East Asian Mathematical Journal
Vol. 25 (2009), No. 1, pp. 1–10

COMMON FIXED POINTS OF GENERALIZED CONTRACTIVE MAPPINGS

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Abstract. In this paper, we prove some common fixed point theorems of compatible mappings under the generalized contractive type in metric spaces and also give some examples to illustrate our main theorems. This results extend the results of several authors.

1. Introduction

The most well-known fixed point theorem is so called *Banach's fixed point theorem,* which asserts that, if a mapping T from a complete metric space (X, d) into itself is contractive, then T has a unique fixed point in a complete metric space, that is, there exists a unique $z \in X$ such that $Tz = z$. For an extension of Banach's fixed point theorem, Hardy-Rogers [2] and many others introduced a more generalized contractive mappings.

In 1976, Jungck [3] initially proved a common fixed point theorem for commuting mappings, which generalizes the well-known Banach's fixed point theorem. This result has been generalized, extended and improved by many authors ([1], [4]-[6], [8]-[12]) in various ways.

On the other hand, in 1982, Sessa [11] introduced a generalization of commutativity, which is called the weak commutativity, and proved some common fixed point theorems for weakly commuting mappings which generalize the results of Das-Naik [1]. Recently, Jungck [4] introduced the concept of the more generalized commutativity, so called compatibility, which is more general than that of weak commutativity. The utility of compatibility in the context of fixed point theory was initially demonstrated in extending a theorem of Park-Bae [9]. By employing compatible mappings instead of commuting mappings and using four mappings instead of three mappings, Jungck [5] extended the results of Khan-Imdad [7], Singh-Singh [12] and, recently, also obtained an interesting result related to his concept in his consecutive paper ([6]). Further, Kang-Kim [7] proved some fixed point theorems for compatible mappings.

Received January 3, 2008; Accepted September 10, 2008.
2000 *Mathematics Subject Classification.* 47H10, 54H25.
Key words and phrases. common fixed points, compatible, generalized contractive mappings.
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In this paper, we prove some common fixed point theorems of compatible mappings under the generalized contractive type in metric spaces and also give some examples to illustrate our main theorems. This results extend the results of Kang-Kim [7] and several others.

2. Preliminaries

The following was introduced by Sessa [11].

Definition 2.1. Let A and B be mappings from a metric space (X, d) into itself. Then A and B are said to be *weakly commuting mappings* on X if $d(ABx, BAx) \leq d(Ax, Bx)$ for all $x \in X$.

Clearly, commuting mappings ($ABx = BAx$ for all $x \in X$) are weakly commuting, but the converse is not necessarily true as in the following example.

Example 2.1. Let $X = [\,0, 1\,]$ with the Euclidean metric d. Define the mappings A, $B : X \to X$ by

$$Ax = \frac{1}{2}\,x, \quad Bx = \frac{x}{2+x}$$

for all $x \in X$, respectively.

The following was given by Jungck [4].

Definition 2.2. Let A and B be mappings from a metric space (X, d) into itself. Then A and B are said to be *compatible mappings* on X if $\lim_{n\to\infty} d(ABx_n, BAx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = t$ for some point $t \in X$.

Obviously, weakly commuting mappings are compatible, but the converse is not necessarily true as in the following example.

Example 2.2. Let $X = (-\infty, \infty)$ with the Euclidean metric d. Define the mappings A, $B : X \to X$ by

$$Ax = x^3, \quad Bx = 2 - x$$

for all $x \in X$, respectively.

We need the following lemmas for our main theorems, which were proved by Jungck [3] and [4].

Lemma 2.1. *Let $\{y_n\}$ be a sequence in a metric space (X, d) satisfying the following condition*

$$d(y_{n+1}, y_n) \leq h\, d(y_n, y_{n-1})$$

for $n = 1, 2, \cdots$, where $0 < h < 1$. Then $\{y_n\}$ is a Cauchy sequence in X.

Lemma 2.2. *Let A and B be compatible mappings from a metric space (X, d) into itself. Suppose that $At = Bt$ for some $t \in X$. Then $d(ABt, BAt) = 0$, that is, $ABt = BAt$.*

Lemma 2.3. *Let A and B be compatible mappings from a metric space (X, d) into itself. Suppose that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = t$ for some $t \in X$. Then $\lim_{n\to\infty} BAx_n = At$ if A is continuous.*

3. Fixed point theorems

Now, let A, B, S and T be mappings from a metric space (X, d) into itself satisfying the following conditions

$$A(X) \subset T(X), \quad B(X) \subset S(X), \tag{3.1}$$

$$\begin{aligned} d(Ax, By) \le p \max\big\{ &d(Ax, Sx), d(By, Ty), \\ &\frac{1}{2}[\, d(Ax, Ty) + d(By, Sx)\,], d(Sx, Ty)\big\} \\ &+ q \max\big\{ d(Ax, Ty), d(By, Sx)\big\} \end{aligned} \tag{3.2}$$

for all $x, y \in X$, where $0 < p + 2q < 1$ (p and q are non-negative real numbers). Then, for an arbitrary point x_0 in X, by (3.1), we choose a point x_1 in X such that $Tx_1 = Ax_0$ and, for this point x_1, there exists a point x_2 in X such that $Sx_2 = Bx_1$ and so on. Continuing in this manner, we can define a sequence $\{y_n\}$ in X such that, for $n = 0, 1, 2, \cdots$,

$$\begin{cases} y_{2n+1} = Tx_{2n+1} = Ax_{2n}, \\ y_{2n} = Sx_{2n} = Bx_{2n-1}. \end{cases} \tag{3.3}$$

Lemma 3.1. *Let A, B, S and T be mappings from a metric space (X, d) into itself satisfying the conditions* (3.1) *and* (3.2). *Then the sequence $\{y_n\}$ defined by* (3.3) *is a Cauchy sequence in X.*

Proof. Let $\{y_n\}$ be the sequence in X defined by (3.3). From (3.2), we have

$$\begin{aligned} &d(y_{2n+1}, y_{2n+2}) \\ &= d(Ax_{2n}, Bx_{2n+1}) \\ &\le p \max\big\{ d(y_{2n}, y_{2n+1}), d(y_{2n+1}, y_{2n+2}), \\ &\qquad \frac{1}{2}[d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n+2})], d(y_{2n}, y_{2n+1})\big\} \\ &\quad + q \max\big\{ d(y_{2n+1}, y_{2n+1}), d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n+2})\big\}, \end{aligned} \tag{3.4}$$

where $0 < h = p + 2q < 1$. In (3.4), if $d(y_{2n+1}, y_{2n+2}) > d(y_{2n}, y_{2n+1})$ for some positive integer n, then we have

$$d(y_{2n+1}, y_{2n+2}) \le hd(y_{2n+1}, y_{2n+2}),$$

which is a contradiction. Thus we have

$$d(y_{2n+1}, y_{2n+2}) \le hd(y_{2n}, y_{2n+1}).$$

Similarly, we obtain

$$d(y_{2n}, y_{2n+1}) \le hd(y_{2n-1}, y_{2n}).$$

It follows from the above facts that

$$d(y_n, y_{n+1}) \leq h\, d(y_{n-1}, y_n)$$

for $n = 1, 2, \cdots$, where $0 < h < 1$. By Lemma 2.1, $\{y_n\}$ is a Cauchy sequence in X. □

Now, we are ready to give our main theorems.

Theorem 3.1. *Let A, B, S and T be mappings from a complete metric space (X, d) into itself satisfying the conditions* (3.1) *and* (3.2)*. Suppose that*
(3.5) *one of A, B, S and T is continuous,*
(3.6) *the pairs A, S and B, T are compatible on X.*
Then A, B, S and T have a unique common fixed point in X.

Proof. Let $\{y_n\}$ be the sequence in X defined by (3.3). By Lemma 3.1, $\{y_n\}$ is a Cauchy sequence and hence it converges to some point $z \in X$. Consequently, the subsequences $\{Ax_{2n}\}$, $\{Sx_{2n}\}$, $\{Bx_{2n-1}\}$ and $\{Tx_{2n-1}\}$ of $\{y_n\}$ also converge to the point z.

Now, suppose that S is continuous. Since A and S are compatible on X, Lemma 2.3 gives that

$$S^2x_{2n} \longrightarrow Sz, \quad ASx_{2n} \longrightarrow Sz \quad \text{as } n \to \infty.$$

By (3.2), we obtain

$$\begin{aligned} &d(ASx_{2n}, Bx_{2n-1}) \\ &\leq p \max\big\{d(ASx_{2n}, S^2x_{2n}), d(Bx_{2n-1}, Tx_{2n-1}), \\ &\qquad \frac{1}{2}[\, d(ASx_{2n}, Tx_{2n-1}) + d(Bx_{2n-1}, S^2x_{2n})\,], \\ &\qquad d(S^2x_{2n}, Tx_{2n-1})\big\} \\ &\quad + q \max\big\{d(ASx_{2n}, Tx_{2n-1}), d(Bx_{2n-1}, S^2x_{2n})\big\}. \end{aligned} \tag{3.7}$$

Letting $n \to \infty$ in (3.7), we have

$$\begin{aligned} d(Sz, z) &\leq p \max\big\{0, 0, \frac{1}{2}[\, d(Sz, z) + d(z, Sz)\,], d(Sz, z)\big\} \\ &\quad + q\, d(Sz, z), \end{aligned}$$

so that $z = Sz$. By (3.2), we also obtain

$$\begin{aligned} &d(Az, Bx_{2n-1}) \\ &\leq p \max\big\{d(Az, Sz), d(Bx_{2n-1}, Tx_{2n-1}), \\ &\qquad \frac{1}{2}[\, d(Az, Tx_{2n-1}) + d(Bx_{2n-1}, Sz)\,], d(Sz, Tx_{2n-1})\big\} \\ &\quad + q \max\big\{d(Az, Tx_{2n-1}), d(Bx_{2n-1}, Sz)\big\}. \end{aligned} \tag{3.8}$$

Letting $n \to \infty$ in (3.8), we have

$$d(Az,z) \le p\max\big\{d(Az,Sz),0,\frac{1}{2}[\,d(Az,z)+d(z,Sz)\,],d(Sz,z)\big\} + q\max\big\{d(Az,z),d(z,Sz)\big\},$$

so that $z = Az$. Since $A(X) \subset T(X)$, we have $z \in T(X)$ and hence there exists a point $u \in X$ such that $z = Az = Tu$.

$$\begin{aligned} d(z,Bu) &= d(Az,Bu) \\ &\le p\max\big\{0,d(Bu,Tu),\frac{1}{2}[\,d(Az,Tu)+d(Bu,z)\,],d(Sz,Tu)\big\} \\ &\quad + q\max\{d(Az,Tu),d(Bu,z)\big\}, \end{aligned}$$

which implies that $z = Bu$. Since B and T are compatible on X and $Tu = Bu = z$, we have $d(TBu,BTu) = 0$ by Lemma 2.2 and hence $Tz = TBu = BTu = Bz$. Moreover, by (3.2), we obtain

$$\begin{aligned} d(z,Tz) &= d(Az,Bz) \\ &\le p\max\big\{0,d(Bz,Tz),\frac{1}{2}[\,d(z,Tz)+d(Bz,z)\,],d(z,Tz)\big\} \\ &\quad + q\max\{d(z,Tz),d(Bz,z)\big\}, \end{aligned}$$

so that $z = Tz$. Therefore, z is a common fixed point of A, B, S and T. Similarly, we can also complete the proof when T is continuous.

Next, suppose that A is continuous. Since A and S are compatible on X, it follows from Lemma 2.3 that

$$A^2x_{2n} \longrightarrow Az, \quad SAx_{2n} \longrightarrow Az \quad \text{as } n \to \infty.$$

By (3.2), we have

$$\begin{aligned} &d(A^2x_{2n},Bx_{2n-1}) \\ &\le p\max\big\{d(A^2x_{2n},SAx_{2n}),d(Bx_{2n-1},Tx_{2n-1}), \\ &\qquad\qquad \frac{1}{2}[\,d(A^2x_{2n},Tx_{2n-1})+d(Bx_{2n-1},SAx_{2n})\,], \\ &\qquad\qquad d(SAx_{2n},Tx_{2n-1})\big\} \\ &\quad + q\max\big\{d(A^2x_{2n},Tx_{2n-1}),d(Bx_{2n-1},SAx_{2n})\big\}. \end{aligned} \tag{3.9}$$

Letting $n \to \infty$ in (3.9), we obtain

$$d(Az,z) \le p\max\big\{0,0,\frac{1}{2}[\,d(Az,z)+d(z,Az)\,],d(Az,z)\big\} + q\,d(Az,z),$$

so that $z = Az$. Hence there exists a point $v \in X$ such that $z = Az = Tv$. By (3.2), we also obtain

$$\begin{aligned} & d(A^2x_{2n}, Bv) \\ & \leq p\max\big\{d(A^2x_{2n}, SAx_{2n}), d(Bv, Tv), \\ & \qquad \frac{1}{2}[\,d(A^2x_{2n}, Tv) + d(Bv, SAx_{2n})\,], d(SAx_{2n}, Tv)\big\} \\ & \quad + q\max\big\{d(A^2x_{2n}, Tv), d(Bv, SAx_{2n})\big\}. \end{aligned} \tag{3.10}$$

Letting $n \to \infty$ in (3.10), we have

$$\begin{aligned} d(z, Bv) \leq\ & p\max\big\{0, d(Bv, Tv), \frac{1}{2}[\,d(Az, Tv) + d(Bv, z)\,], d(z, Tv)\big\} \\ & + q\max\{d(Az, Tv), d(Bv, z)\big\}, \end{aligned}$$

which implies that $z = Bv$. Since B and T are compatible on X and $Tv = Bv = z$, we have $d(TBv, BTv) = 0$ by Lemma 2.2 and hence $Tz = TBv = BTv = Bz$. Moreover, by (3.2), we have

$$\begin{aligned} & d(Ax_{2n}, Bz) \\ & \leq p\max\big\{d(Ax_{2n}, Sx_{2n}), d(Bz, Tz), \\ & \qquad \frac{1}{2}[\,d(Ax_{2n}, Tz) + d(Bz, Sx_{2n})\,], d(Sx_{2n}, Tz)\big\} \\ & \quad + q\max\big\{d(Ax_{2n}, Tz), d(Bz, Sx_{2n})\big\}. \end{aligned} \tag{3.11}$$

Letting $n \to \infty$ in (3.11), we obtain

$$\begin{aligned} d(z, Bz) \leq\ & p\max\big\{0, d(Bz, Tz), \frac{1}{2}[\,d(z, Tz) + d(Bz, z)\,], d(z, Tz)\big\} \\ & + q\max\{d(z, Tz), d(Bz, z)\big\}, \end{aligned}$$

so that $z = Bz$. Since $B(X) \subset S(X)$, there exists a point $w \in X$ such that $z = Bz = Sw$ and so, by (3.2),

$$\begin{aligned} d(Aw, z) &= d(Aw, Bz) \\ & \leq p\max\big\{d(Aw, Sw), 0, \frac{1}{2}[\,d(Aw, z) + d(z, Sw)\,], d(Sw, z)\big\} \\ & \quad + q\max\big\{d(Aw, z), d(z, Sw)\big\}, \end{aligned}$$

so that $Aw = z$. Since A and S are compatible on X and $Aw = Sw = z$, we have $d(SAw, ASw) = 0$ and hence $Sz = SAw = ASw = Az$. Therefore, z is a common fixed point of A, B, S and T. Similarly, we can also complete the proof when B is continuous.

It follows easily from (3.2) that z is is a unique common fixed point of A, B, S and T. This completes the proof. □

The following corollary follows immediately from Theorem 3.1.

Corollary 3.1. *Let A, B, S and T be mappings from a complete metric space (X,d) into itself satisfying the conditions* (3.1), (3.5) *and* (3.6). *Suppose that*

$$\begin{aligned} d(Ax,By) \le p \max\big\{ & d(Ax,Sx), d(By,Ty), \\ & \frac{1}{2}\, d(Ax,Ty), \frac{1}{2}\, d(By,Sx), d(Sx,Ty)\big\} \\ & + q \max\big\{d(Ax,Ty), d(By,Sx)\big\} \end{aligned}$$

for all $x,y \in X$, where $0 < p + 2q < 1$. Then A, B, S and T have a unique common fixed point in X.

Remark 3.1. If we put $q = 0$ in Theorem 3.1 and Corollary 3.1, we obtain the results of Kang-Kim [7] and several others.

4. Examples

In this section, we give some examples to illustrate our main theorems.

In the following example, we show the existence of a common fixed point of mappings which are compatible, but not commuting.

Example 4.1. Let $X = [1,\infty)$ with the Euclidean metric d. Define the mappings A, B, S, $T : X \to X$ by

$$Ax = x^3, \quad Bx = x^2, \quad Sx = 2x^6 - 1, \quad Tx = 2x^4 - 1$$

for all $x \in X$, respectively. Now, $A(X) = B(X) = S(X) = T(X) = X$. Moreover, since

$$d(Ax_n, Sx_n) = \left|2x_n^3 + 1\right| \left|x_n^3 - 1\right| \to 0$$

if and only if $x_n \to 1$, we have

$$\lim_{n\to\infty} d(ASx_n, SAx_n) = \lim_{n\to\infty} 6x_n^6 (x_n^6 - 1)^2 = 0 \quad \text{as } x_n \to 1.$$

Thus A and S are compatible on X, but they are not commuting mappings at $x = 2$. Likewise, since

$$d(Bx_n, Tx_n) = (2x_n^2 + 1)\left|x_n^2 - 1\right| \to 0$$

if and only if $x_n \to 1$, we have

$$\lim_{n\to\infty} d(BTx_n, TBx_n) = \lim_{n\to\infty} 2(x_n^4 - 1)^2 = 0 \quad \text{as } x_n \to 1.$$

Furthermore, we obtain, where $0 < q < \frac{3}{8}$,

$$\begin{aligned} d(Ax,By) & \le \frac{1}{4}\, d(Sx,Ty) \\ & \le \frac{1}{4} \max\big\{d(Ax,Sx), d(By,Ty), \\ & \qquad \frac{1}{2}[\, d(Ax,Ty) + d(By,Sx)\,], d(Sx,Ty)\big\} \\ & \quad + q \max\big\{d(Ax,Ty), d(By,Sx)\big\} \end{aligned}$$

since

$$d(Sx,Ty) = 2\left|x^3 - y^2\right|\left|x^3 + y^2\right| \geq 4\,d(Ax,By)$$

for all $x, y \in X$. Therefore, we see that the hypotheses of Theorem 3.1 except the commutativity of A and S are satisfied, but A, B, S and T have a unique common fixed point in X.

Now, we show that the condition (3.1) is necessary in Theorem 3.1.

Example 4.2. Let $X = [\,0,1\,]$ with the Euclidean metric d. Define the mappings A, B, S, $T : X \to X$ by

$$Ax = \begin{cases} \dfrac{1}{4} & \text{if } x = 0, \\ \dfrac{1}{4}\,x & \text{if } x \neq 0, \end{cases} \qquad Bx = 0, \qquad Sx = \begin{cases} 1 & \text{if } x = 0, \\ x & \text{if } x \neq 0, \end{cases} \qquad Tx = x$$

for all $x \in X$, respectively. Since each of the pairs A, S and B, T is commutative, they are compatible pairs. Furthermore, we have

$$\begin{aligned} d(Ax,By) &= \begin{cases} \dfrac{1}{4} = \dfrac{1}{3}\,d(Ax,Sx) & \text{if } x = 0, \\ \dfrac{1}{4}\,x = \dfrac{1}{3}\,d(Ax,Sx) & \text{if } x \neq 0 \end{cases} \\ &\leq \frac{1}{3}\max\Big\{d(Ax,Sx), d(By,Ty), \\ &\qquad\qquad \frac{1}{2}[\,d(Ax,Ty) + d(By,Sx)\,], d(Sx,Ty)\Big\} \\ &\quad + q\max\big\{d(Ax,Ty), d(By,Sx)\big\} \end{aligned}$$

for all $x, y \in X$, where $0 < q < \frac{1}{3}$. All the hypotheses of Theorem 3.1 are satisfied except the condition $B(X) \subset S(X)$, but A does not have a fixed point in X.

We give an example showing that Theorem 3.1 is no longer true if we do not assume that any one of mappings is continuous.

Example 4.3. Let $X = [\,0,1\,]$ with the Euclidean metric d. Define the mappings A, B, S, $T : X \to X$ by

$$Ax = Bx = \begin{cases} \dfrac{1}{8} & \text{if } x = 0, \\ \dfrac{1}{8}\,x & \text{if } x \neq 0, \end{cases} \qquad Sx = Tx = \begin{cases} 1 & \text{if } x = 0, \\ \dfrac{1}{2}\,x & \text{if } x \neq 0 \end{cases}$$

for all $x \in X$, respectively. $A(X) = (0, \frac{1}{8}\,] \subset (0, \frac{1}{2}\,] \subset S(X)$. Moreover, we obtain

$$d(AS0, SA0) = \frac{1}{8} - \frac{1}{16} = \frac{1}{16} < \frac{7}{8} = 1 - \frac{1}{8} = d(S0, A0)$$

and $ASx = SAx = \frac{1}{16}x$ for all $x \in X - \{0\}$. So, A and S are compatible on X. Furthermore, we obtain

$$d(Ax, Ay) = \begin{cases} 0 & \text{if } x = y = 0, \\ \dfrac{1}{8}(1-x) < \dfrac{1}{4}\Big(1 - \dfrac{1}{2}x\Big) = \dfrac{1}{4}\, d(Sy, Sx) & \text{if } x > y = 0, \\ \dfrac{1}{8}(1-y) < \dfrac{1}{4}\Big(1 - \dfrac{1}{2}y\Big) = \dfrac{1}{4}\, d(Sx, Sy) & \text{if } y > x = 0, \\ \dfrac{1}{8}\,|x-y| = \dfrac{1}{4}\, d(Sx, Sy) & \text{if } x, y \neq 0, \end{cases}$$

$$\begin{aligned} &\leq \frac{1}{4}\max\big\{d(Ax, Sx), d(Ay, Sy), \\ &\qquad\qquad \frac{1}{2}[\, d(Ax, Sy) + d(Ay, Sx)\,], d(Sx, Sy)\big\} \\ &\quad + q\max\big\{d(Ax, Sy), d(Ay, Sx)\big\} \end{aligned}$$

for all $x, y \in X$, where $0 < q < \frac{3}{8}$. We find that all the hypotheses of Theorem 3.1 are satisfied except the continuity of A and S, but none of mappings A, S has a fixed point in X.

We show that the condition of the compatibility is also necessary in Theorem 3.1.

Example 4.4. Let $X = [\,0, \infty)$ with the Euclidean metric d. Define the mappings A, B, S, $T : X \to X$ by

$$Ax = Bx = \frac{1}{8}\,x + 1, \quad Sx = Tx = \frac{1}{2}\,x + 1$$

for all $x \in X$, respectively. Obviously, the sequences $\{Ax_n\}$ and $\{Sx_n\}$ converge to 1 if and only if $\{x_n\}$ converges to 0, but

$$\lim_{n\to\infty} d(ASx_n, SAx_n) = \frac{1}{2} - \frac{1}{8} = \frac{3}{8}.$$

Therefore, the pair A, S is not compatible. Furthermore, we have

$$\begin{aligned} d(Ax, Ay) &= \frac{1}{4}\, d(Sx, Sy) \\ &\leq \frac{1}{4}\max\big\{d(Ax, Sx), d(Ay, Sy), \\ &\qquad\qquad \frac{1}{2}[\, d(Ax, Sy) + d(Ay, Sx)\,], d(Sx, Sy)\big\} \\ &\quad + q\max\big\{d(Ax, Sy), d(Ay, Sx)\big\} \end{aligned}$$

for all $x, y \in X$, where $0 < q < \frac{3}{8}$. We see that all the hypotheses of Theorem 3.1 are satisfied except the compatibility of the pair A, S, but A and S don't have a common fixed point in X.

Acknowledgement. This work was supported by Korea Research Foundation Grant (KRF-1999-005-D00003).

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