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GENERALIZED HÖLDER ESTIMATES FOR THE $\bar{\partial}$-EQUATION ON CONVEX DOMAINS IN $\mathbb{C}^2$

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Abstract. In this paper, we introduce the generalized Hölder space with a majorant function and prove the Hölder regularity for solutions of the Cauchy-Riemann equation in the generalized Hölder spaces on a bounded convex domain in $\mathbb{C}^2$.

1. Introduction and regular majorant

Let D be a bounded domain in $\mathbb{C}^n$. The Hölder space of order α, $\Lambda_\alpha(D)$ $(0 < \alpha < 1)$, is defined by the set of all functions g on D such that there exists a constant $C = C_g > 0$ satisfying

$$|g(z) - g(\zeta)| \lesssim C|z - \zeta|^\alpha, \quad z, \zeta \in D.$$

We first introduce some generalized Hölder space with a majorant function. A continuous increasing function ω on $[0, \infty)$, satisfying that $\omega(0) = 0$, $\omega(t)/t$ is non-increasing, and in addition, there is a constant $C = C(\omega)$ such that

$$\int_0^\delta \frac{\omega(t)}{t}\,dt + \delta \int_\delta^\infty \frac{\omega(t)}{t^2}\,dt \le C\omega(\delta), \quad \text{for any } 0 < \delta < 1, \tag{1}$$

is called a *regular majorant*. Given a regular majorant, the generalized Hölder space, $\Lambda_\omega(D)$, is defined by the family of all functions g on D such that

$$|g(z) - g(\zeta)| \le C\omega(|z - \zeta|), \quad z, \zeta \in D. \tag{2}$$

The norm $||g||_\omega$ of $g \in \Lambda_\omega(D)$ is given by $C_g + ||g||_\infty$, where $C_g \ge 0$ is the smallest constant satisfying (2) and $||g||_\infty$ is the L^∞ norm in D. Note that with this norm $\Lambda_\omega(D)$ is a Banach space and $\Lambda_\omega(D) \subset L^\infty(D)$. The generalized Hölder spaces have been studied by many authors (see [2], [3], [4], [5], [6], [7], [8], and references in their papers).

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Theorem 1.1. *Let D be a bounded convex domain in $\mathbb{C}^2$ with C^2 boundary bD. Let ω be a regular majorant. There is a bounded linear operator $S : C_{0,1}(\bar{D}) \to C(D)$ such that $\bar{\partial}(Sf) = f$ on D and*

$$\|Sf\|_{\Lambda_\omega(D)} \leq C\|f\|_{\Lambda_\omega(D)} \quad \textit{for all} \quad f \quad \textit{with} \quad \bar{\partial}f = 0.$$

As convention we use the notation $A \lesssim B$ or $A \gtrsim B$ if there are constants c_1, c_2, independent of the quantities under consideration, satisfying $A \leq c_1 B$ and $A \geq c_2 B$, respectively.

Remark 1. Before proving Theorem 1.1, we give some examples of regular majorants.

(i) Typical example of the regular majorant is a function $\omega(t) = t^\alpha$ $(0 < \alpha < 1)$.
(ii) Non-trivial example is the type of the function, $\omega(t) = t^\alpha |\log t|^\beta$, where $0 < \alpha < 1$ and $-\infty < \beta < \infty$.
(iii) Let $m(t) = 1/|\log t|^\beta$, $\beta > 0$. Then $m(t)$ is continuous and increasing near 0, but it is not a regular majorant.

2. Henkin's solution operator of the $\bar{\partial}$-equation

Let D be a bounded and convex domain in $\mathbb{C}^2$ with C^2 boundary bD. We choose a C^2 defining function ρ for D, so that in a neighborhood U of bD

$$\rho(z) = \begin{cases} -\text{dist}(z, bD) & \text{for} \quad z \in U \cap \bar{D} \\ +\text{dist}(z, bD) & \text{for} \quad z \in U \setminus D. \end{cases}$$

We define

$$\phi(\zeta, z) = \sum_{j=1}^{2} \frac{\partial \rho}{\partial \zeta_j}(\zeta)(\zeta_j - z_j).$$

The following estimate-for a sufficiently small neighborhood U of bD-is a well-known consequence of the convexity of D :

$$|\phi(\zeta, z)| \gtrsim |\text{Im } \phi(\zeta, z)| + \rho(\zeta) - \rho(z) \quad \text{for} \quad \zeta, z \in U \tag{3}$$

and $d_\zeta \phi(\zeta, z)|_{z=\zeta} = \partial\rho(\zeta)$.

Lemma 2.1. *([9]) Let $(\zeta_0, z_0) \in \partial D \times \partial D$ such that $\phi(\zeta_0, z_0) = 0$. Then there exist neighborhoods V of ζ_0 and W of z_0 such that for each $z \in W$, there exists a C^1 local coordinate system $\zeta \mapsto t^{(z)}(\zeta) = (t_1, t_2, t_3, t_4)$ on V with the following properties:*

$$t_1(\zeta) = \rho(\zeta), \quad t_2(\zeta) = \text{Im}\, \phi(\zeta, z), \quad t_3(z) = t_4(z) = 0;$$

$$|t^{(z)}(\zeta) - t^{(z)}(\zeta')| \sim |\zeta - \zeta'|$$

for all $\zeta, \zeta' \in V$ with the constants in (3) independent of $z \in W$.

We have the Henkin's solution operator $\mathbb{S}f = \mathbb{H}f + \mathbb{K}f$ of the $\bar\partial$-equation, where

$$\mathbb{H}f(z) = c\int_{\zeta\in bD} f(\zeta)\wedge \frac{\frac{\partial\rho}{\partial\zeta_1}(\bar\zeta_2-\bar z_2)-\frac{\partial\rho}{\partial\zeta_2}(\bar\zeta_1-\bar z_1)}{\phi(\zeta,z)|\zeta-z|^2}d\zeta_1\wedge d\zeta_2, \tag{4}$$

and

$$\mathbb{K}f(z) = \int_{\zeta\in bD} f(\zeta)\wedge K(\zeta,z),$$

where $K(\zeta,z)$ is the Bochner-Martinelli kernel [10].

It is well-known that the Bochner-Martinelli integral, $\mathbb{K}f$ has a good regularity such that $\mathbb{K}$ is a bounded operator from bounded forms to the forms whose coefficients are in $\Lambda_\alpha(D)$ for any $0<\alpha<1$ [10]. However, it is not at all clear that this kind of generalized Hölder regularity for the Bochner-Martinelli integral still holds.

Proposition 2.2. Let ω be a regular majorant. Then $\mathbb{K}: L^\infty_{0,1}(D)\to\Lambda_\omega(D)$ is a bounded operator, where $L^\infty_{0,1}(D)$ is the space of bounded forms of type $(0,1)$.

Proof. It suffices to show that for arbitrary $z, z'\in D$,

$$|\mathbb{K}f(z)-\mathbb{K}f(z')| \lesssim \omega(|z-z'|)||f||_\infty. \tag{5}$$

The proof of the regularity of the Bochner-Martinelli integral in the classical Hölder space implies that

$$|\mathbb{K}f(z)-\mathbb{K}f(z')| \lesssim |z-z'|(1+|\log|z-z'||)||f||_\infty$$

(for the details, see the chapter 4 of Range's book [?]). Since $\omega(t)/t$ is nonincreasing, it follows that $t\lesssim\omega(t)$ for t with $0<t<R$, where $R=\sup_{z,\zeta\in D}|z-\zeta|$. Thus we have $|\mathbb{K}f(z)-\mathbb{K}f(z')|\lesssim\omega(|z-z'|)||f||_\infty$. □

3. Proof of Theorem 1.1

To prove that a function g belongs to the generalized Hölder space $\Lambda_\omega(D)$, we need a variant of the Hardy-Littlewood Lemma.

Lemma 3.1. *([1]) Let $D\subset\mathbb{R}^n$ be a bounded domain with the C^1-boundary defining function ρ. If g is a $C^1(D)$-function and ω is a regular majorant such that for some constant c_g depending on g,*

$$|dg(x)|\le c_g\frac{\omega(|\rho(x)|)}{|\rho(x)|},\quad x\in D,$$

then we have

$$|g(x)-g(y)|\le c\cdot c_g\,\omega(|x-y|).$$

Lemma 3.2. *For sufficiently small $\epsilon>0$, $\omega(t)/t^{1-\epsilon}$ is decreasing.*

Proof. Put

$$F(x)=\int_x^\infty \frac{\omega(t)}{t^2}dt.$$

Since ω is a regular majorant, by the second term of the left hand side of (**??**), it follows that

$$F(x)=\int_x^\infty \frac{\omega(t)}{t^2}dt \lesssim \frac{\omega(x)}{x}.$$

Since ω is increasing, we have

$$\begin{aligned} F(x) &= \int_x^\infty \frac{\omega(t)}{t^2}dt \\ &\geq \omega(x)\int_x^\infty \frac{1}{t^2}dt = \frac{\omega(x)}{x}. \end{aligned}$$

Thus it follows that

$$F(x)\sim \frac{\omega(x)}{x} = -xF'(x).$$

We will show that, for sufficiently small $\epsilon > 0$, $x^\epsilon F(x)$ is decreasing. We choose sufficiently small $\epsilon > 0$ such that

$$\begin{aligned} F(x) &\leq \frac{1}{\epsilon}\frac{\omega(x)}{x} \\ &= -\frac{1}{\epsilon}xF'(x). \end{aligned}$$

Thus it follows that

$$(x^\epsilon F(x))' = \epsilon x^{\epsilon-1}F(x) + x^\epsilon F'(x) \leq 0$$

and so that we get the result. □

By Lemma 3.1, for the proof of Theorem 1.1, it is enough to prove the following result.

Proposition 3.3. There exists a constant $C_\omega > 0$ such that

$$|d_z\mathbb{H}f(z)| \leq C_\omega \|f\|_{\Lambda_\omega(D)} \frac{\omega(|\rho(z)|)}{|\rho(z)|} \quad \text{for } z\in D. \tag{6}$$

Proof. By differentiating under the integral sign in (4), one obtains $d\mathbb{H}f(z) = \mathbb{G}_1 f(z) + \mathbb{G}_2 f(z)$, where

$$\begin{aligned} \mathbb{G}_1 f(z) &= \int_{bD} f\wedge \frac{A_1(\zeta,z)}{(\phi(\zeta,z))^2|\zeta-z|^2}, \\ \mathbb{G}_2 f(z) &= \int_{bD} f\wedge \frac{A_2(\zeta,z)}{\phi(\zeta,z)|\zeta-z|^4}, \end{aligned}$$

where $A_j(\zeta,z)$ are smooth double forms, of degree 1 in z and type (2,1) in ζ, which satisfy $A_j(\zeta,z)| \lesssim |\zeta-z|^j, j=1,2$. The compactness of bD, and a partition of unity, the estimation of $\mathbb{G}_1 f(z)$ is reduced to proving the estimate

$$|I(z)| \le C_\omega \|f\|_{\Lambda_\omega(D)} \frac{\omega(|\rho(z)|)}{|\rho(z)|} \quad \text{for} \quad z \in W \cap D,$$

where

$$I(z) = \int_{bD\cap V} f(\zeta) \frac{\chi(\zeta) A_1(\zeta, z)}{(\phi(\zeta, z))^2 |\zeta - z|^2},$$

where V, W are neighborhoods as given in Lemma 2.1, and χ has compact support in V. Note that the new coordinate system in Lemma 2.1 satisfies $t^{(z)}(\zeta) = (0, t')$ for $\zeta \in V \cap bD$, where $t' = (t_2, t_3, t_4)$. We choose $\zeta' \in V \cap bD$ satisfying $t^{(z)}(\zeta') = (0, 0, t_3, t_4)$. Then we have $I(z) \le I_1(z) + I_2(z)$, where

$$I_1(z) = \left| \int_{bD\cap V} \frac{(f(\zeta) - f(\zeta'))\chi(\zeta) A_1(\zeta, z)}{(\phi(\zeta, z))^2 |\zeta - z|^2} \, d\sigma(\zeta) \right|,$$
$$I_2(z) = \left| \int_{bD\cap V} \frac{f(\zeta')\chi(\zeta) A_1(\zeta, z)}{(\phi(\zeta, z))^2 |\zeta - z|^2} \, d\sigma(\zeta) \right|.$$

It follows from the definition of $||\cdot||_{\Lambda_\omega(D)}$ and the inequality $|A_1(\zeta, z)| \lesssim |\zeta - z|$ that

$$I_1(z) \lesssim ||f||_{\Lambda_\omega(D)} \int_{bD\cap V} \frac{\omega(|\zeta - \zeta'|)}{|\phi(\zeta, z)|^2 |\zeta - z|} \, d\sigma(\zeta). \tag{7}$$

To estimate the integral of the right hand side of (7), we use the coordinate system t, the inequality (3). We introduce polar coordinates in $t'' = (t_3, t_4) \in \mathbb{R}^2$, and set $r = |t''|$. Then we have

$$\begin{aligned} I_1(z) &\lesssim ||f||_{\Lambda_\omega(D)} \int_{|t'|<1} \frac{\omega(|t_2|)}{(|t_2| + |\rho(z)|)^2 |t'|} \, dt' \\ &\lesssim ||f||_{\Lambda_\omega(D)} \int_{|t_2|<1} \omega(|t_2|) \left[\int_0^1 \frac{r dr}{(|t_2| + |\rho(z)|)^2 r} \right] dt_2 \\ &\lesssim ||f||_{\Lambda_\omega(D)} \int_0^1 \frac{\omega(t_2)}{(t_2 + |\rho(z)|)^2} \, dt_2. \end{aligned}$$

We may assume that $0 < |\rho(z)| < 1$, since $z \in D$ is close to the boundary. We decompose the integral as follows:

$$\int_0^1 \frac{\omega(t_2)}{(t_2 + |\rho(z)|)^2} \, dt_2 = \int_0^{|\rho(z)|} \frac{\omega(t_2)}{(t_2 + |\rho(z)|)^2} \, dt_2 + \int_{|\rho(z)|}^1 \frac{\omega(t_2)}{(t_2 + |\rho(z)|)^2} \, dt_2.$$

By the first term of the left hand side of (1), we have

$$\int_0^{|\rho(z)|} \frac{\omega(t_2)}{(t_2 + |\rho(z)|)^2} \, dt_2 \lesssim \frac{1}{|\rho(z)|} \int_0^{|\rho(z)|} \frac{\omega(t_2)}{t_2} \, dt_2 \lesssim \frac{\omega(|\rho(z)|)}{|\rho(z)|}.$$

By the second term of the left hand side of (1), one obtains

$$\int_{|\rho(z)|}^{1} \frac{\omega(t_2)}{(t_2+|\rho(z)|)^2}\,dt_2 \lesssim \int_{|\rho(z)|}^{1} \frac{\omega(t_2)}{t_2^2}dt_2 \lesssim \frac{\omega(|\rho(z)|)}{|\rho(z)|}.$$

These imply

$$I_1(z) \lesssim ||f||_{\Lambda_\omega(D)} \frac{\omega(|\rho(z)|)}{|\rho(z)|}.$$

For $I_2(z)$, we need somewhat different method. An integration by parts allows one to lower the singularity order of the Henkin kernel. This kind of method was used in [11].

We see that

$$\frac{1}{\phi^2} = -\left(\frac{\partial\phi}{\partial t_2}\right)^{-1} \frac{\partial}{\partial t_2}\left(\frac{1}{\phi}\right).$$

Therefore, by the integration by parts, we have

$$\begin{aligned} I_2(z) &\lesssim \left| \int_{|t'|\le 1} -\left(\frac{\partial\phi}{\partial t_2}\right)^{-1} \frac{\partial}{\partial t_2}\left(\frac{1}{\phi}\right) \frac{f(0,0,t'')\chi(t')A_1(t',z)}{|t'|^2}\,dt' \right| \\ &= \left| \int_{|t'|\le 1} f(0,0,t'')\frac{1}{\phi}\frac{\partial}{\partial t_2}\left[\left(\frac{\partial\phi}{\partial t_2}\right)^{-1} \frac{\chi(t')A_1(t',z)}{|t'|^2}\right] dt' \right| \qquad (8) \\ &= \left| \int_{|t'|\le 1} f(0,0,t'')\frac{1}{\phi}\left(\frac{\partial\phi}{\partial t_2}\right)^{-2} B(t',z)dt' \right|, \end{aligned}$$

where

$$B(t',z) = -\frac{\partial^2\phi}{\partial t_2^2}\frac{\chi(t')A_1(t',z)}{|t'|^2} + \frac{\partial\phi}{\partial t_2}\frac{\partial}{\partial t_2}\left(\frac{\chi(t')A(t',z)}{|t'|^2}\right).$$

In the second equality of (8), we have use the fact that $f(0,0,t'')$ does not depend on t_2. Since $t_2 = \text{Im}\ \phi$, we have $|\partial\phi/\partial t_2| \geq 1$. Therefore we have

$$I_2(z) \lesssim ||f||_\infty \int_{|t'|\le 1} \frac{dt'}{|\phi||t'|^2}.$$

In [1], we proved that

$$\int_{|t'|\le 1} \frac{dt'}{|\phi||t'|^2} \lesssim \|f\|_\infty |\rho(z)|^{-\epsilon}.$$

Thus we have

$$I_2(z) \lesssim \|f\|_\infty |\rho(z)|^{-\epsilon}.$$

By Lemma 3.2, for sufficiently small $\epsilon > 0$ we get

$$|\rho(z)|^{-\epsilon} = \frac{|\rho(z)|^{1-\epsilon}}{|\rho(z)|} \lesssim \frac{\omega(|\rho(z)|)}{|\rho(z)|}.$$

Thus it follows that

$$I_2(z) \lesssim \|f\|_\infty \frac{\omega(|\rho(z)|)}{|\rho(z)|}$$

and so that

$$|\mathbb{G}_1 f(z)| \lesssim \|f\|_\infty \frac{\omega(|\rho(z)|)}{|\rho(z)|}.$$

The estimate for $\mathbb{G}_2 f(z)$ is exactly like the last part of the estimate for I_2.

Thus we complete the proof of Theorem 1.1 by using Proposition 3 and Lemma 3.1. □

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