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FIXED POINT THEOREM IN $\mathcal{L}^*_{\mathcal{M}}$-FUZZY METRIC SPACES FOR TWO MAPS

Jong Kyu Kim, S. Sedghi and N. Shobe

Abstract. In this paper, we give some new definitions of $\mathcal{L}^*_{\mathcal{M}}$-fuzzy metric spaces and we prove a common fixed point theorem for two mappings in complete $\mathcal{L}^*_{\mathcal{M}}$-fuzzy metric spaces. We get some improved versions of several fixed point theorems in complete $\mathcal{L}^*_{\mathcal{M}}$-fuzzy metric spaces.

1. Introduction and preliminaries

The concept of fuzzy sets was introduced initially by Zadeh [23] in 1965. Since then, to use this concept in topology and analysis many authors have expansively developed the theory of fuzzy sets and application. George and Veeramani [11] and Kramosil and Michalek [14] have introduced the concept of fuzzy topological spaces induced by fuzzy metric which have very important applications in quantum particle physics particularly in connections with both string and E-infinity theory which were given and studied by El Naschie [6, 7, 8, 10, 20]. Many authors [13, 16, 18] have proved fixed point theorem in fuzzy (probabilistic) metric spaces. Vasuki [21] obtained the fuzzy version of common fixed point theorem which had extra conditions. In fact, Vasuki proved fuzzy common fixed point theorem by a strong definition of Cauchy sequence (see Note 3.13 and Definition 3.15 of [11] also [19, 22]).

In this paper, we prove a common fixed point theorem in fuzzy metric spaces for arbitrary t-norms and modified definition of Cauchy sequence in George and Veeramani's sense. There have been a number of generalizations of metric spaces. One such generalization is generalized metric space or D-metric space initiated by Dhage [9] in 1992. He proved some results on fixed points for a self-map satisfying a contraction for complete and bounded D-metric spaces. Rhoades [15] generalized Dhage's contractive condition by increasing the number of factors and proved the existence of unique fixed point of a self-map in D-metric space. Recently, motivated by the concept of compatibility for metric space, Singh and Sharma [17] introduced the concept of D-compatibility of

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The corresponding author: jongkyuk@kyungnam.ac.kr.

maps in D-metric space and proved some fixed point theorems using a contractive condition.

In what follows $\mathbb{N}$ the set of all natural numbers, and $\mathbb{R}^+$ the set of all positive real numbers.

Definition 1.1. Let X be a nonempty set. A generalized metric (or D-metric) on X is a function: $D : X^3 \longrightarrow \mathbb{R}^+$ that satisfies the following conditions for each $x, y, z, a \in X$.

(1) $D(x, y, z) \geq 0$,
(2) $D(x, y, z) = 0$ if and only if $x = y = z$,
(3) $D(x, y, z) = D(p\{x, y, z\})$ (symmetry), where p is a permutation function,
(4) $D(x, y, z) \leq D(x, y, a) + D(a, z, z)$.

The pair (X, D) is called a generalized metric (or D-metric) space.

It is easy to show that the following function D are D-metric.

(a) $D(x, y, z) = \max\{d(x, y), d(y, z), d(z, x)\}$.
(b) D(x,y,z)=d(x,y)+d(y,z)+d(z,x),
where d is the ordinary metric on X.
(c) If $X = \mathbb{R}^n$ then we define

$$D(x, y, z) = (||x - y||^p + ||y - z||^p + ||z - x||^p)^{\frac{1}{p}}$$

for every $p \in \mathbb{R}^+$.
(d) If $X = \mathbb{R}^+$ then we define

$$D(x, y, z) = \begin{cases} 0 & \text{if } x = y = z, \\ \max\{x, y, z\} & \text{otherwise.} \end{cases}$$

Remark 1.2. Let D be a D-metric. Then we have $D(x, x, y) = D(x, y, y)$. Since

(i) $D(x, x, y) \leq D(x, x, x) + D(x, y, y) = D(x, y, y)$

and

(ii) $D(y, y, x) \leq D(y, y, y) + D(y, x, x) = D(y, x, x)$,

we get $D(x, x, y) = D(x, y, y)$,

Let (X, D) be a D-metric space. For $r > 0$ define

$$B_D(x, r) = \{y \in X : D(x, y, y) < r\}.$$

Example 1.3. Let $X = \mathbb{R}$ and $D(x, y, z) = |x - y| + |y - z| + |z - x|$ for all $x, y, z \in \mathbb{R}$. Then

$$\begin{aligned} B_D(1, 2) = \{y \in \mathbb{R} : D(1, y, y) < 2\} &= \{y \in \mathbb{R} : |y - 1| + |y - 1| < 2\} \\ &= \{y \in \mathbb{R} : |y - 1| < 1\} = (0, 2). \end{aligned}$$

Definition 1.4. Let (X, D) be a D-metric space and $A \subset X$.

(1) If for every $x \in A$ there exist $r > 0$ such that $B_D(x, r) \subset A$, then A is called open subset of X.

(2) A is said to be D-bounded if there exists $r > 0$ such that $D(x, y, y) < r$ for all $x, y \in A$.

(3) A sequence $\{x_n\}$ in X converges to x if and only if $D(x_n, x_n, x) = D(x, x, x_n) \to 0$ as $n \to \infty$. That is for each $\epsilon > 0$ there exist $n_0 \in \mathbb{N}$ such that

$$\forall n \geq n_0 \Longrightarrow D(x, x, x_n) < \epsilon. \qquad (*)$$

This is equivalent with, for each $\epsilon > 0$ there exist $n_0 \in \mathbb{N}$ such that

$$\forall n, m \geq n_0 \Longrightarrow D(x, x_n, x_m) < \epsilon. \qquad (**)$$

Suppose that (*) holds. Then

$$D(x_n, x_m, x) = D(x_n, x, x_m) \leq D(x_n, x, x) + D(x, x_m, x_m) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \varepsilon.$$

Conversely, set $m = n$ in $(**)$ we have $D(x_n, x_n, x) < \epsilon$.

(4) Sequence $\{x_n\}$ in X is called a Cauchy sequence if for each $\epsilon > 0$, there exits $n_0 \in \mathbb{N}$ such that $D(x_n, x_n, x_m) < \epsilon$ for each $n, m \geq n_0$. The D-metric space (X, D) is said to be complete if every Cauchy sequence in X is convergent.

Let τ be the set of all open subsets of X. Then τ is a topology on X.

Lemma 1.5. *Let (X, D) be a D-metric space. If $r > 0$, then ball $B_D(x, r)$ with center $x \in X$ and radius r is open.*

Proof. Let $z \in B_D(x, r)$. Then $D(x, z, z) < r$. If set $D(x, z, z) = \delta$ and $r' = r - \delta$ then we prove that $B_D(z, r') \subseteq B_D(x, r)$. Let $y \in B_D(z, r')$. Then, by triangular inequality we have $D(x, y, y) = D(y, y, x) \leq D(y, y, z) + D(z, x, x) < r' + \delta = r$. Hence $B_D(z, r') \subseteq B_D(x, r)$. That is ball $B_D(x, r)$ is open. □

Lemma 1.6. *Let (X, D) be a D-metric space. If the sequence $\{x_n\}$ in X converges to x, then it is unique.*

Proof. Let $x_n \longrightarrow y$ and $y \neq x$. Since $\{x_n\}$ converges to x and y, for each $\epsilon > 0$ there exist

$$n_1 \in \mathbb{N} \text{ such that for every } n \geq n_1 \Longrightarrow D(x, x, x_n) < \frac{\epsilon}{2}$$

and

$$n_2 \in \mathbb{N} \text{ such that for every } n \geq n_2 \Longrightarrow D(y, y, x_n) < \frac{\epsilon}{2}.$$

If set $n_0 = \max\{n_1, n_2\}$, then for every $n \geq n_0$ by triangular inequality we have

$$D(x, x, y) \leq D(x, x, x_n) + D(x_n, y, y) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \varepsilon.$$

Hence $D(x, x, y) = 0$ is a contradiction. So, $x = y$. □

Lemma 1.7. *Let (X, D) be a D-metric space. If the sequence $\{x_n\}$ in X is convergent to x, then it is a Cauchy sequence.*

Proof. Since $x_n \longrightarrow x$ for each $\epsilon > 0$ there exists

$$n_1 \in \mathbb{N} \text{ such that for every } n \geq n_1 \Longrightarrow D(x_n, x_n, x) < \frac{\epsilon}{2}$$

and

$$n_2 \in \mathbb{N} \text{ such that for every } m \geq n_2 \Longrightarrow D(x, x_m, x_m) < \frac{\epsilon}{2}.$$

If set $n_0 = \max\{n_1, n_2\}$, then for every $n, m \geq n_0$ by triangular inequality we have

$$D(x_n, x_n, x_m) \leq D(x_n, x_n, x) + D(x, x_m, x_m) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence sequence $\{x_n\}$ is a Cauchy sequence. □

Definition 1.8. A 3-tuple $(X, \mathcal{M}, *)$ is called a $\mathcal{M}$-fuzzy metric space if X is an arbitrary (non-empty) set, $*$ is a continuous t-norm, and $\mathcal{M}$ is a fuzzy set on $X^3 \times (0, \infty)$, satisfying the following conditions for each $x, y, z, a \in X$ and $t, s > 0$,

(1) $\mathcal{M}(x, y, z, t) > 0$,
(2) $\mathcal{M}(x, y, z, t) = 1$ if and only if $x = y = z$,
(3) $\mathcal{M}(x, y, z, t) = \mathcal{M}(p\{x, y, z\}, t)$ (symmetry), where p is a permutation function,
(4) $\mathcal{M}(x, y, a, t) * \mathcal{M}(a, z, z, s) \leq \mathcal{M}(x, y, z, t + s)$,
(5) $\mathcal{M}(x, y, z, \cdot) : (0, \infty) \longrightarrow [0, 1]$ is continuous.

Remark 1.9. Let $(X, \mathcal{M}, *)$ be a $\mathcal{M}$-fuzzy metric space. We prove that for every $t > 0$, $\mathcal{M}(x, x, y, t) = \mathcal{M}(x, y, y, t)$. Because for each $\epsilon > 0$ by triangular inequality we have

(i) $\mathcal{M}(x, x, y, \epsilon + t) \geq \mathcal{M}(x, x, x, \epsilon) * \mathcal{M}(x, y, y, t) = \mathcal{M}(x, y, y, t)$,
(ii) $\mathcal{M}(y, y, x, \epsilon + t) \geq \mathcal{M}(y, y, y, \epsilon) * \mathcal{M}(y, x, x, t) = \mathcal{M}(y, x, x, t)$.

By taking limits of (i) and (ii) when $\epsilon \longrightarrow 0$, we obtain $\mathcal{M}(x, x, y, t) = \mathcal{M}(x, y, y, t)$.

Let $(X, \mathcal{M}, *)$ be a $\mathcal{M}$-fuzzy metric space. For $t > 0$, the open ball $B_{\mathcal{M}}(x, r, t)$ with center $x \in X$ and radius $0 < r < 1$ is defined by

$$B_{\mathcal{M}}(x, r, t) = \{y \in X : \mathcal{M}(x, y, y, t) > 1 - r\}.$$

A subset A of X is called open set if for each $x \in A$ there exist $t > 0$ and $0 < r < 1$ such that $B_{\mathcal{M}}(x, r, t) \subseteq A$.

A sequence $\{x_n\}$ in X converges to x if and only if $\mathcal{M}(x, x, x_n, t) \longrightarrow 1$ as $n \longrightarrow \infty$, for each $t > 0$. It is called a Cauchy sequence if for each $0 < \epsilon < 1$ and $t > 0$, there exist $n_0 \in \mathbb{N}$ such that $\mathcal{M}(x_n, x_n, x_m, t) > 1 - \epsilon$ for each $n, m \geq n_0$.

The $\mathcal{M}$-fuzzy metric $(X, \mathcal{M}, *)$ is said to be complete if every Cauchy sequence is convergent.

Example 1.10. Let X is a nonempty set and D is the D-metric on X. Denote $a * b = a \cdot b$ for all $a, b \in [0,1]$. For each $t \in]0, \infty[$, define

$$\mathcal{M}(x, y, z, t) = \frac{t}{t + D(x, y, z)}$$

for all $x, y, z \in X$. It is easy to see that $(X, \mathcal{M}, *)$ is a $\mathcal{M}$-fuzzy metric space.

Lemma 1.11. *Let $(X, M, *)$ is a fuzzy metric space. If we define $\mathcal{M} : X^3 \times (0, \infty) \longrightarrow [0,1]$ by*

$$\mathcal{M}(x, y, z, t) = M(x, y, t) * M(y, z, t) * M(z, x, t)$$

for every x, y, z in X, then $(X, \mathcal{M}, *)$ is a $\mathcal{M}$-fuzzy metric space.

Proof. (1) It is easy to see that for every $x, y, z \in X$, $\mathcal{M}(x, y, z, t) > 0$, $\forall t > 0$.

(2) $\mathcal{M}(x, y, z, t) = 1$ if and only if $M(x, y, t) = M(y, z, t) = M(z, x, t) = 1$ if and only if $x = y = z$.

(3) $\mathcal{M}(x, y, z, t) = \mathcal{M}(p\{x, y, z\}, t)$, where p is a permutation function.

$$\begin{aligned}(4)\ \mathcal{M}(x, y, z, t+s) &= M(x, y, t+s) * M(y, z, t+s) * M(z, x, t+s) \\ &\geq M(x, y, t) * M(y, a, t) * M(a, z, s) * M(z, a, s) * M(a, x, t) \\ &= \mathcal{M}(x, y, a, t) * M(a, z, s) * M(z, a, s) * M(z, z, s) \\ &= \mathcal{M}(x, y, a, t) * \mathcal{M}(a, z, z, s).\end{aligned}$$

□

Lemma 1.12. *Let $(X, \mathcal{M}, *)$ be a $\mathcal{M}$-fuzzy metric space. Then $\mathcal{M}(x, y, z, t)$ is nondecreasing with respect to t, for all x, y, z in X.*

Proof. By Definition 1.8(4), for each $x, y, z, a \in X$ and $t, s > 0$ we have $\mathcal{M}(x, y, a, t) * \mathcal{M}(a, z, z, s) \leq \mathcal{M}(x, y, z, t+s)$. If set $a = z$ we get $\mathcal{M}(x, y, z, t) * \mathcal{M}(z, z, z, s) \leq \mathcal{M}(x, y, z, t+s)$, that is $\mathcal{M}(x, y, z, t+s) \geq \mathcal{M}(x, y, z, t)$. □

Definition 1.13. Let $(X, \mathcal{M}, *)$ be a $\mathcal{M}$-fuzzy metric space. $\mathcal{M}$ is said to be continuous function on $X^3 \times (0, \infty)$ if

$$\lim_{n \to \infty} \mathcal{M}(x_n, y_n, z_n, t_n) = \mathcal{M}(x, y, z, t),$$

for a sequence $\{(x_n, y_n, z_n, t_n)\}$ in $X^3 \times (0, \infty)$ converges to a point $(x, y, z, t) \in X^3 \times (0, \infty)$, i.e.,

$$\lim_{n \to \infty} x_n = x,\ \lim_{n \to \infty} y_n = y,\ \lim_{n \to \infty} z_n = z \text{ and } \lim_{n \to \infty} \mathcal{M}(x, y, z, t_n) = \mathcal{M}(x, y, z, t).$$

Lemma 1.14. *Let $(X, \mathcal{M}, *)$ be a $\mathcal{M}$-fuzzy metric space. Then $\mathcal{M}$ is continuous function on $X^3 \times (0, \infty)$.*

Proof. Let $x, y, z \in X$ and $t > 0$, and let $\{(x'_n, y'_n, z'_n, t'_n)\}$ be a sequence in $X^3 \times (0, \infty)$ that converges to (x, y, z, t). Since $\{(\mathcal{M}(x'_n, y'_n, z'_n, t'_n))\}$ is a sequence in $(0, 1]$, there is a subsequence $\{(x_n, y_n, z_n, t_n)\}$ of sequence $\{(x'_n, y'_n, z'_n, t'_n)\}$ such that sequence $\{(\mathcal{M}(x_n, y_n, z_n, t_n))\}$ converges to some point of $[0, 1]$.

Fix $\delta > 0$ such that $\delta < \frac{t}{2}$. Then, there is $n_0 \in \mathbb{N}$ such that $|t - t_n| < \delta$ for every $n \geq n_0$. Hence,

$$\begin{aligned}
&\mathcal{M}(x_n, y_n, z_n, t_n)\\
&\geq \mathcal{M}(x_n, y_n, z_n, t - \delta)\\
&\geq \mathcal{M}(x_n, y_n, z, t - \frac{4\delta}{3}) * \mathcal{M}(z, z_n, z_n, \frac{\delta}{3})\\
&\geq \mathcal{M}(x_n, z, y, t - \frac{5\delta}{3}) * \mathcal{M}(y, y_n, y_n, \frac{\delta}{3}) * \mathcal{M}(z, z_n, z_n, \frac{\delta}{3})\\
&\geq \mathcal{M}(z, y, x, t - 2\delta) * \mathcal{M}(x, x_n, x_n, \frac{\delta}{3}) * \mathcal{M}(y, y_n, y_n, \frac{\delta}{3}) * \mathcal{M}(z, z_n, z_n, \frac{\delta}{3})
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{M}(x, y, z, t + 2\delta)\\
&\geq \mathcal{M}(x, y, z, t_n + \delta)\\
&\geq \mathcal{M}(x, y, z_n, t_n + \frac{2\delta}{3}) * \mathcal{M}(z_n, z, z, \frac{\delta}{3})\\
&\geq \mathcal{M}(x, z_n, y_n, t_n + \frac{\delta}{3}) * \mathcal{M}(y_n, y, y, \frac{\delta}{3}) * \mathcal{M}(z_n, z, z, \frac{\delta}{3})\\
&\geq \mathcal{M}(z_n, y_n, x_n, t_n) * \mathcal{M}(x_n, x, x, \frac{\delta}{3}) * \mathcal{M}(y_n, y, y, \frac{\delta}{3}) * \mathcal{M}(z_n, z, z, \frac{\delta}{3}),
\end{aligned}$$

for all $n \geq n_0$. By taking limits when $n \longrightarrow \infty$, we obtain

$$\lim_{n \to \infty} \mathcal{M}(x_n, y_n, z_n, t_n) \geq \mathcal{M}(x, y, z, t - 2\delta) * 1 * 1 * 1 = \mathcal{M}(x, y, z, t - 2\delta)$$

and

$$\mathcal{M}(x, y, z, t + 2\delta) \geq \lim_{n \to \infty} \mathcal{M}(x_n, y_n, z_n, t_n) 1 * 1 * 1 = \lim_{n \to \infty} \mathcal{M}(x_n, y_n, z_n, t_n),$$

respectively. So, by continuity of the function $t \longmapsto \mathcal{M}(x, y, z, t)$, we immediately deduce that

$$\lim_{n \to \infty} \mathcal{M}(x_n, y_n, z_n, t_n) = \mathcal{M}(x, y, z, t).$$

Therefore $\mathcal{M}$ is continuous on $X^3 \times (0, \infty)$. □

Lemma 1.15. *Let $(X, \mathcal{M}, *)$ be a $\mathcal{M}$-fuzzy metric space. If we define $E_{\lambda,\mathcal{M}} : X^3 \to \mathbb{R}^+ \cup \{0\}$ by*

$$E_{\lambda,\mathcal{M}}(x, y, z) = \inf\{t > 0 \ : \ \mathcal{M}(x, y, z, t) > 1 - \lambda\}$$

for every $\lambda \in (0, 1)$, then

(i) *for each $\mu \in (0, 1)$ there exists $\lambda \in (0, 1)$ such that*

$$\begin{aligned}E_{\mu,\mathcal{M}}(x_1, x_1, x_n) \leq\ & E_{\lambda,\mathcal{M}}(x_1, x_1, x_2) + E_{\lambda,\mathcal{M}}(x_2, x_2, x_3) + \cdots\\ &+ E_{\lambda,\mathcal{M}}(x_{n-1}, x_{n-1}, x_n)\end{aligned}$$

for any $x_1, x_2, \cdots, x_n \in X$.

(ii) *The sequence* $\{x_n\}$ *is convergent in* $\mathcal{M}$*-fuzzy metric space* $(X, \mathcal{M}, *)$ *if and only if* $E_{\lambda,\mathcal{M}}(x_n, x_n, x) \to 0$*. Also the sequence* $\{x_n\}$ *is Cauchy sequence if and only if it is Cauchy with* $E_{\lambda,\mathcal{M}}$*.*

Proof. (i) For every $\mu \in (0,1)$, we can find a $\lambda \in (0,1)$ such that

$$\overbrace{(1-\lambda) * (1-\lambda) * \cdots * (1-\lambda)}^{n} \geq 1-\mu.$$

By triangular inequality, we have

$$\begin{aligned}
&\mathcal{M}(x_1, x_1, x_n, E_{\lambda,\mathcal{M}}(x_1, x_1, x_2) + E_{\lambda,\mathcal{M}}(x_2, x_2, x_3) + \cdots \\
&\quad + E_{\lambda,\mathcal{M}}(x_{n-1}, x_{n-1}, x_n) + n\delta) \\
&\geq \mathcal{M}(x_1, x_1, x_2, E_{\lambda,\mathcal{M}}(x_1, x_1, x_2) + \delta) * \cdots \\
&\quad * \mathcal{M}(x_{n-1}, x_{n-1}, x_n, E_{\lambda,\mathcal{M}}(x_{n-1}, x_{n-1}, x_n) + \delta) \\
&\geq \overbrace{(1-\lambda) * (1-\lambda) * \cdots * (1-\lambda)}^{n} \\
&\geq 1-\mu,
\end{aligned}$$

for very $\delta > 0$, which implies that

$$\begin{aligned}
E_{\mu,\mathcal{M}}(x_1, x_1, x_n) \leq{}& E_{\lambda,\mathcal{M}}(x_1, x_1, x_2) + E_{\lambda,\mathcal{M}}(x_2, x_2, x_3) + \cdots \\
&+ E_{\lambda,\mathcal{M}}(x_{n-1}, x_{n-1}, x_n) + n\delta.
\end{aligned}$$

Since $\delta > 0$ is arbitrary, we have

$$\begin{aligned}
E_{\mu,\mathcal{M}}(x_1, x_1, x_n) \leq{}& E_{\lambda,\mathcal{M}}(x_1, x_1, x_2) + E_{\lambda,\mathcal{M}}(x_2, x_2, x_3) + \cdots \\
&+ E_{\lambda,\mathcal{M}}(x_{n-1}, x_{n-1}, x_n).
\end{aligned}$$

(ii) Note that since $\mathcal{M}$ is continuous in its third item and

$$E_{\lambda,\mathcal{M}}(x, x, y) = \inf\{t > 0 \ : \ \mathcal{M}(x, x, y, t) > 1-\lambda\}.$$

Hence, we have

$$\mathcal{M}(x_n, x, x, \eta) > 1-\lambda \Longleftrightarrow E_{\lambda,\mathcal{M}}(x_n, x, x) < \eta$$

for every $\eta > 0$. □

Lemma 1.16. *Let* $(X, \mathcal{M}, *)$ *be a* $\mathcal{M}$*-fuzzy metric space. If*

$$\mathcal{M}(x_n, x_n, x_{n+1}, t) \geq \mathcal{M}(x_0, x_0, x_1, k^n t)$$

for some $k > 1$ *and for every* $n \in \mathbb{N}$*. Then sequence* $\{x_n\}$ *is Cauchy.*

Proof. For every $\lambda \in (0,1)$ and $x_n, x_{n+1} \in X$, we have

$$\begin{aligned} E_{\lambda,\mathcal{M}}(x_n, x_n, x_{n+1}) &= \inf\{t > 0 \ : \ \mathcal{M}(x_n, x_n, x_{n+1}, t) > 1 - \lambda\} \\ &\leq \inf\{t > 0 \ : \ \mathcal{M}(x_0, x_0, x_1, k^n t) > 1 - \lambda\} \\ &= \inf\{\frac{t}{k^n} > 0 \ : \ \mathcal{M}(x_0, x_0, x_1, t) > 1 - \lambda\} \\ &= \frac{1}{k^n} \inf\{t > 0 \ : \ \mathcal{M}(x_0, x_0, x_1, t) > 1 - \lambda\} \\ &= \frac{1}{k^n} E_{\lambda,\mathcal{M}}(x_0, x_0, x_1). \end{aligned}$$

By Lemma 1.15, for every $\mu \in (0,1)$ there exists $\lambda \in (0,1)$ such that

$$\begin{aligned} & E_{\mu,\mathcal{M}}(x_n, x_n, x_m) \\ \leq \ & E_{\lambda,\mathcal{M}}(x_n, x_n, x_{n+1}) + E_{\lambda,\mathcal{M}}(x_{n+1}, x_{n+1}, x_{n+2}) + \cdots \\ & + E_{\lambda,\mathcal{M}}(x_{m-1}, x_{m-1}, x_m) \\ \leq \ & \frac{1}{k^n} E_{\lambda,\mathcal{M}}(x_0, x_0, x_1) + \frac{1}{k^{n+1}} E_{\lambda,\mathcal{M}}(x_0, x_0, x_1) + \cdots \\ & + \frac{1}{k^{m-1}} E_{\lambda,\mathcal{M}}(x_0, x_0, x_1) \\ = \ & E_{\lambda,\mathcal{M}}(x_0, x_0, x_1) \sum_{j=n}^{m-1} \frac{1}{k^j} \\ \longrightarrow \ & 0. \end{aligned}$$

Hence the sequence $\{x_n\}$ is Cauchy. □

Lemma 1.17. ([5]) *Consider the set L^* and operation $\leq_{L^*}$ defined by*

$$L^* = \{(x_1, x_2) : (x_1, x_2) \in [0,1]^2 \text{ and } x_1 + x_2 \leq 1\},$$

$(x_1, x_2) \leq_{L^*} (y_1, y_2) \Longleftrightarrow x_1 \leq y_1$ *and* $x_2 \geq y_2$, *for every* $(x_1, x_2), (y_1, y_2) \in L^*$. *Then* $(L^*, \leq_{L^*})$ *is a complete lattice.*

Definition 1.18. ([1]) An intuitionistic fuzzy set $\mathcal{A}_{\zeta,\eta}$ in a universe U is an object $\mathcal{A}_{\zeta,\eta} = \{(\zeta_{\mathcal{A}}(u), \eta_{\mathcal{A}}(u)) | u \in U\}$, where, for all $u \in U$, $\zeta_{\mathcal{A}}(u) \in [0,1]$ and $\eta_{\mathcal{A}}(u) \in [0,1]$ are called the membership degree and the non-membership degree, respectively, of u in $\mathcal{A}_{\zeta,\eta}$, and furthermore they satisfy $\zeta_{\mathcal{A}}(u) + \eta_{\mathcal{A}}(u) \leq 1$.

For every $z_i = (x_i, y_i) \in L^*$ if $c_i \in [0,1]$ such that $\sum_{j=1}^n c_j = 1$, then it is easy to show that

$$c_1(x_1, y_1) + \cdots + c_n(x_n, y_n) = \sum_{j=1}^n c_j(x_j, y_j) = (\sum_{j=1}^n c_j x_j, \sum_{j=1}^n c_j y_j) \in L^*. \quad (1.1)$$

We denote its units by $0_{L^*} = (0,1)$ and $1_{L^*} = (1,0)$. Classically, a triangular norm $* = T$ on $[0,1]$ is defined as an increasing, commutative, associative mapping $T : [0,1]^2 \longrightarrow [0,1]$ satisfying $T(1,x) = 1 * x = x$, for all $x \in [0,1]$. A triangular conorm $S = \diamond$ is defined as an increasing, commutative, associative mapping $S : [0,1]^2 \longrightarrow [0,1]$ satisfying $S(0,x) = 0 \diamond x = x$, for all $x \in [0,1]$. Using the lattice $(L^*, \leq_{L^*})$, these definitions can be straightforwardly extended.

Definition 1.19. ([3, 4]) A triangular norm (t–norm) on L^* is a mapping $\mathcal{T} : (L^*)^2 \longrightarrow L^*$ satisfying the following conditions:

(1) $(\forall x \in L^*)(\mathcal{T}(x, 1_{L^*}) = x)$ (boundary condition)

(2) $(\forall (x,y) \in (L^*)^2)(\mathcal{T}(x,y) = \mathcal{T}(y,x))$ (commutativity)

(3) $(\forall (x,y,z) \in (L^*)^3)(\mathcal{T}(x, \mathcal{T}(y,z)) = \mathcal{T}(\mathcal{T}(x,y), z))$ (associativity)

(4) $(\forall (x,x',y,y') \in (L^*)^4)(x \leq_{L^*} x'$ and $y \leq_{L^*} y' \Longrightarrow \mathcal{T}(x,y) \leq_{L^*} \mathcal{T}(x',y'))$ (monotonicity).

Definition 1.20. ([2]) A continuous t–norm $\mathcal{T}$ on L^* is called continuous *t–representable* if and only if there exist a continuous t–norm $*$ and a continuous t–conorm $\diamond$ on $[0,1]$ such that, for all $x = (x_1, x_2), y = (y_1, y_2) \in L^*$,

$$\mathcal{T}(x,y) = (x_1 * y_1, x_2 \diamond y_2).$$

Now define a sequence $\mathcal{T}^n$ recursively by $\mathcal{T}^1 = \mathcal{T}$ and

$$\mathcal{T}^n(x^{(1)}, \cdots, x^{(n+1)}) = \mathcal{T}(\mathcal{T}^{n-1}(x^{(1)}, \cdots, x^{(n)}), x^{(n+1)})$$

for $n \geq 2$ and $x^{(i)} \in L^*$.

Definition 1.21. ([3, 4]) A negator on L^* is any decreasing mapping $\mathcal{N} : L^* \longrightarrow L^*$ satisfying $\mathcal{N}(0_{L^*}) = 1_{L^*}$ and $\mathcal{N}(1_{L^*}) = 0_{L^*}$. If $\mathcal{N}(\mathcal{N}(x)) = x$, for all $x \in L^*$, then $\mathcal{N}$ is called an involutive negator. A negator on $[0,1]$ is a decreasing mapping $N : [0,1] \longrightarrow [0,1]$ satisfying $N(0) = 1$ and $N(1) = 0$. N_s denotes the standard negator on $[0,1]$ defined as, for all $x \in [0,1]$, $N_s(x) = 1-x$.

Definition 1.22. Let M, N are fuzzy sets from $X^3 \times (0, +\infty)$ to $[0,1]$ such that $M(x,y,z,t) + N(x,y,z,t) \leq 1$ for all $x,y,z \in X$ and $t > 0$. The 3-tuple $(X, \mathcal{M}_{M,N}, \mathcal{T})$ is said to be an *intuitionistic fuzzy metric space* if X is an arbitrary (non-empty) set, $\mathcal{T}$ is a continuous t–representable and $\mathcal{M}_{M,N}$ is a mapping $X^3 \times (0, +\infty) \to L^*$ (an intuitionistic fuzzy set, see Definition 1.18) satisfying the following conditions:

For every $x, y, z \in X$ and $t, s > 0$:

(a) $\mathcal{M}_{M,N}(x,y,z,t) >_{L^*} 0_{L^*}$;

(b) $\mathcal{M}_{M,N}(x,y,z,t) = 1_{L^*}$ if and only if $x = y = z$;

(c) $\mathcal{M}_{M,N}(x,y,z,t) = \mathcal{M}_{M,N}(p\{x,y,z\}, t)$ (symmetry), where p is a permutation function;

(d) $\mathcal{M}_{M,N}(x,y,z,t+s) \geq_{L^*} \mathcal{T}(\mathcal{M}_{M,N}(x,y,a,t), \mathcal{M}_{M,N}(a,z,z,s))$;

(e) $\mathcal{M}_{M,N}(x,y,z,\cdot) : (0,\infty) \longrightarrow L^*$ is continuous.

In this case $\mathcal{M}_{M,N}$ is called an *intuitionistic $\mathcal{M}$-fuzzy metric*, where

$$\mathcal{M}_{M,N}(x,y,z,t) = (M(x,y,z,t), N(x,y,z,t)).$$

Example 1.23. Let (X, D) be a D-metric space. Denote $\mathcal{T}(a,b) = (a_1b_1, \min(a_2+b_2, 1))$ for all $a = (a_1, a_2)$ and $b = (b_1, b_2) \in L^*$ and let M and N be $\mathcal{M}$-fuzzy sets on $X^3 \times (0, \infty)$ defined as follows:

$$\begin{aligned}\mathcal{M}_{M,N}(x,y,z,t) &= (M(x,y,z,t), N(x,y,z,t)) \\ &= (\frac{ht^n}{ht^n + mD(x,y,z)}, \frac{mD(x,y,z)}{ht^n + mD(x,y,z)}),\end{aligned}$$

for all $t, h, m, n \in \mathbb{R}^+$. Then $(X, \mathcal{M}_{M,N}, \mathcal{T})$ is an intuitionistic $\mathcal{M}$-fuzzy metric space.

Definition 1.24. A sequence $\{x_n\}$ is Cauchy in an intuitionistic $\mathcal{M}$-fuzzy metric space $(X, \mathcal{M}_{M,N}, \mathcal{T})$ if for each $0 < \varepsilon < 1$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\mathcal{M}_{M,N}(x_n, x_n, x_m, t) >_{L^*} (N_s(\varepsilon), \varepsilon),$$

and for each $n, m \geq n_0$, where N_s is the standard negator. The sequence $\{x_n\}$ is said to be *convergent* to $x \in X$ in the intuitionistic $\mathcal{M}$-fuzzy metric space $(X, \mathcal{M}_{M,N}, \mathcal{T})$ and denoted by $x_n \stackrel{\mathcal{M}_{M,N}}{\longrightarrow} x$ if $\mathcal{M}_{M,N}(x_n, x_n, x, t) \longrightarrow 1_{L^*}$ whenever $n \longrightarrow \infty$ for every $t > 0$. An intuitionistic $\mathcal{M}$-fuzzy metric space is said to be *complete* if and only if every Cauchy sequence is convergent.

Lemma 1.25. *Let $\mathcal{M}_{M,N}$ be an intuitionistic $\mathcal{M}$-fuzzy metric. Then, for any $t > 0$, $\mathcal{M}_{M,N}(x,y,z,t)$ is nondecreasing with respect to t, in $(L^*, \leq_{L^*})$, for all x, y, z in X.*

Proof. The proof is same as $\mathcal{M}$-fuzzy metric spaces (see Lemma 1.12). □

Definition 1.26. Let $(X, \mathcal{M}_{M,N}, \mathcal{T})$ be an intuitionistic $\mathcal{M}$-fuzzy metric space. For $t > 0$, define the *open ball* $B_{\mathcal{M}_{M,N}}(x, r, t)$ with center $x \in X$ and radius $0 < r < 1$, as

$$B_{\mathcal{M}_{M,N}}(x, r, t) = \{y \in X : \mathcal{M}_{M,N}(x, y, y, t) >_{L^*} (N_s(r), r)\}.$$

A subset $A \subseteq X$ is called *open* if for each $x \in A$, there exist $t > 0$ and $0 < r < 1$ such that $B_{\mathcal{M}_{M,N}}(x, r, t) \subseteq A$. Let $\tau_{\mathcal{M}_{M,N}}$ denote the family of all open subset of X. $\tau_{\mathcal{M}_{M,N}}$ is called the *topology induced by intuitionistic $\mathcal{M}$-fuzzy metric.*

Definition 1.27. Let $(X, \mathcal{M}_{M,N}, \mathcal{T})$ be an intuitionistic $\mathcal{M}$-fuzzy metric space. A subset A of X is said to be *IF-bounded* if there exist $t > 0$ and $0 < r < 1$ such that $\mathcal{M}_{M,N}(x, y, y, t) >_{L^*} (N_s(r), r)$ for each $x, y \in A$.

Definition 1.28. Let $(X, \mathcal{M}_{M,N}, \mathcal{T})$ be an intuitioistic $\mathcal{M}$-fuzzy metric space. $\mathcal{M}$ is said to be continuous on $X^3 \times]0, \infty[$ if

$$\lim_{n \to \infty} \mathcal{M}_{M,N}(x_n, y_n, z_n, t_n) = \mathcal{M}_{M,N}(x, y, z, t),$$

whenever a sequence $\{(x_n, y_n, z_n, t_n)\}$ in $X^3\times]0,\infty[$ converges to a point $(x, y, z, t) \in X^3\times]0,\infty[$ i.e.,

$$\lim_n \mathcal{M}_{M,N}(x_n, x, z, t) = \lim_n \mathcal{M}_{M,N}(x, y_n, z, t) = \lim_n \mathcal{M}_{M,N}(x, y, z_n, t) = 1_{\mathcal{L}^*}$$

and

$$\lim_n \mathcal{M}_{M,N}(x, y, z, t_n) = \mathcal{M}_{M,N}(x, y, z, t).$$

Lemma 1.29. *Let $(X, \mathcal{M}_{M,N}, \mathcal{T})$ be an intuitioistic $\mathcal{M}$-fuzzy metric space. Then $\mathcal{M}$ is continuous function on $X^3\times]0,\infty[$.*

Proof. The proof is same as $\mathcal{M}$-fuzzy metric spaces (see Lemma 1.14). □

Lemma 1.30. *Let $(X, \mathcal{M}_{M,N}, \mathcal{T})$ be an intuitioistic $\mathcal{M}$-fuzzy metric space. Define $E_{\lambda,\mathcal{M}} : X^3 \longrightarrow \mathbb{R}^+ \cup \{0\}$ by*

$$E_{\lambda,\mathcal{M}}(x, y, z) = \inf\{t > 0 : \mathcal{M}_{M,N}(x, y, z, t) >_{L^*} (N_s(\lambda), \lambda)$$

for each $0 < \lambda < 1$ and $x, y, z \in X$. Then we have

(i) *for any $0 < \mu < 1$ there exists $0 < \lambda < 1$ such that*

$$E_{\mu,\mathcal{M}}(x_1, x_1, x_n) \leq E_{\lambda,\mathcal{M}}(x_1, x_1, x_2) + E_{\lambda,\mathcal{M}}(x_2, x_2, x_3) + \cdots + E_{\lambda,\mathcal{M}}(x_{n-1}, x_{n-1}, x_n)$$

for any $x_1, ..., x_n \in X$;

(ii) *the sequence $\{x_n\}$ is convergent in the intuitioistic $\mathcal{M}$-fuzzy metric $(X, \mathcal{M}_{M,N}, \mathcal{T})$ if and only if $E_{\lambda,\mathcal{M}}(x_n, x_n, x) \overset{\mathcal{M}_{M,N}}{\longrightarrow} 0$. Also the sequence $\{x_n\}$ is Cauchy if and only if it is Cauchy with $E_{\lambda,\mathcal{M}}$.*

Proof. The proof is same as fuzzy metric spaces (see Lemma 1.15) □

Lemma 1.31. *Let $(X, \mathcal{M}_{M,N}, \mathcal{T})$ be an intuitioistic $\mathcal{M}$-fuzzy metric space. If*

$$\mathcal{M}_{M,N}(x_n, x_n, x_{n+1}, t) \geq_{L^*} \mathcal{M}_{M,N}(x_0, x_0, x_1, k^n t)$$

for some $k > 1$ and $n \in \mathbb{N}$, then $\{x_n\}$ is a Cauchy sequence.

Proof. For every $\lambda \in (0, 1)$ and $x_n \in X$, we have

$$\begin{aligned} E_{\lambda,\mathcal{M}}(x_n, x_n, x_{n+1}) &= \inf\{t > 0 : \mathcal{M}_{M,N}(x_n, x_n, x_{n+1}, t) >_{L^*} (N_s(\lambda), \lambda)\} \\ &\leq \inf\{t > 0 : \mathcal{M}_{M,N}(x_0, x_0, x_1, k^n t) >_{L^*} (N_s(\lambda), \lambda)\} \\ &= \inf\{\frac{t}{k^n} : \mathcal{M}_{M,N}(x_0, x_0, x_1, t) >_{L^*} (N_s(\lambda), \lambda)\} \\ &= \frac{1}{k^n} \inf\{t > 0 : \mathcal{M}_{M,N}(x_0, x_0, x_1, t) >_{L^*} (N_s(\lambda), \lambda)\} \\ &= \frac{1}{k^n} E_{\lambda,\mathcal{M}}(x_0, x_0, x_1). \end{aligned}$$

From Lemma 1.30, for every $\mu \in (0,1)$ there exists $\lambda \in (0,1)$ such that

$$\begin{aligned} & E_{\mu,\mathcal{M}}(x_n, x_n, x_m) \\ \leq \quad & E_{\lambda,\mathcal{M}}(x_n, x_n, x_{n+1}) + E_{\lambda,\mathcal{M}}(x_{n+1}, x_{n+1}, x_{n+2}) + \cdots \\ & + E_{\lambda,\mathcal{M}}(x_{m-1}, x_{m-1}, x_m) \\ \leq \quad & \frac{1}{k^n} E_{\lambda,\mathcal{M}}(x_0, x_0, x_1) + \frac{1}{k^{n+1}} E_{\lambda,\mathcal{M}}(x_0, x_0, x_1) + \cdots \\ & + \frac{1}{k^{m-1}} E_{\lambda,\mathcal{M}}(x_0, x_0, x_1) \\ = \quad & E_{\lambda,\mathcal{M}}(x_0, x_0, x_1) \sum_{j=n}^{m-1} \frac{1}{k^j} \\ \longrightarrow \quad & 0. \end{aligned}$$

Hence sequence $\{x_n\}$ is Cauchy in an intuitimistic $\mathcal{M}$-fuzzy metric space. □

2. The main results

Theorem 2.1. *Let $(X, \mathcal{M}_{M,N}, \mathcal{T})$ be a complete intuitionistic $\mathcal{M}$-fuzzy metric space with $\mathcal{T}(t,t) = t$ for all $t \in L^*$. Let $S, T : X \to X$ be mappings satisfying the following condition: there exists a constant $k \in (0,1)$ such that*

$$\begin{aligned} \mathcal{M}_{M,N}(Sx, Ty, Tz, kt) \geq_{L^*} \; & a(t)\mathcal{M}_{M,N}(x, Sx, Sx, t) \\ & + b(t)\mathcal{M}_{M,N}(y, Ty, Tz, t) + c(t)\mathcal{M}_{M,N}(x, Ty, Tz, \alpha t) \\ & + h(t)\mathcal{M}_{M,N}(y, Sx, Sx, (2-\alpha)t) \\ & + p(t)\mathcal{M}_{M,N}(x, y, z, t) \end{aligned}$$

for every $x, y, z \in X$ and for all $\alpha \in (0,2)$, where $a, b, c, h, p : [0,\infty) \longrightarrow [0,1]$ are five functions such that

$$a(t) + b(t) + c(t) + h(t) + p(t) = 1 \; for \; every \; t \in [0,\infty).$$

Then S and T have a unique common fixed point in X.

Proof. Let $x_0 \in X$ be an arbitrary point and there exists $x_1, x_2 \in X$ such that $x_1 = Sx_0$ and $x_2 = Tx_1$. Inductively, construct sequence $\{x_n\}$ in X such that $x_{2n+1} = Sx_{2n}$ and $x_{2n+2} = Tx_{2n+1}$, for $n = 0, 1, 2, \cdots$. We show that the sequence $\{x_n\}$ is Cauchy. Let

$$d_m(t) = \mathcal{M}_{M,N}(x_m, x_{m+1}, x_{m+1}, t), t > 0.$$

Then, we prove $\{d_m(t)\}$ is increasing w.r.t m. For, $m = 2n+1$, we have

$$\begin{aligned}
d_{2n+1}(kt) &= \mathcal{M}_{M,N}(x_{2n+1}, x_{2n+2}, x_{2n+2}, kt)\\
&= \mathcal{M}_{M,N}(Sx_{2n}, Tx_{2n+1}, Tx_{2n+1}, kt)\\
&\geq_{L^*} a(t)\mathcal{M}_{M,N}(x_{2n}, Sx_{2n}, Sx_{2n}, t)\\
&\quad + b(t)\mathcal{M}_{M,N}(x_{2n+1}, Tx_{2n+1}, Tx_{2n+1}, t)\\
&\quad + c(t)\mathcal{M}_{M,N}(x_{2n}, Tx_{2n+1}, Tx_{2n+1}, \alpha t)\\
&\quad + h(t)\mathcal{M}_{M,N}(x_{2n+1}, Sx_{2n}, Sx_{2n}, (2-\alpha)t)\\
&\quad + p(t)\mathcal{M}_{M,N}(x_{2n}, x_{2n+1}, x_{2n+1}, t)\\
&= a(t)\mathcal{M}_{M,N}(x_{2n}, x_{2n+1}, x_{2n+1}, t)\\
&\quad + b(t)\mathcal{M}_{M,N}(x_{2n+1}, x_{2n+2}, x_{2n+2}, t)\\
&\quad + c(t)\mathcal{M}_{M,N}(x_{2n}, x_{2n+2}, x_{2n+2}, \alpha t)\\
&\quad + h(t)\mathcal{M}_{M,N}(x_{2n+1}, x_{2n+1}, x_{2n+1}, (2-\alpha)t)\\
&\quad + p(t)\mathcal{M}_{M,N}(x_{2n}, x_{2n+1}, x_{2n+1}, t).
\end{aligned}$$

Hence,

$$\begin{aligned}
d_{2n+1}(kt) \geq_{L^*} & \ a(t)d_{2n}(t) + b(t)d_{2n+1}(t) + c(t)\mathcal{T}(d_{2n}(t), d_{2n+1}(qt))\\
& + h(t) + p(t)d_{2n}(t).
\end{aligned} \tag{2.1}$$

The last equality is true, because if set $\alpha = 1+q$, for $q \in (k,1)$, then

$$\begin{aligned}
& \mathcal{M}_{M,N}(x_{2n}, x_{2n+2}, x_{2n+2}, (1+q)t)\\
= \ & \mathcal{M}_{M,N}(x_{2n}, x_{2n}, x_{2n+2}, (1+q)t)\\
\geq_{L^*} \ & \mathcal{T}(\mathcal{M}_{M,N}(x_{2n}, x_{2n}, x_{2n+1}, t), \mathcal{M}_{M,N}(x_{2n+1}, x_{2n+2}, x_{2n+2}, qt))\\
= \ & \mathcal{T}(d_{2n}(t), d_{2n+1}(qt)).
\end{aligned}$$

We claim that for every $n \in \mathbb{N}$, $d_{2n+1}(t) \geq_{L^*} d_{2n}(t)$. If $d_{2n+1}(t) <_{L^*} d_{2n}(t)$ then, since for some $n \in \mathbb{N}$, $\mathcal{T}(d_{2n+1}(qt), d_{2n}(t)) >_{L^*} \mathcal{T}(d_{2n+1}(qt), d_{2n+1}(qt))$ $= d_{2n+1}(qt)$ in inequality (2.1), we have

$$\begin{aligned}
d_{2n+1}(kt) &>_{L^*} a(t)d_{2n+1}(qt) + b(t)d_{2n+1}(qt) + c(t)d_{2n+1}(qt)\\
&\quad + h(t)d_{2n+1}(qt) + p(t)d_{2n+1}(qt)\\
&= d_{2n+1}(qt).
\end{aligned}$$

This is a contradiction. Hence $d_{2n+1}(t) \geq_{L^*} d_{2n}(t)$ for every $n \in \mathbb{N}$ and $\forall t > 0$. Similarly, we have $d_{2n}(t) \geq_{L^*} d_{2n-1}(t)$. Thus $\{d_n(t)\}$ is an increasing sequence in L^*. By inequality (2.1), we have

$$\begin{aligned}
d_{2n+1}(kt) &\geq_{L^*} a(t)d_{2n}(qt) + b(t)d_{2n}(qt) + c(t)\mathcal{T}(d_{2n}(qt), d_{2n}(qt))\\
&\quad + h(t)d_{2n}(qt) + p(t)d_{2n}(qt)\\
&= d_{2n}(qt).
\end{aligned}$$

Now, if $m = 2n$, then by hypothesis, we have

$$\begin{aligned} d_{2n}(kt) &= \mathcal{M}_{M,N}(x_{2n}, x_{2n+1}, x_{2n+1}, kt) \\ &= \mathcal{M}_{M,N}(Sx_{2n-1}, Tx_{2n}, Tx_{2n}, kt) \\ &\geq_{L^*} a(t)\mathcal{M}_{M,N}(x_{2n-1}, Sx_{2n-1}, Sx_{2n-1}, t) \\ &\quad + b(t)\mathcal{M}_{M,N}(x_{2n}, Tx_{2n}, Tx_{2n}, t) \\ &\quad + c(t)\mathcal{M}_{M,N}(x_{2n-1}, Tx_{2n}, Tx_{2n}, \alpha t) \\ &\quad + h(t)\mathcal{M}_{M,N}(x_{2n}, Sx_{2n-1}, Sx_{2n-1}, (2-\alpha)t) \\ &\quad + p(t)\mathcal{M}_{M,N}(x_{2n-1}, x_{2n}, x_{2n}, t) \\ &= a(t)\mathcal{M}_{M,N}(x_{2n-1}, x_{2n}, x_{2n}, t) \\ &\quad + b(t)\mathcal{M}_{M,N}(x_{2n}, x_{2n+1}, x_{2n+1}, t) \\ &\quad + c(t)\mathcal{M}_{M,N}(x_{2n-1}, x_{2n+1}, x_{2n+1}, \alpha t) \\ &\quad + h(t)\mathcal{M}_{M,N}(x_{2n}, x_{2n}, x_{2n}, (2-\alpha)t) \\ &\quad + p(t)\mathcal{M}_{M,N}(x_{2n-1}, x_{2n}, x_{2n}, t). \end{aligned}$$

Hence

$$\begin{aligned} d_{2n}(kt) \geq_{L^*} & \ a(t)d_{2n-1}(t) + b(t)d_{2n}(t) + c(t)\mathcal{T}(d_{2n-1}(t), d_{2n}(qt)) \\ & + h(t) + p(t)d_{2n-1}(t). \end{aligned} \tag{2.2}$$

The last equality is true, because if set $\alpha = 1 + q$, for $q \in (k, 1)$, then

$$\begin{aligned} & \mathcal{M}_{M,N}(x_{2n-1}, x_{2n+1}, x_{2n+1}, (1+q)t) \\ = \ & \mathcal{M}_{M,N}(x_{2n-1}, x_{2n-1}, x_{2n+1}, (1+q)t) \\ \geq_{L^*} & \ \mathcal{T}(\mathcal{M}_{M,N}(x_{2n-1}, x_{2n-1}, x_{2n}, t), \mathcal{M}_{M,N}(x_{2n}, x_{2n+1}, x_{2n+1}, qt)) \\ = \ & \mathcal{T}(d_{2n-1}(t), d_{2n}(qt)). \end{aligned}$$

We claim that for every $n \in \mathbb{N}$, $d_{2n}(t) \geq_{L^*} d_{2n-1}(t)$. If $d_{2n}(t) <_{L^*} d_{2n-1}(t)$, then since $\mathcal{T}(d_{2n}(qt), d_{2n-1}(t)) \geq_{L^*} \mathcal{T}(d_{2n}(qt), d_{2n}(qt)) = d_{2n}(qt)$ in inequality (2.2), we have

$$\begin{aligned} d_{2n}(kt) >_{L^*} & \ a(t)d_{2n}(qt) + b(t)d_{2n}(qt) + c(t)d_{2n}(qt) + h(t)d_{2n}(qt) \\ & + p(t)d_{2n}(qt) \\ = \ & d_{2n}(qt). \end{aligned}$$

This is a contradiction. Hence $d_{2n}(t) \geq_{L^*} d_{2n-1}(t)$ for every $n \in \mathbb{N}$ and $\forall t > 0$. Similarly, we have $d_{2n-1}(t) \geq_{L^*} d_{2n-2}(t)$. Thus $\{d_n(t)\}$ is an increasing sequence in L^*. By inequality (2.2), we have

$$\begin{aligned} d_{2n}(kt) \geq_{L^*} & \ a(t)d_{2n-1}(qt) + b(t)d_{2n-1}(qt) + c(t)\mathcal{T}(d_{2n-1}(qt), d_{2n-1}(qt)) \\ & + h(t)d_{2n-1}(qt) + p(t)d_{2n-1}(qt) \\ = \ & d_{2n-1}(qt). \end{aligned}$$

Hence we have $d_{2n}(kt) \geq_{L^*} d_{2n-1}(qt)$. Thus $d_n(kt) \geq_{L^*} d_{n-1}(qt)$, for every $n \in \mathbb{N}$. That is,

$$\begin{aligned}\mathcal{M}_{M,N}(x_n, x_{n+1}, x_{n+1}, t) &\geq_{L^*} \mathcal{M}_{M,N}(x_{n-1}, x_n, x_n, \frac{q}{k}t)\\ &\geq_{L^*} \cdots\\ &\geq_{L^*} \mathcal{M}_{M,N}(x_0, x_1, x_1, (\frac{q}{k})^n t).\end{aligned}$$

Hence by Lemma 1.31 $\{x_n\}$ is a Cauchy sequence in X, and so $\{x_n\}$ converges to x in X. That is, $\lim_{n\to\infty} x_n = x$, hence

$$\begin{aligned}\lim_{n\to\infty} x_{2n+1} &= \lim_{n\to\infty} Sx_{2n} = \lim_{n\to\infty} Tx_{2n+1}\\ &= \lim_{n\to\infty} x_{2n+2} = x.\end{aligned}$$

We prove that $Sx = x$.

For if $\alpha = 1$, setting $x = x$ and $y = z = x_{2n+1}$ in inequality (2.1), we obtain

$$\begin{aligned}&\mathcal{M}_{M,N}(Sx, Tx_{2n+1}, Tx_{2n+1}, kt)\\ \geq_{L^*}\ & a(t)\mathcal{M}_{M,N}(x, Sx, Sx, t) + b(t)\mathcal{M}_{M,N}(x_{2n+1}, Tx_{2n+1}, Tx_{2n+1}, t)\\ &+c(t)\mathcal{M}_{M,N}(x, Tx_{2n+1}, Tx_{2n+1}, t) + h(t)\mathcal{M}_{M,N}(x_{2n+1}, Sx, Sx, t)\\ &+p(t)\mathcal{M}_{M,N}(x, x_{2n+1}, x_{2n+1}, t).\end{aligned}$$

If $Sx \neq x$, then by taking $n \longrightarrow \infty$, we have

$$\begin{aligned}&\mathcal{M}_{M,N}(Sx, x, x, kt)\\ \geq_{L^*}\ & a(t)\mathcal{M}_{M,N}(x, Sx, Sx, t) + b(t)\mathcal{M}_{M,N}(x, x, x, t)\\ &+c(t)\mathcal{M}_{M,N}(x, x, x, t) + h(t)\mathcal{M}_{M,N}(x, Sx, Sx, t)\\ &+p(t)\mathcal{M}_{M,N}(x, x, x, t)\\ >_{L^*}\ & \mathcal{M}_{M,N}(x, x, Sx, t),\end{aligned}$$

which is a contradiction. It follows that $Sx = x$.

Similarly we prove that $Tx = x$. Again, replacing x by x_{2n} and y, z by x in (2.1), for $\alpha = 1$, we have

$$\begin{aligned}&\mathcal{M}_{M,N}(Sx_{2n}, Tx, Tx, kt)\\ \geq_{L^*}\ & a(t)\mathcal{M}_{M,N}(x_{2n}, Sx_{2n}, Sx_{2n}, t) + b(t)\mathcal{M}_{M,N}(x, Tx, Tx, t)\\ &+c(t)\mathcal{M}_{M,N}(x_{2n}, Tx, Tx, t) + h(t)\mathcal{M}_{M,N}(x, Sx_{2n}, Sx_{2n}, t)\\ &+p(t)\mathcal{M}_{M,N}(x_{2n}, x, x, t)\end{aligned}$$

and so if $Tx \neq x$, taking $n \longrightarrow \infty$, we have

$$\begin{aligned}&\mathcal{M}_{M,N}(x, Tx, Tx, kt)\\ \geq_{L^*}\ & a(t)\mathcal{M}_{M,N}(x, x, x, t) + b(t)\mathcal{M}_{M,N}(x, Tx, Tx, t)\\ &+c(t)\mathcal{M}_{M,N}(x, Tx, Tx, t) + h(t)\mathcal{M}_{M,N}(x, x, x, t)\\ &+p(t)\mathcal{M}_{M,N}(x, x, x, t)\\ >_{L^*}\ & \mathcal{M}_{M,N}(x, Tx, Tx, t),\end{aligned}$$

which implies that, $Tx = x$. Therefore, $Sx = Tx = x$, this is, x is a common fixed point of self-maps S and T. Now, we have to prove the uniqueness of the common fixed point of S and T. If x' is another fixed point of S and T, then for $\alpha = 1$ we have

$$\begin{aligned} & \mathcal{M}_{M,N}(x, x', x', kt) \\ = \quad & \mathcal{M}_{M,N}(Sx, Tx', Tx', kt) \\ \geq_{L^*} \quad & a(t)\mathcal{M}_{M,N}(x, Sx, Sx, t) + b(t)\mathcal{M}_{M,N}(x', Tx', Tx', t) \\ & +c(t)\mathcal{M}_{M,N}(x, Tx', Tx', t) + h(t)\mathcal{M}_{M,N}(x', Sx, Sx, t) \\ & +p(t)\mathcal{M}_{M,N}(x, x', x', t) \\ >_{L^*} \quad & \mathcal{M}_{M,N}(x, x', x', t) \end{aligned}$$

and so, $x = x'$. □

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Jong kyu kim
Department of Mathematics, Education
Kyungnam University
Masan, 631-701, Korea
E-mail address: jongkyuk@kyungnam.ac.kr

S. Sedghi
Department of Mathematics
Islamic Azad University-Ghaemshar Branch, Iran
E-mail address: sedghi_gh@yahoo.com

N. Shobe
Department of Mathematics
Islamic Azad University-Babol Branch, Iran
E-mail address: nabi_shobe@yahoo.com