $k \geq 1$. Similarly we conclude that $d_{k2n}^{s+t} \leq d_{k2n-1}^{s+t} - w(d_{k2n-1}^{s+t})$ for each $n \geq 1$ and $k \geq 1$. It follows that (2.1) holds.

At present, we demonstrate that

$$\lim_{n\to\infty} d_{kn} = 0, \quad \forall k \geq 1. \tag{2.3}$$

In view of (2.1) we deduce that

$$\sum_{i=1}^{n} w(d_{ki}^{s+t}) \leq d_{k1}^{s+t} - d_{kn+1}^{s+t} \leq d_{k1}^{s+t}, \quad \forall n \geq 1,\ k \geq 1,$$

which yields that the series of nonnegative terms $\sum_{n=1}^{\infty} w(d_{kn}^{s+t})$ for all $k \geq 1$ is convergent. Therefore

$$\lim_{n\to\infty} w(d_{kn}^{s+t}) = 0, \quad \forall k \geq 1. \tag{2.4}$$

Due to (2.1) we get that

$$0 \leq d_{kn}^{s+t} \leq d_{kn-1}^{s+t} \leq \cdots \leq d_{k1}^{s+t}, \quad \forall n \geq 1,\ k \geq 1,$$

which implies that $\lim_{n\to\infty} d_{kn}^{s+t} = a$ for some $a \in \mathbb{R}^+$ and for each $k \geq 1$. By (2.4) and the continuity of $w$ we have

$$w(a) = \lim_{n\to\infty} w(d_{kn}^{s+t}) = 0, \quad \forall k \geq 1,$$

thereupon $a = 0$. It follows that (2.3) holds.

In order to show that $\{y_n\}_{n\geq 1}$ is a Cauchy sequence, in view of (2.3), it is sufficient to show that $\{y_{2n}\}_{n\geq 1}$ is a Cauchy sequence. Suppose that $\{y_{2n}\}_{n\geq 1}$ is not a Cauchy sequence. Hence there exists a positive number $\varepsilon$ and $k \geq 1$ on condition that for each even integer $2r$, there are even integers $2m(r)$ and $2n(r)$ such that $2m(r) > 2n(r) > 2r$, and

$$d_k(y_{2m(r)}, y_{2n(r)}) > \varepsilon.$$

For each even integer $2r$, let $2m(r)$ be the least even integer exceeding $2n(r)$ satisfying the above inequality, therefore we obtain that

$$d_k(y_{2m(r)-2}, y_{2n(r)}) \leq \varepsilon \quad \text{and} \quad d_k(y_{2m(r)}, y_{2n(r)}) > \varepsilon. \tag{2.5}$$

It follows that for each even integer $2r$,

$$d_k(y_{2m(r)}, y_{2n(r)}) \leq d_{k2m(r)-1} + d_{k2m(r)-2} + d_k(y_{2m(r)-2}, y_{2n(r)}).$$

Using (2.3), (2.5) and the above inequality we obtain that

$$\lim_{r\to\infty} d_k(y_{2m(r)}, y_{2n(r)}) = \varepsilon. \tag{2.6}$$

It is apparent that for any $r \geq 1$,

$$|d_k(y_{2m(r)}, y_{2n(r)+1}) - d_k(y_{2m(r)}, y_{2n(r)})| \leq d_{k2n(r)},$$

$$|d_k(y_{2m(r)+1}, y_{2n(r)+1}) - d_k(y_{2m(r)}, y_{2n(r)+1})| \leq d_{k2m(r)}$$

and

$$|d_k(y_{2m(r)+1}, y_{2n(r)+2}) - d_k(y_{2m(r)+1}, y_{2n(r)+1})| \leq d_{k2n(r)+1}.$$

In light of (2.3), (2.6) and the above inequalities we draw that

$$\begin{aligned}\varepsilon &= \lim_{r\to\infty} d_k(y_{2m(r)}, y_{2n(r)+1}) = \lim_{r\to\infty} d_k(y_{2m(r)+1}, y_{2n(r)+1}) \\ &= \lim_{r\to\infty} d_k(y_{2m(r)+1}, y_{2n(r)+2}).\end{aligned}$$

By means of (1.6) we arrive at

$$\begin{aligned}& d_k^{s+t}(fx_{2m(r)}, gx_{2n(r)+1}) \\ & \leq \max\Big\{\{p^s q^t : p, q \in \{d_k(lx_{2m(r)}, hx_{2n(r)+1}), d_k(fx_{2m(r)}, lx_{2m(r)}), \\ & \qquad d_k(gx_{2n(r)+1}, hx_{2n(r)+1}), d_k(fx_{2m(r)}, gx_{2n(r)+1})\}\} \\ & \qquad \bigcup\Big\{p^s\Big(\frac{q}{2}\Big)^t, p^t\Big(\frac{q}{2}\Big)^s : p \in \{d_k(lx_{2m(r)}, hx_{2n(r)+1}), \\ & \qquad\quad d_k(fx_{2m(r)}, lx_{2m(r)}), d_k(gx_{2n(r)+1}, hx_{2n(r)+1}), \\ & \qquad\quad d_k(fx_{2m(r)}, gx_{2n(r)+1})\}, q \in \{d_k(fx_{2m(r)}, hx_{2n(r)+1}), \\ & \qquad\quad d_k(gx_{2n(r)+1}, lx_{2m(r)})\}\Big\}\Big\} \\ & \quad - w\Big(\max\Big\{\{p^s q^t : p, q \in \{d_k(lx_{2m(r)}, hx_{2n(r)+1}), d_k(fx_{2m(r)}, lx_{2m(r)}), \\ & \qquad\quad d_k(gx_{2n(r)+1}, hx_{2n(r)+1}), d_k(fx_{2m(r)}, gx_{2n(r)+1})\}\} \\ & \qquad\quad \bigcup\Big\{p^s\Big(\frac{q}{2}\Big)^t, p^t\Big(\frac{q}{2}\Big)^s : p \in \{d_k(lx_{2m(r)}, hx_{2n(r)+1}), \\ & \qquad\qquad d_k(fx_{2m(r)}, lx_{2m(r)}), d_k(gx_{2n(r)+1}, hx_{2n(r)+1}), \\ & \qquad\qquad d_k(fx_{2m(r)}, gx_{2n(r)+1})\}, q \in \{d_k(fx_{2m(r)}, hx_{2n(r)+1}), \\ & \qquad\qquad d_k(gx_{2n(r)+1}, lx_{2m(r)})\}\Big\}\Big\}\Big).\end{aligned}$$

Letting $r \to \infty$, we get that

$$\begin{aligned}\varepsilon^{s+t} &\leq \max\Big\{\varepsilon^{s+t}, 0, \varepsilon^s\Big(\frac{\varepsilon}{2}\Big)^t, \varepsilon^t\Big(\frac{\varepsilon}{2}\Big)^s\Big\} - w\Big(\max\Big\{\varepsilon^{s+t}, 0, \varepsilon^s\Big(\frac{\varepsilon}{2}\Big)^t, \varepsilon^t\Big(\frac{\varepsilon}{2}\Big)^s\Big\}\Big) \\ &= \varepsilon^{s+t} - w(\varepsilon^{s+t}) < \varepsilon^{s+t},\end{aligned}$$

which is a contradiction. Thus $\{y_n\}_{n\geq 1}$ is a Cauchy sequence and it converges to a point $u \in X$ by completeness of $X$. Suppose that $l$ is continuous. Consequently, we infer that

$$\lim_{n\to\infty} llx_{2n} = \lim_{n\to\infty} ly_{2n} = lu.$$

$(B2)$ means that

$$\lim_{n\to\infty} fy_{2n} = \lim_{n\to\infty} flx_{2n} = lu.$$

We subsequently produce that $u$ is a common fixed point of $f$, $g$, $h$ and $l$. It follows from (1.6) that

$$\begin{aligned}
&d_k^{s+t}(fy_{2n}, gx_{2n+1})\\
&\le \max\Big\{\{p^s q^t : p, q \in \{d_k(ly_{2n}, hx_{2n+1}), d_k(fy_{2n}, ly_{2n}),\\
&\qquad d_k(gx_{2n+1}, hx_{2n+1}), d_k(fy_{2n}, gx_{2n+1})\}\}\\
&\qquad \bigcup\Big\{p^s\Big(\frac{q}{2}\Big)^t, p^t\Big(\frac{q}{2}\Big)^s : p \in \{d_k(ly_{2n}, hx_{2n+1}),\\
&\qquad\quad d_k(fy_{2n}, ly_{2n}), d_k(gx_{2n+1}, hx_{2n+1}), d_k(fy_{2n}, gx_{2n+1})\},\\
&\qquad\quad q \in \{d_k(fy_{2n}, hx_{2n+1}), d_k(gx_{2n+1}, ly_{2n})\}\Big\}\Big\}\\
&\quad - w\Big(\max\Big\{\{p^s q^t : p, q \in \{d_k(ly_{2n}, hx_{2n+1}), d_k(fy_{2n}, ly_{2n}),\\
&\qquad d_k(gx_{2n+1}, hx_{2n+1}), d_k(fy_{2n}, gx_{2n+1})\}\}\\
&\qquad \bigcup\Big\{p^s\Big(\frac{q}{2}\Big)^t, p^t\Big(\frac{q}{2}\Big)^s : p \in \{d_k(ly_{2n}, hx_{2n+1}),\\
&\qquad\quad d_k(fy_{2n}, ly_{2n}), d_k(gx_{2n+1}, hx_{2n+1}), d_k(fy_{2n}, gx_{2n+1})\},\\
&\qquad\quad q \in \{d_k(fy_{2n}, hx_{2n+1}), d_k(gx_{2n+1}, ly_{2n})\}\Big\}\Big\}\Big), \quad \forall k \ge 1.
\end{aligned}$$

As $n \to \infty$ in above inequality, we know that

$$d_k^{s+t}(lu, u) \le d_k^{s+t}(lu, u) - w(d_k^{s+t}(lu, u)) \le d_k^{s+t}(lu, u), \quad \forall k \ge 1,$$

which signifies that $d_k(lu, u) = 0$ for any $k \ge 1$, that is, $lu = u$. Using (1.6) we see that

$$\begin{aligned}
&d_k^{s+t}(fu, gx_{2n+1})\\
&\le \max\Big\{\{p^s q^t : p, q \in \{d_k(lu, hx_{2n+1}), d_k(fu, lu),\\
&\qquad d_k(gx_{2n+1}, hx_{2n+1}), d_k(fu, gx_{2n+1})\}\}\\
&\qquad \bigcup\Big\{p^s\Big(\frac{q}{2}\Big)^t, p^t\Big(\frac{q}{2}\Big)^s : p \in \{d_k(lu, hx_{2n+1}),\\
&\qquad\quad d_k(fu, lu), d_k(gx_{2n+1}, hx_{2n+1}), d_k(fu, gx_{2n+1})\},\\
&\qquad\quad q \in \{d_k(fu, hx_{2n+1}), d_k(gx_{2n+1}, lu)\}\Big\}\Big\}
\end{aligned}$$

$$
\begin{aligned}
- w\Big( \max \Big\{ & \{p^s q^t : p, q \in \{d_k(lu, hx_{2n+1}), d_k(fu, lu), \\
& d_k(gx_{2n+1}, hx_{2n+1}), d_k(fu, gx_{2n+1})\}\} \\
& \bigcup \Big\{ p^s \Big(\frac{q}{2}\Big)^t, p^t \Big(\frac{q}{2}\Big)^s : p \in \{d_k(lu, hx_{2n+1}), \\
& \quad d_k(fu, lu), d_k(gx_{2n+1}, hx_{2n+1}), d_k(fu, gx_{2n+1})\}, \\
& \quad q \in \{d_k(fu, hx_{2n+1}), d_k(gx_{2n+1}, lu)\}\Big\}\Big\}\Big), \quad \forall k \geq 1.
\end{aligned}
$$

As $n \to \infty$ in the above inequality, we gain that

$$
d_k^{s+t}(fu, u) \leq d_k^{s+t}(fu, u) - w(d_k^{s+t}(fu, u)) \leq d_k^{s+t}(fu, u), \quad \forall k \geq 1,
$$

which implies that $d_k(fu, u) = 0$ for any $k \geq 1$. Therefore $fu = u$. In view of $(B3)$, there exists a point $z \in X$ such that $u = fu = hz$. It follows from (1.6) that

$$
\begin{aligned}
& d_k^{s+t}(u, gz) \\
& = d_k^{s+t}(fu, gz) \\
& \leq \max \Big\{ \{p^s q^t : p, q \in \{d_k(lu, hz), d_k(fu, lu), d_k(gz, hz), d_k(fu, gz)\}\} \\
& \qquad \bigcup \Big\{ p^s \Big(\frac{q}{2}\Big)^t, p^t \Big(\frac{q}{2}\Big)^s : p \in \{d_k(lu, hz), d_k(fu, lu), d_k(gz, hz), \\
& \qquad\quad d_k(fu, gz)\}, q \in \{d_k(fu, hz), d_k(gz, lu)\}\Big\}\Big\} \\
& \quad - w\Big( \max \Big\{ \{p^s q^t : p, q \in \{d_k(lu, hz), d_k(fu, lu), d_k(gz, hz), d_k(fu, gz)\}\} \\
& \qquad \bigcup \Big\{ p^s \Big(\frac{q}{2}\Big)^t, p^t \Big(\frac{q}{2}\Big)^s : p \in \{d_k(lu, hz), d_k(fu, lu), d_k(gz, hz), \\
& \qquad\quad d_k(fu, gz)\}, q \in \{d_k(fu, hz), d_k(gz, lu)\}\Big\}\Big\}\Big) \\
& = d_k^{s+t}(u, gz) - w(d_k^{s+t}(u, gz)) \leq d_k^{s+t}(u, gz), \quad \forall k \geq 1,
\end{aligned}
$$

which implies that $d_k(u, gz) = 0$ for each $k \geq 1$. Thereupon $u = gz$. By means of $(B2), gz = hz = u$ and Lemma 2.1, we come into $d_k(gu, hu) = 0$ for any $k \geq 1$, which gives that $gu = hu$. In a similar manner, by (1.6) we get that

$$
\begin{aligned}
& d_k^{s+t}(fu, gu) \\
& \leq \max \Big\{ \{p^s q^t : p, q \in \{d_k(lu, hu), d_k(fu, lu), d_k(gu, hu), d_k(fu, gu)\}\} \\
& \qquad \bigcup \Big\{ p^s \Big(\frac{q}{2}\Big)^t, p^t \Big(\frac{q}{2}\Big)^s : p \in \{d_k(lu, hu), d_k(fu, lu), d_k(gu, hu), \\
& \qquad\quad d_k(fu, gu)\}, q \in \{d_k(fu, hu), d_k(gu, lu)\}\Big\}\Big\}
\end{aligned}
$$

$$- w\Big( \max\Big\{ \{p^s q^t : p, q \in \{d_k(lu, hu), d_k(fu, lu), d_k(gu, hu), d_k(fu, gu)\}\} \bigcup\Big\{ p^s\Big(\frac{q}{2}\Big)^t, p^t\Big(\frac{q}{2}\Big)^s : p \in \{d_k(lu, hu), d_k(fu, lu), d_k(gu, hu), d_k(fu, gu)\}, q \in \{d_k(fu, hu), d_k(gu, lu)\}\Big\}\Big\}\Big), \quad \forall k \geq 1,$$

which deduces that

$$d_k^{s+t}(fu, gu) \leq d_k^{s+t}(fu, gu) - w(d_k^{s+t}(fu, gu)) \leq d_k^{s+t}(fu, gu), \quad \forall k \geq 1,$$

which implies that $d_k(fu, gu) = 0$ for each $k \geq 1$. Therefore, $fu = gu$. Hence we come to a conclusion that $fu = gu = hu = lu = u$. Similarly, we can obtain this result in the case of the continuity of $f$ or $g$ or $h$.

We finally demonstrate that $u$ is a unique common fixed point of $f$, $g$, $h$ and $l$. If $v \in X \setminus \{u\}$ is another common fixed point of $f$, $g$, $h$ and $l$, it is easy to see that from (1.6)

$$d_k^{s+t}(u, v) = d_k^{s+t}(fu, gv) \leq d_k^{s+t}(u, v) - w(d_k^{s+t}(u, v)) \leq d_k^{s+t}(u, v), \quad \forall k \geq 1,$$

which implies that $d_k(u, v) = 0$ for all $k \geq 1$. That is, $u = v$, which is a contradiction. This completes the proof. □

**Theorem 2.2.** *Let the induced metric space $(X, d)$ be complete and $f$, $g$, $h$ and $l$ be four self mappings of $(X, d)$ such that $(B1), (B3), (B4)$ and*

*$(B5)$ the pairs $f, l$ and $g, h$ are compatible mappings of type $(P)$.*

*Then $f$, $g$, $h$ and $l$ have a unique common fixed point in $X$.*

*Proof.* As in the proof of Theorem 2.1, we arrive at $\{y_n\}_{n\geq 1}$ is a Cauchy sequence and it converges to a point $u \in X$ by completeness of $X$. Suppose that $l$ is continuous. It follows that

$$\lim_{n\to\infty} llx_{2n} = \lim_{n\to\infty} ly_{2n} = lu.$$

In view of $(B5)$ we obtain that

$$\lim_{n\to\infty} fy_{2n+1} = \lim_{n\to\infty} ffx_{2n} = lu.$$

We subsequently produce that $u$ is a common fixed point of $f$, $g$, $h$ and $l$. It follows from (1.6) that

$$d_k^{s+t}(fy_{2n+1}, gx_{2n+1}) \leq \max\Big\{ \{p^s q^t : p, q \in \{d_k(ly_{2n+1}, hx_{2n+1}), d_k(fy_{2n+1}, ly_{2n+1}), d_k(gx_{2n+1}, hx_{2n+1}), d_k(fy_{2n+1}, gx_{2n+1})\}\} \bigcup\Big\{ p^s\Big(\frac{q}{2}\Big)^t, p^t\Big(\frac{q}{2}\Big)^s : p \in \{d_k(ly_{2n+1}, hx_{2n+1}), d_k(fy_{2n+1}, ly_{2n+1}), d_k(gx_{2n+1}, hx_{2n+1}), d_k(fy_{2n+1}, gx_{2n+1})\}, q \in \{d_k(fy_{2n+1}, hx_{2n+1}), d_k(gx_{2n+1}, ly_{2n+1})\}\Big\}\Big\}$$

$$- w\Big( \max \Big\{ \{p^s q^t : p, q \in \{d_k(ly_{2n+1}, hx_{2n+1}), d_k(fy_{2n+1}, ly_{2n+1}), d_k(gx_{2n+1}, hx_{2n+1}), d_k(fy_{2n+1}, gx_{2n+1})\}\} \bigcup \Big\{ p^s \Big(\frac{q}{2}\Big)^t, p^t \Big(\frac{q}{2}\Big)^s : p \in \{d_k(ly_{2n+1}, hx_{2n+1}), d_k(fy_{2n+1}, ly_{2n+1}), d_k(gx_{2n+1}, hx_{2n+1}), d_k(fy_{2n+1}, gx_{2n+1})\}, q \in \{d_k(fy_{2n+1}, hx_{2n+1}), d_k(gx_{2n+1}, ly_{2n+1})\} \Big\} \Big\} \Big), \quad \forall k \geq 1.$$

As $n \to \infty$ in the above inequality, we infer that

$$d_k^{s+t}(lu, u) \leq d_k^{s+t}(lu, u) - w(d_k^{s+t}(lu, u)) \leq d_k^{s+t}(lu, u), \quad \forall k \geq 1,$$

which signifies that $d_k(lu, u) = 0$ for any $k \geq 1$, that is, $lu = u$. The rest of the proof is the similar as that of Theorem 2.1. This completes the proof. □

**Remark 2.1.** Theorems 2.1 and 2.2 both unify and improve Theorem 2.1 of Huang, Lee and Kang [4] and some results of Theorem 2.1, 2.2 and 2.3 of Liu [7]. The condition $(B1)$ weakens the corresponding condition in Theorem 3.1 of Pathak, Cho, Kang and Lee [17].

## 3. Applications

In this section, let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|_1)$ be both real Banach spaces, $S \subseteq X$ be the state space, and $D \subseteq Y$ be the decision space. Denote by $BB(S)$ the set of all real-value mappings on $S$ that are bounded on bounded subsets of $S$. It is easy to verify that $BB(S)$ is a linear space over $\mathbb{R}$ under usual definitions of addition and multiplication by scalars. For $k \geq 1$ and $f, g \in BB(S)$, let

$$d_k(f, g) = \sup\{|f(x) - g(x)| : x \in \overline{B}(0, k)\},$$

$$d(f, g) = \sum_{k=1}^{\infty} 2^{-k} \frac{d_k(f, g)}{1 + d_k(f, g)},$$

where $\overline{B}(0, k) = \{x : x \in S \text{ and } \|x\| \leq k\}$. Clearly, $\{d_k\}_{k\geq 1}$ is a countable family of pseudometrics on $BB(S)$ and $(BB(S), d)$ is a complete metric space.

Now we study the existence and uniqueness of common solution for the system of functional equations (1.7) in the deduced complete metric space $(BB(S), d)$.

**Theorem 3.1.** *=labelTheorem 3.1. Let $u : S \times D \to S, T : S \times D \to S$ and $H_1, H_2, G_1, G_2 : S \times D \times \mathbb{R} \to \mathbb{R}$ satisfy the following conditions:*

*$(C1)$ for given $k \geq 1$ and $a \in BB(S)$, there exists $r(k, a) > 0$ such that*

$$|u(x, y)| + \max\{|H_i(x, y, a(T(x, y)))|, |G_i(x, y, a(T(x, y)))| : i \in \{1, 2\}\} \leq r(k, a), \quad \forall (x, y) \in \overline{B}(0, k) \times D;$$

$(C2)$ *there exist* $w \in W, s, t \in \mathbb{R}^+$ *and* $s+t \geq 1$ *such that*

$$
\begin{aligned}
&|H_1(x,y,a(c)) - H_2(x,y,b(c))|^{s+t} \\
&\leq \max\Big\{\{p^s q^t : p, q \in \{d_k(J_1a, J_2b), d_k(A_1a, J_1a), d_k(A_2b, J_2b), \\
&\qquad d_k(A_1a, A_2b)\}\}\bigcup\Big\{p^s\Big(\frac{q}{2}\Big)^t, p^t\Big(\frac{q}{2}\Big)^s : p \in \{d_k(J_1a, J_2b), \\
&\qquad d_k(A_1a, J_1a), d_k(A_2b, J_2b), d_k(A_1a, A_2b)\}, \\
&\qquad q \in \{d_k(A_1a, J_2b), d_k(A_2b, J_1a)\}\Big\}\Big\} \\
&\quad - w\Big(\max\Big\{\{p^s q^t : p, q \in \{d_k(J_1a, J_2b), d_k(A_1a, J_1a), \\
&\qquad\qquad d_k(A_2b, J_2b), d_k(A_1a, A_2b)\}\} \\
&\qquad\qquad \bigcup\Big\{p^s\Big(\frac{q}{2}\Big)^t, p^t\Big(\frac{q}{2}\Big)^s : p \in \{d_k(J_1a, J_2b), d_k(A_1a, J_1a), \\
&\qquad\qquad d_k(A_2b, J_2b), d_k(A_1a, A_2b)\}, q \in \{d_k(A_1a, J_2b), \\
&\qquad\qquad d_k(A_2b, J_1a)\}\Big\}\Big\}\Big), \\
&\qquad\qquad \forall (x,y,c) \in \overline{B}(0,k) \times D \times S,\ k \geq 1,
\end{aligned}
\tag{3.1}
$$

*where* $A_1, A_2, J_1$ *and* $J_2$ *are defined as follows:* $\forall (x,a) \in S \times BB(S)$, $i \in \{1,2\}$,

$$
\begin{aligned}
A_i a(x) &= \operatorname*{opt}_{y \in D}\{u(x,y) + H_i(x,y,a(T(x,y)))\}, \\
J_i a(x) &= \operatorname*{opt}_{y \in D}\{u(x,y) + G_i(x,y,a(T(x,y)))\};
\end{aligned}
\tag{3.2}
$$

$(C3)$ *either there exists* $A_i \in \{A_1, A_2\}$ *such that for any sequence* $\{a_n\}_{n\geq 1} \subset BB(S), a \in BB(S)$ *and* $k \geq 1$,

$$
\lim_{n\to\infty} \sup_{x\in\overline{B}(0,k)} |a_n(x) - a(x)| = 0 \Longrightarrow \lim_{n\to\infty} \sup_{x\in\overline{B}(0,k)} |A_i a_n(x) - A_i a(x)| = 0
$$

*or there exists* $J_i \in \{J_1, J_2\}$ *such that for any sequence* $\{a_n\}_{n\geq 1} \subset BB(S), a \in BB(S)$ *and* $k \geq 1$,

$$
\lim_{n\to\infty} \sup_{x\in\overline{B}(0,k)} |a_n(x) - a(x)| = 0 \Longrightarrow \lim_{n\to\infty} \sup_{x\in\overline{B}(0,k)} |J_i a_n(x) - J_i a(x)| = 0;
$$

$(C4)$ $A_1(BB(S)) \subseteq J_2(BB(S))$ *and* $A_2(BB(S)) \subseteq J_1(BB(S))$;
$(C5)$ *for any* $k \geq 1$ *and* $i \in \{1,2\}$,

$$
\begin{aligned}
&\lim_{n\to\infty} \sup_{x\in\overline{B}(0,k)} |A_i J_i a_n(x) - J_i J_i a_n(x)| \\
&= \lim_{n\to\infty} \sup_{x\in\overline{B}(0,k)} |J_i A_i a_n(x) - A_i A_i a_n(x)| = 0,
\end{aligned}
$$

*whenever* $\{a_n\}_{n\geq 1} \subset BB(S)$ *is a sequence in* $BB(S)$ *such that*

$$\lim_{n\to\infty} \sup_{x\in\overline{B}(0,k)} |A_i a_n(x) - a(x)| = \lim_{n\to\infty} \sup_{x\in\overline{B}(0,k)} |J_i a_n(x) - a(x)| = 0$$

*for some* $a \in BB(S)$.
*Then the system of functional equations* (1.7) *possesses a unique common solution in* $BB(S)$.

*Proof.* It follows from $(C1)$ and $(C2)$ that $A_1, A_2$ $J_1$ and $J_2$ are self mappings in $BB(S)$. Clearly, $(C3)$ implies that one of $A_1, A_2, J_1$ and $J_2$ is continuous, and $(C5)$ means that $A_1, J_1$ and $A_2, J_2$ are compatible mappings of type $(A)$.

Presume that $\mathrm{opt}_{y\in D} = \inf_{y\in D}$. Let $a, b \in BB(S),\ k \geq 1, x \in \overline{B}(0,k)$ and $\varepsilon > 0$. In terms of (3.2) we deduce that there exist $y, z \in D$ such that

$$A_1 a(x) > u(x,y) + H_1(x,y,a(T(x,y))) - \varepsilon, \tag{3.3}$$

$$A_2 b(x) > u(x,z) + H_2(x,z,b(T(x,z))) - \varepsilon. \tag{3.4}$$

It is clear that

$$A_1 a(x) \leq u(x,z) + H_1(x,z,a(T(x,z))), \tag{3.5}$$

$$A_2 b(x) \leq u(x,y) + H_2(x,y,b(T(x,y))). \tag{3.6}$$

(3.4) together with (3.5) leads to

$$\begin{aligned} &A_1 a(x) - A_2 b(x) \\ &< H_1(x,z,a(T(x,z))) - H_2(x,z,b(T(x,z))) + \varepsilon \\ &\leq |H_1(x,z,a(T(x,z))) - H_2(x,z,b(T(x,z)))| + \varepsilon. \end{aligned} \tag{3.7}$$

From (3.3) and (3.6) we know that

$$\begin{aligned} &A_1 a(x) - A_2 b(x) \\ &> H_1(x,y,a(T(x,y))) - H_2(x,y,b(T(x,y))) - \varepsilon \\ &\geq -|H_1(x,y,a(T(x,y))) - H_2(x,y,b(T(x,y)))| - \varepsilon. \end{aligned} \tag{3.8}$$

It follows from (3.7) and (3.8) that

$$\begin{aligned} &|A_1 a(x) - A_2 b(x)| \\ &\leq \max\{|H_1(x,y,a(T(x,y))) - H_2(x,y,b(T(x,y)))|, \\ &\qquad |H_1(x,z,a(T(x,z))) - H_2(x,z,b(T(x,z)))|\} + \varepsilon. \end{aligned}$$

By means of (3.1), the above inequality and the mean value theorem, we have

$$\begin{aligned}
&|A_1a(x) - A_2b(x)|^{s+t}\\
&\le (\max\{|H_1(x, y, a(T(x, y))) - H_2(x, y, b(T(x, y)))|,\\
&\qquad |H_1(x, z, a(T(x, z))) - H_2(x, z, b(T(x, z)))|\} + \varepsilon)^{s+t}\\
&\le (\max\{|H_1(x, y, a(T(x, y))) - H_2(x, y, b(T(x, y)))|,\\
&\qquad |H_1(x, z, a(T(x, z))) - H_2(x, z, b(T(x, z)))|\})^{s+t}\\
&\quad + (s + t)(2r(k, a) + \varepsilon)^{s+t-1}\varepsilon\\
&\le \max\Big\{\{p^s q^t : p, q \in \{d_k(J_1a, J_2b), d_k(A_1a, J_1a), d_k(A_2b, J_2b), d_k(A_1a, A_2b)\}\}\\
&\qquad \bigcup\Big\{p^s\Big(\frac{q}{2}\Big)^t, p^t\Big(\frac{q}{2}\Big)^s : p \in \{d_k(J_1a, J_2b), d_k(A_1a, J_1a), d_k(A_2b, J_2b),\\
&\qquad\quad d_k(A_1a, A_2b)\}, q \in \{d_k(A_1a, J_2b), d_k(A_2b, J_1a)\}\Big\}\Big\}\\
&\quad - w\Big(\max\Big\{\{p^s q^t : p, q \in \{d_k(J_1a, J_2b), d_k(A_1a, J_1a), d_k(A_2b, J_2b),\\
&\qquad\quad d_k(A_1a, A_2b)\}\}\bigcup\Big\{p^s\Big(\frac{q}{2}\Big)^t, p^t\Big(\frac{q}{2}\Big)^s : p \in \{d_k(J_1a, J_2b),\\
&\qquad\quad d_k(A_1a, J_1a), d_k(A_2b, J_2b), d_k(A_1a, A_2b)\},\\
&\qquad\quad q \in \{d_k(A_1a, J_2b), d_k(A_2b, J_1a)\}\Big\}\Big\}\Big)\\
&\quad + (s + t)(2r(k, a) + \varepsilon)^{s+t-1}\varepsilon,
\end{aligned}$$

which gives that

$$\begin{aligned}
&d_k^{s+t}(A_1a, A_2b)\\
&\le \max\Big\{\{p^s q^t : p, q \in \{d_k(J_1a, J_2b), d_k(A_1a, J_1a), d_k(A_2b, J_2b),\\
&\qquad d_k(A_1a, A_2b)\}\}\bigcup\Big\{p^s\Big(\frac{q}{2}\Big)^t, p^t\Big(\frac{q}{2}\Big)^s : p \in \{d_k(J_1a, J_2b),\\
&\qquad d_k(A_1a, J_1a), d_k(A_2b, J_2b), d_k(A_1a, A_2b)\},\\
&\qquad q \in \{d_k(A_1a, J_2b), d_k(A_2b, J_1a)\}\Big\}\Big\}\\
&\quad - w\Big(\max\Big\{\{p^s q^t : p, q \in \{d_k(J_1a, J_2b), d_k(A_1a, J_1a), d_k(A_2b, J_2b),\\
&\qquad\quad d_k(A_1a, A_2b)\}\}\bigcup\Big\{p^s\Big(\frac{q}{2}\Big)^t, p^t\Big(\frac{q}{2}\Big)^s : p \in \{d_k(J_1a, J_2b),\\
&\qquad\quad d_k(A_1a, J_1a), d_k(A_2b, J_2b), d_k(A_1a, A_2b)\},\\
&\qquad\quad q \in \{d_k(A_1a, J_2b), d_k(A_2b, J_1a)\}\Big\}\Big\}\Big)\\
&\quad + (s + t)(2r(k, a) + \varepsilon)^{s+t-1}\varepsilon.
\end{aligned} \tag{3.9}$$

Similarly we get that (3.9) also holds for $\mathrm{opt}_{y\in D} = \sup_{y\in D}$. Letting $\varepsilon \to 0$ in (3.9), we deduce that

$$
\begin{aligned}
&d_k^{s+t}(A_1a, A_2b)\\
&\le \max\Big\{\{p^sq^t : p,q \in \{d_k(J_1a,J_2b), d_k(A_1a,J_1a), d_k(A_2b,J_2b),\\
&\qquad d_k(A_1a,A_2b)\}\}\bigcup\Big\{p^s\Big(\frac{q}{2}\Big)^t, p^t\Big(\frac{q}{2}\Big)^s : p \in \{d_k(J_1a,J_2b), d_k(A_1a,J_1a),\\
&\qquad d_k(A_2b,J_2b), d_k(A_1a,A_2b)\}, q \in \{d_k(A_1a,J_2b), d_k(A_2b,J_1a)\}\Big\}\Big\}\\
&\quad - w\Big(\max\Big\{\{p^sq^t : p,q \in \{d_k(J_1a,J_2b), d_k(A_1a,J_1a), d_k(A_2b,J_2b),\\
&\qquad\quad d_k(A_1a,A_2b)\}\}\bigcup\Big\{p^s\Big(\frac{q}{2}\Big)^t, p^t\Big(\frac{q}{2}\Big)^s : p \in \{d_k(J_1a,J_2b),\\
&\qquad\quad d_k(A_1a,J_1a), d_k(A_2b,J_2b), d_k(A_1a,A_2b)\},\\
&\qquad\quad q \in \{d_k(A_1a,J_2b), d_k(A_2b,J_1a)\}\Big\}\Big\}\Big).
\end{aligned}
$$

Theorem 2.1 ensures that $A_1, A_2, J_1$ and $J_2$ have a unique common fixed point $j \in BB(S)$. That is, $j$ is a unique common solution of the system of functional equations (1.7). This completes the proof. □

As in the proof of Theorem 3.1, we have the following result.

**Theorem 3.2.** *Let $u : S \times D \to S, T : S \times D \to S$ and $H_1, H_2, G_1, G_2 : S \times D \times \mathbb{R} \to \mathbb{R}$ satisfy* $(C1)$, $(C2)$, $(C3)$, $(C4)$ *and*
$(C6)$ *for any $k \ge 1$ and $i \in \{1,2\}$,*

$$\lim_{n\to\infty} \sup_{x\in\overline{B}(0,k)} |A_iA_ia_n(x) - J_iJ_ia_n(x)| = 0,$$

*whenever $\{a_n\}_{n\ge1} \subset BB(S)$ is a sequence in $BB(S)$ such that*

$$\lim_{n\to\infty} \sup_{x\in\overline{B}(0,k)} |A_ia_n(x) - a(x)| = \lim_{n\to\infty} \sup_{x\in\overline{B}(0,k)} |J_ia_n(x) - a(x)| = 0$$

*for some $a \in BB(S)$.*
*Then the system of functional equations in* (1.7) *possesses a unique common solution in $BB(S)$.*

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