

BIPARTITE STEINHAUS GRAPHS WITH CONNECTIVITY TWO

DAEKEUN LIM

ABSTRACT. In this paper, we investigate the generating strings and the number of 2-(edge)connected bipartite Steinhaus graphs.

1. Introduction

Let $T = a_{11}a_{12}\cdots a_{1n}$ be an n -long string of zeros and ones with $a_{11} = 0$. The *Steinhaus graph* G , generated by T has as its adjacency matrix, the *Steinhaus matrix*, $A(G) = [a_{ij}]$ which is obtained from the following, called the *Steinhaus property*:

$$a_{ij} = \begin{cases} 0 & \text{if } 1 \leq i = j \leq n; \\ a_{i-1,j-1} + a_{i-1,j} \pmod{2} & \text{if } 1 < i < j \leq n; \\ a_{ji} & \text{if } 1 \leq j < i \leq n. \end{cases}$$

In this case, T is called the *generating string* of G . It is obvious that there are exactly 2^{n-1} Steinhaus graphs of order n . The vertices of a Steinhaus graph are usually labeled by their corresponding row numbers. In Figure 1, the Steinhaus graph generated by 00110110 is pictured. For each $1 \leq i \leq n$, the n -long string $a_{i1} \cdots a_{ii}a_{i,i+1} \cdots a_{i,n}$ in $A(G)$ generates $A(G)$ by Steinhaus property. Thus, the generating string is the general generating string with respect to 1.

The *partner* $P(G)$ of G is the Steinhaus graph generated by the reverse of the last column of $A(G)$, i.e., $a_{n,n}a_{n-1,n} \cdots a_{1,n}$. Note that G is isomorphic to its partner $P(G)$ and the correspondence assigns vertex i to $n - i + 1$ for $i = 1, 2, \dots, n$. If $G = P(G)$, then G is called *doubly symmetric*. The graph in Figure 1 is doubly symmetric.

We now present Pascal's rectangle modulo two (see Figure 2). The rows of the rectangle are labelled R_1^*, R_2^*, \dots , and so the k th element of R_n^* is 0 if $k > n$ and is $\binom{n-1}{k-1} \pmod{2}$ if $1 \leq k \leq n$. We denote by $R_{n,k}$ the string formed by the first k elements of R_n^* and we set $R_n = R_{n,n}$ (See Figure 2).

Received January 5, 2009; Revised April 22, 2009; Accepted April 28, 2009.

2000 *Mathematics Subject Classification*. 05C40.

Key words and phrases. Steinhaus graph, bipartite graph, generating string, partner, doubly symmetric, Pascal's rectangle.

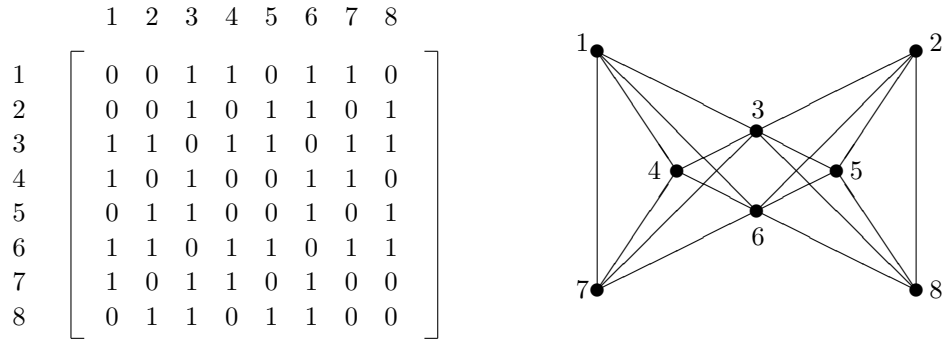


FIGURE 1. Steinhaus graph with the generating string 00110110

$$\begin{aligned}
 R_{1,8} &\rightarrow 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \\
 R_{2,8} &\rightarrow 1\ 1\ 0\ 0\ 0\ 0\ 0\ 0 \\
 R_{3,8} &\rightarrow 1\ 0\ 1\ 0\ 0\ 0\ 0\ 0 \\
 R_{4,8} &\rightarrow 1\ 1\ 1\ 1\ 0\ 0\ 0\ 0 \\
 R_{5,8} &\rightarrow 1\ 0\ 0\ 0\ 1\ 0\ 0\ 0 \\
 R_{6,8} &\rightarrow 1\ 1\ 0\ 0\ 1\ 1\ 0\ 0 \\
 R_{7,8} &\rightarrow 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0 \\
 R_{8,8} &\rightarrow 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1
 \end{aligned}$$

FIGURE 2. Pascal’s rectangle of length 8

In this paper, $\lfloor x \rfloor$ is the floor of x and $\lceil x \rceil$ is the ceiling of x . We denote $\log_2(x)$ by $\lg(x)$. Also, if k is a positive integer, then let $K = 2^{\lceil \lg(k) \rceil}$ and $T = R_{K-k+1, K}$. If T is a string of zeros and ones, then T^k is the string T concatenated with itself $k - 1$ times. For example, if $T = 01$, then $T^4 = 01010101$. In [4], the generating strings for bipartite Steinhaus graphs were described as follows.

Theorem 1.1 ([4]). *A Steinhaus graph is bipartite if and only if its generating string is a prefix of either $0^k T^{i2^m} 0^{K2^m}$ or $0^k T^{2^j} 0^m$ for each positive integer k , odd positive integer i larger than 1, non-negative integers j, m .*

Theorem 1.2 ([4]). *Let $b(n)$ be the number of bipartite Steinhaus graph with n vertices. The recurrence for $b(n)$ is given as follows: $b(2) = 2, b(3) = 4$, and for $k \geq 2$,*

$$\begin{aligned}
 b(2k + 1) &= 2b(k + 1) + 1 \\
 b(2k) &= b(k) + b(k + 1).
 \end{aligned}$$

In [8], [9] and [3], connectivity of Steinhaus graphs were investigated. In fact, they studied connection between minimum degree of Steinhaus graphs and connectivity of Steinhaus graphs which is either 2 or 3.

Theorem 1.3 ([8]). *Let $n > 5$ and let G be a nonempty Steinhaus graph of order n . Then the following statements are equivalent.*

- (1) G is 2-connected.
- (2) G is 2-edge-connected.
- (3) its minimum degree $\delta(G)$ is larger than one.

Kim and Lim in [9] showed that the number of Steinhaus graphs of order n having pendent vertices (vertices of degree one), $p(n)$, is given by

$$p(n) = 2 \sum_{i=1}^{n-1} \delta_i - \sum_{j=2}^{\lfloor \frac{n+2}{2} \rfloor} \epsilon_j,$$

where $\delta_i = \min\{2^m, n - i\}$ for the nonnegative integer m such that $2^{m-1} < i \leq 2^m$ and where

$$\epsilon_j = \begin{cases} 1 & \text{if } 2^{\lceil \lg(j-1) \rceil} \text{ divides } n - j + 1; \\ 0 & \text{otherwise.} \end{cases}$$

2. Connectivity two in bipartite Steinhaus graphs

From now on, assume that G is a bipartite Steinhaus graph with generating string $a_{11} \cdots a_{1n}$. So, $a_{11} \cdots a_{1n}$ is a prefix of either $0^k T^i 2^m 0^k 2^m$ or $0^k T^j 0^m$ for each positive integer k , odd positive integer i larger than 1, non-negative integers j, m . Note that k is the largest number among l 's that the submatrix (a_{ij}) for $1 \leq i, j \leq l$ in $A(G)$ is a zero matrix.

Since the partner, $P(G)$, of G is isomorphic to G , $P(G)$ is also bipartite. By Steinhaus property, we have the following two general generating strings of vertex $k, k + 1$.

If $a_{11} a_{12} \cdots a_{1n}$ is $0^k T^j 0^m$ for some $j \geq 1, m > 0$, then the general generating strings of vertices k and $k + 1$ are $0^k 1^j \bar{k} 0^m, 1^k 0^j \bar{k} 1 0^{m-1}$ respectively.

If $a_{11} a_{12} \cdots a_{1n}$ is a prefix of $0^k T^j$ for some $j \geq 1$, then the general generating strings of vertices k and $k + 1$ are $0^k 1^{n-k}, 1^k 0^{n-k}$ respectively.

By Theorem 1.3, to count the number of 2-connected bipartite Steinhaus graphs, it suffices to find the number of bipartite Steinhaus graphs with minimum degree one. Let $n = 2^r + s$ where $0 \leq s < 2^r$. Now we divide all generating strings into two types, $0^k T^j 0^m$ for some $j \geq 1, m > 0$ and $0^k T^j$ for some $j \geq 1$.

Lemma 2.1. *Let $a_{11} a_{12} \cdots a_{1n}$ be $0^k T^j 0^m$ for some $j \geq 1, m > 0$. Then pendent vertices of G may be either 1 or n . Moreover, G has a pendent vertex 1 or n if and only if at least one of generating strings of $G, P(G)$ satisfies that k is a power of 2 and $j = 1$.*

Proof. Suppose that G has a pendent vertex $1 < p < n$. Let q be adjacent to p . First, assume $2 \leq p \leq n$. If $q < p$, from the general generating string of q , $a_{q,q+1}a_{q,q+2} \cdots a_{q,n}$ is 1^{n-q} . So, $a_{q+1,q+2}a_{q+1,q+3} \cdots a_{q+1,n}$ is 0^{n-q-1} . This gives a contradiction by the structure of $A(G)$. If $q > p$, then $a_{11} \cdots a_{1,q-1}$ is 0^{q-1} and $a_q=1$. So, $k = q$. From the general generating string of k , $k + 1 \leq p \leq k + jK$. Since the string $a_{k+1,k+1+jK} a_{k+2,k+1+jK} \cdots a_{k+jK,k+1+jK}$ is 1^{jK} , $a_{k,p} = a_{l,k+1+jK} = 1$. Thus p is not a pendent vertex, which gives a contradiction. Thus pendent vertices of G may be either 1 or n . Without loss of generality, we assume that G has the pendent vertex 1 by considering its partner $P(G)$. Then j should be 1 and $T = R_{1,K}$. This gives that $k = K$ is a power of 2 because $K - k + 1 = 1$. The converse is clear because $k = K$ and $T = R_{1,k}$. \square

Lemma 2.2. *Let $n = 2^r + s$, where $0 \leq s < 2^r$. Let $1 \leq k \leq \lceil \frac{n}{2} \rceil$. Let $a_{11}a_{12} \cdots a_{1n}$ be a prefix of $0^k T^j$ for some $j \geq 1$. Then $\delta(G) = 1$ if and only if $k + K \leq n$. Moreover, $k + K$ is the smallest pendent vertex in G .*

Proof. Suppose that G has a pendent vertex. Since $k + 2^{r-1} \leq \lceil \frac{n}{2} \rceil + 2^{r-1} \leq n$, vertex i is of degree at least 2 by examining the general generating string of vertex i for $1 \leq i \leq k$. Let $l \geq k + 1$ be the smallest pendent vertex in G . Since $a_{k,k+1}a_{k,k+2} \cdots a_{k,n}$ in the general generating string of k is 1^{n-k} , $a_{k+1,k+2}a_{k+1,k+3} \cdots a_{k+1,n}$ is 0^{n-k-1} . So the general generating string of $k + K$ is $0^{k-1}10^{n-k}$. Since $a_{k,l} = 1$, the general generating string of l , $a_{1,l}a_{2,l} \cdots a_{l,l}a_{l,l+1} \cdots a_{l,n}$ is $0^{k-1}10^{n-k}$. Also, it is clear that none of the vertices $k + 1, k + 2, \dots, k + K$ are of degree one. So, $k + K = l \leq n$. The converse is clear because the vertex $k + K$ is a pendent vertex. \square

Let $a_{11}a_{12} \cdots a_{1n}$ be a prefix of $0^k T^j$ for $\lceil \frac{n}{2} \rceil < k < n$, $j \geq 1$. Then the generating string of its partner has at most $\lceil \frac{n}{2} \rceil$ leading zeroes.

Theorem 2.3. *G have pendent vertices if and only if either G or the partner $P(G)$ has a generating string described in Lemmas 2.1 and 2.2.*

Finally, we want to count the number of bipartite Steinhaus graphs with a pendent vertex from Theorem 2.3.

(a): G have generating strings described in Lemma 2.1.

The number of k satisfying the condition in Lemma 2.1 is given

$$\begin{cases} r - 2 & \text{if } s = 0; \\ r - 1 & \text{if } 0 < s < 2^r. \end{cases}$$

Let $n = 2^j l$, where l is odd. If G in Lemma 2.1, is doubly symmetric, then the generating string $a_{n,n}a_{n,n-1} \cdots a_{n,1}$ of $P(G)$ is $a_{11}a_{12} \cdots a_{1n}$, which is $0^k T^m$, k is a power of 2. This gives that k is a divisor of 2^j . Therefore, the number of graphs that either G or $P(G)$ have the

generating strings in described in Lemma 2.1 is

$$\begin{cases} 2(r-2) - (j+1) = 2r - j - 5 & \text{if } s = 0; \\ 2(r-1) - (j+1) = 2r - j - 3 & \text{if } 0 < s < 2^r. \end{cases}$$

(b): Either G or $P(G)$ have generating strings described in Lemma 2.2.

First, let us count the number of bipartite Steinhaus graphs G that G have generating string in Lemma 2.2. So, we need to find all k 's satisfying $k + K \leq n$. For $1 \leq k \leq 2^{r-1}$, k satisfies the inequality. For $2^{r-1} < k \leq \frac{n}{2}$, $K = 2^r$. So, $k + K = k + 2^r \leq n = 2^r + s$. i.e. $k \leq s$. Since $2^{r-1} \leq k$, $2^{r-1} \leq s$. Therefore, only if $2^{r-1} \leq s$, then k satisfies the inequality $k + K$ for $2^{r-1} < k \leq \frac{n}{2}$. Next, the number of bipartite Steinhaus graphs $P(G)$ that $P(G)$ have generating string in Lemma 2.2 is exactly same as previous case except few graphs. In fact, if n is even, the graph with $\frac{n}{2}$ leading zeroes is doubly symmetric. If n is odd, the graph with $\lceil \frac{n}{2} \rceil$ leading zeroes is the partner of the graph with $\lceil \frac{n}{2} \rceil - 1$ leading zeroes. Note that when $k = n$, G is the trivial graph. Combining above argument, we have the following.

$$\begin{cases} 2^r - 1 & \text{if } s = 0; \\ 2^r & \text{if } 0 < s < 2^{r-1}; \\ 2s & \text{if } 2^{r-1} \leq s < 2^r. \end{cases}$$

Let $b_1(n)$ be the number of bipartite Steinhaus graphs with a pendent vertex. Note that the family of generating strings in (a) and the family of generating strings in (b) are disjoint. By combining (a) and (b), we get to the following Theorem.

Theorem 2.4. For $n = 2^r + s \geq 4$ where $0 \leq s < 2^r$ and $n = 2^j l$ where l is odd. Then the number of $b_1(n)$ is given by

$$\begin{cases} 2^r + r - 6 & \text{if } s = 0; \\ 2^r + 2r - j - 3 & \text{if } 0 < s < 2^{r-1}; \\ 2s + 2r - j - 3 & \text{if } 2^{r-1} \leq s < 2^r. \end{cases}$$

Therefore, the number of 2-(edge)connected bipartite Steinhaus graphs is $b(n) - b_1(n)$.

Theorem 2.5. For $n = 2^r + s \geq 4$ where $0 \leq s < 2^r$ and $n = 2^j l$ where l is odd. Then the number of $b_1(n)$ is given by

$$\begin{cases} 2^r + r - 6 & \text{if } s = 0; \\ 2^r + 2r - j - 3 & \text{if } 0 < s < 2^{r-1}; \\ 2s + 2r - j - 3 & \text{if } 2^{r-1} \leq s < 2^r. \end{cases}$$

Therefore, the number of 2-(edge)connected bipartite Steinhaus graphs is $b(n) - pb(n)$.

References

- [1] B. Bollobas, *Graph Theory*, Springer-Verlag, New York, 1979.
- [2] W. M. Dymacek, *Bipartite Steinhaus graphs*, Discrete Mathematics, **59**(1986), 9–22.
- [3] W. M. Dymacek, *Connectivity in Steinhaus graphs*, in preperation.
- [4] W. M. Dymacek and T. Whaley, *Generating strings for bipartite Steinhaus graphs*, Discrete Mathematics, **141**(1995), no.1–3, 95–107.
- [5] W. M. Dymacek, M. Koerlin and T. Whaley, *A survey of Steinhaus graphs*, Proceedings of the Eighth Quadrennial Intrrnational Conference on Graph Theory, Combinatorics, Algorithm and Applications, 313–323, **1**(1998).
- [6] G. J. Chang, B. DasGupta, W. M. Dymacek, M. Furer, M. Koerlin, Y. Lee and T. Whaley, *Characterizations of bipartite Steinhaus graphs*, Discrete Mathematics, **199**(1999), 11–25.
- [7] H. Harborth, *Solution of Steinhaus’s problem with plus and minus signs*, J. Combinatorial Theory Ser. A, **12**(1972), 253–259.
- [8] D. J. Kim and D. K. Lim, *2-connected and 2-edge-connected Steinhaus graphs*, Discrete Math., **256**(2002), no.1–2, 257–265.
- [9] D. J. Kim and D. K. Lim, *Steinhaus Graphs with Minimum Degree Two*, Kyungpook Mathematical Journal, **43**(2003), no.4, 567–577.
- [10] H. Steinhaus, *One Hundred Problems in Elementary Mathematics*, Dover, New York, 1979.

DEPARTMENT OF MATHEMATHICS
KEIMYUNG UNIVERSITY
DAEGU 701-704, KOREA
E-mail address: e-mail address: limd@kmu.ac.kr