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APPROXIMATION OF COMMON FIXED POINTS OF NON-SELF ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

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Abstract. Let E be a uniformly convex Banach space and K a nonempty closed convex subset which is also a nonexpansive retract of E. For $i = 1, 2, 3$, let $T_i : K \to E$ be an asymptotically nonexpansive mappings with sequence $\{k_n^{(i)}\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n^{(i)} - 1) < \infty$, $k_n^{(i)} \to 1$, as $n \to \infty$ and $F(T) = \bigcap_{i=1}^{3} F(T_i) \neq \phi$ (the set of all common fixed points of T_i, $i = 1, 2, 3$). Let $\{a_n\}, \{b_n\}$ and $\{c_n\}$ are three real sequences in $[0, 1]$ such that $\epsilon \leq\ a_n,\ b_n,\ c_n \leq 1 - \epsilon$ for $n \in N$ and some $\epsilon \geq 0$. Starting with arbitrary $x_1 \in K$, define sequence $\{x_n\}$ by setting

$$\begin{cases} x_{n+1} = P((1 - a_n)x_n + a_nT_1(PT_1)^{n-1}y_n) \\ \quad y_n = P((1 - b_n)x_n + a_nT_2(PT_2)^{n-1}z_n) \\ \quad z_n = P((1 - c_n)x_n + c_nT_3(PT_3)^{n-1}x_n). \end{cases}$$

Assume that one of the following conditions holds:

(1) E satisfies the Opial property,

(2) E has Frechet differentiable norm,

(3) E^* has Kedec -Klee property, where E^* is dual of E.

Then sequence $\{x_n\}$ converges weakly to some $p \in F(T)$.

1. Introduction

Definition 1.1. Let K be a nonempty subset of a real normed linear space E. A self-mapping $T : K \to K$ is called *asymptotically nonexpansive* if there exists a sequence $\{k_n\} \subset [1, \infty)$, $k_n \to 1$ as $n \to \infty$ such that for all $x,\ y\ \in\ K$, the following inequality holds:

$$\|T^n x - T^n y\| \leq k_n\|x - y\|,\ \forall n \geq 1. \tag{1.1}$$

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T is called *uniformly L-Lipschitzian* if there exists a constant $L > 0$ such that for all $x,\ y\ \in\ K$,

$$\|T^n x - T^n y\| \leq L\|x - y\|,\ \forall n \geq 1. \tag{1.2}$$

The class of asymptotically nonexpansive mappings was introduced by Gobel and Kirk [11] as an important generalization of the class of nonexpansive mappings (i.e., $\|Tx - Ty\| \leq L\|x - y\|, \forall\ x, y \in K$) who proved that if K is a nonempty closed convex subset of a real uniformly convex Banach space and T is an asymptotically nonexpansive self-mapping of K, then T has a fixed point.

Iterative methods for approximating fixed points of nonexpansive mappings and asymptotically nonexpansive mappings have been studied by various authors (see e.g.[1]-[2], [3]-[6], [8]-[10], [20]-[22], [24]-[29], [34]-[38]) using the Mann iteration process [17], the Ishikawa iteration process [12]), the modified Mann iteration process ([28], [29]), the modified Ishikawa iteration process [25], Generalized Ishikawa iteration process of rank 3, [26] and modified Ishikawa iteration process of rank 3, [27].

In [26], Sahu introduced Ishikawa iteration process of rank r as bellow:
Let C be a nonempty subset of a normed space X and $T : C \to X$ be a nonlinear operator. Further, let r be a positive integer and let $\{a_{n,i}\},\ i = 1, 2, \cdots, r$ be a real sequence in $[0, 1]$. For $x_0 \in C$, the generalized Ishikawa iteration process (of rank r) $\{x_n\}$ is given by

$$\begin{cases} x_{n+1} = (1 - a_{n,1})x_n + a_{n,1}Ty_{n,1}, \\ \ \ y_{n,i} = (1 - a_{n,i+1})x_n + a_{n,i+1}Ty_{n,i+1},\ i = 1, 2, \cdots, r - 1, \\ \ \ y_{n,r} = x_n. \end{cases}$$

In particular, we underlying that whenever referring to three step iterative scheme mean that process defined for rank 3 and defined as follows:

$$\begin{cases} x_{n+1} = (1 - a_n)x_n + a_nTy_n, \\ \quad y_n = (1 - b_n)x_n + b_nTz_n, \\ \quad z_n = (1 - c_n)x_n + c_nTx_n,\ \forall n \geq 0, \end{cases}$$

where $\{a_n\}, \{b_n\}$ and $\{c_n\}$ are real sequences in $[0, 1]$.

In [27], Sahu et. al. introduced a modified Ishikawa iteration process of rank r to approximate common fixed points of (L_i, α_i)- Lipschizian type asymptotically quasi-nonexpansive mapping on a compact subset of uniformly convex Banach spaces. More precisely, they proved the following theorem:

Theorem 1.2. ([27]) *Let K be a nonempty compact subset of a uniformly convex Banach space X and for $i = 1, 2, 3$, let $T_i : K \to K$ be uniformly (L_i, α_i)-Lipschitz and asymptotically quasi-nonexpansive mappings with sequence $\{k_n^i\}$*

such that $\sum_{n=1}^{\infty}(k_n^i - 1) < \infty$. *Define a sequence* $\{x_n\}$ *in* K *as follows:*

$$\begin{cases} x_{n+1} = (1-a_n)x_n + a_n T_1^n y_n, \\ y_n = (1-b_n)x_n + b_n T_2^n z_n, \\ z_n = (1-c_n)x_n + c_n T_3^n x_n, \ \forall n \geq 1, \end{cases} \tag{1.3}$$

where $\{a_n\}, \{b_n\}$ *and* $\{c_n\}$ *are sequences in* $[0,1]$ *such that* $0 < \underline{a} \leq a_n < \overline{a} < 1, 0 < \underline{b} \leq b_n < \overline{b} < 1$, $0 < \underline{c} \leq c_n < \overline{c} < 1$. *If* $F(T_1) \bigcap F(T_2) \bigcap F(T_3) \neq \phi$, *then sequence* $\{x_n\}$ *converges strongly to common fixed point of* T_1, T_2, T_3.

In above result T remains self-mapping of a nonempty closed convex subset K of uniformly convex Banach space E, if, however, the domain K of T is a proper subset of E (and this is the cases in several application), and T maps K into E, then the iteration processes of Mann and Ishikawa studied by these authors may fail to be well defined.

In 2003, Chidume [7] studied the following iteration process

$$x_{n+1} = P((1-\alpha_n)x_n + \alpha_n T(PT)^{n-1}x_n)), \tag{1.4}$$

for $x_1 \in K$ in the frame work of uniformly convex Banach space, where K is a closed convex nonexpansive retract of a real uniformly convex Banach space E with P as a nonexpansive retraction, $T : K \to E$ is an asymptotically nonexpansive non-self mapping with sequence $\{k_n\} \subset [0,\infty), k_n \to 1$ as $n \to \infty$, where $\{\alpha_n\}$ is real sequence in $[0,1]$ satisfying the condition $\epsilon \leq \alpha_n \leq 1-\epsilon$. He proved strong and weak convergence theorems for asymptotically nonexpansive non-self mappings.

Recently, Shahzad [31] studied the sequence $\{x_n\}$ defined by

$$x_{n+1} = P((1-\alpha_n)x_n + \alpha_n TP[(1-\beta_n)x_n + \beta_n Tx_n]), \tag{1.5}$$

where K is a closed convex nonexpansive retract of real uniformly convex Banach space E with P as nonexpansive retraction. He proved weak and strong convergence theorem for non-self asymptotically nonexpansive maps in Banach spaces.

Finding common fixed points of mappings acting on a Hilbert space is a problem that often aries in applied mathematics. In fact, many algorithms have been introduced for different class of mappings with nonempty set of common fixed point. Unfortunately, the existence results of common fixed points of mappings are not known in many situations. Therefore, it is natural to consider approximation results for these classes mapspings. Approximation of common fixed point of two or more nonexpansive and asymptotically nonexpansive non-self mappings by iteration has been studied by many authors (see e.g. [3]-[13], [23], [32]-[33]).

Our purpose in this paper is to construct iterative scheme and deal with the problem of approximation of common fixed point for asymptotically nonexpansive non-self mappings.

In this, paper we study weak and strong convergence results for defined iterative scheme.

Our weak convergence results are apply not only Hilbert space and L^p space $(1 < p < \infty)$, but also to the rather large class of space admitting the Kadec-Klee property (see e.g. [13]). We also discuss strong convergence of iterative scheme.

2. Preliminaries

Recall that a Banach space E is said to be *uniformly convex* if for each $r \in [0, 2]$, the modulus of convexity of E given by:

$$\delta(r) = \inf \left\{ 1 - \frac{1}{2}\|x + y\| : \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq r \right\}$$

satisfies the inequality $\delta(r) > 0$ for all $r > 0$.

For sequences, the symbol $\rightarrow$ (*resp.* $\rightharpoonup$) indicates norm (resp. weak) convergence.

Let $S = \{x \in E : \|x\| = 1\}$ and let E^* be the dual of E. The space E has:

(1) *Gateaux differentiable norm* [36] if

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in S$.

(2) *Frechet differentiable norm* [36] if for each $x \in S$, the above limit exists and is attained uniformly for $y \in S$ and in this case, it has been shown in [36] that

$$\langle h, J(x) \rangle + \frac{1}{2}\|x\|^2 \leq \frac{1}{2}\|x + h\|^2 \leq \langle h, J(x) \rangle + \frac{1}{2}\|x\|^2 + b(\|h\|)$$

for all $x, h \in E$, where J is the Frechet derivative of the functional $\frac{1}{2}\| \cdot \|^2$ at $x \in X$, $\langle \cdot, \cdot \rangle$ is the pairing between E, E^* and b is a function defined on $[0, \infty)$ such that $\lim_{t \downarrow 0} \frac{b(t)}{t} = 0$.

(3) *Opial property* [19] if for any sequence $\{x_n\} \in E$, $x_n \rightharpoonup x$ implies that

$$\limsup_{n \to \infty} \|x_n - x\| < \limsup_{n \to \infty} \|x_n - y\|$$

for all $y \in E$ with $y \neq x$.

(4) *Kadec-Klee property* if for every sequence $\{x_n\} \in E$, $x_n \rightharpoonup x$ and $\|x_n\| \rightarrow \|x\|$ together imply $\|x_n - x\| \rightarrow 0$ as $n \rightarrow \infty$.

(5) A mapping T with domain $D(T)$ and range $R(T)$ in E is said to be *demiclosed* at p if whenever $\{x_n\}$ is a sequence in $D(T)$ such that $\{x_n\}$ converges weakly to $x^* \in D(T)$ and $\{Tx_n\}$ converges strongly to p then $Tx^* = p$.

We recall the following useful definitions and lemmas for the development of our results.

Definition 2.1. A subset K of E is said to be retract of E if there exists a continuous mappings $P : E \to K$ such that $Px = x$, for all $x \in K$.

Every closed convex subset of a uniformly convex Banach space is retract. A mapping $P : E \to E$ is said to be retraction if $P^2 = P$. It follows that if a mapping P is a retraction, then $Py = y$ for all y in the range of P.

Definition 2.2. ([7]) Let E be a real normed linear space, K a nonempty subset of E. Let $P : E \to K$ be the nonexpansive retraction of E onto K. A map $T : K \to E$ is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ and $k_n \to 1$ as $n \to \infty$ such that the following inequality holds:

$$\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq k_n\|x - y\|, \ \forall x, y \in K, n \geq 1. \tag{2.1}$$

T is called uniformly L-Lipschitzian if there exists $L > 0$ such that

$$\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq L\|x - y\|, \ \forall x, y \in K, n \geq 1. \tag{2.2}$$

Next, we define Generalized Ishikawa iterative scheme as follows:

Let K be a nonempty closed convex subset of real Banach space E. Let $P : E \to K$ be a nonexpansive retraction of E onto K. For $i = 1, 2, 3$, a mapping $T_i : K \to E$ be an asymptotically nonexpansive defined in Definition (2.2). The *Generalized Ishikawa iterative scheme of rank* 3 is defined as follows.

$$\begin{cases} x_{n+1} = P((1 - a_n)x_n + a_nT_1(PT_1)^{n-1}y_n) \\ \quad y_n = P((1 - b_n)x_n + a_nT_2(PT_2)^{n-1}z_n) \\ \quad z_n = P((1 - c_n)x_n + c_nT_3(PT_3)^{n-1}x_n) \end{cases} \tag{2.3}$$

where $\{a_n\}, \{b_n\}$ and $\{c_n\}$ are real sequences in $[0, 1]$.

Remark 2.3. If T is a self-mapping, then P becomes the identity mapping so that (2.1) and (2.2) coincide with (1.1) and (1.2) respectively. Moreover (2.3) reduces to modified Ishikawa iteration scheme with rank 3 (see [27]).

Definition 2.4. ([28]) Let K be a nonempty subset of a Banach space E. A mapping $T : K \to K$ with $F(T) \neq \phi$, is said to satisfy Condition (A) [30] if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that

$$\|x - Tx\| \geq f(d(x, F(T)))$$

for all $x \in C$ where $d(x, F(T)) = \inf\{\|x - p\| : p \in F(T)\}$.

For approximating fixed points of nonexpansive mapping, Senter and Doston [30] introduced Condition (A). Letter on Mati and Ghosh [18] and Tan and Xu [36] pointed out the Condition (A) is weaker than the requirement that semi-compactness.

In this sequel, we shall need the following lemmas.

Lemma 2.5. ([29]) *Suppose that E is a uniformly convex Banach space and $0 < p \le t_n \le q < 1$ for all $n \in N$. Let $\{x_n\}$ and $\{y_n\}$ be sequences of E such that*

$$\limsup_{n\to\infty} \|x_n\| \le r, \ \limsup_{n\to\infty} \|y_n\| \le r$$

and

$$\lim_{n\to\infty} \|t_n x_n + (1-t_n) y_n\| = r$$

hold for some $r \ge 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Lemma 2.6. ([36]) *Let $\{r_n\}, \{s_n\}$ and $\{t_n\}$ be three nonnegative sequences satisfying the following condition:*

$$r_{n+1} \le r_n + t_n$$

for all $n \in N$. If $\sum_{n=1}^{\infty} t_n < \infty$, then $\lim_{n\to\infty} r_n$ exists.

Lemma 2.7. ([7]) *Let E be a uniformly convex Banach space, K be a nonempty closed convex subset of E. Let $T : K \to E$ be an asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1, \infty)$ and $k_n \to 1$ as $n \to \infty$. Then $(I - T)$ is a demiclosed at zero.*

Lemma 2.8. ([13]) *Let E be a reflexive Banach space such that its dual E^* has the Kadec-Klee property. Let $\{x_n\}$ be a bounded sequence in E and $x^*, y^* \in \omega_w(x_n)$ (weak w-limit set of $\{x_n\}$). Suppose*

$$\lim_{n\to\infty} \|t x_n + (1-t) x^* - y^*\|$$

exists for all $t \in [0, 1]$. Then $x^* = y^*$.

3. Preparatory lemmas

In this section, we prove some lemmas which play key roll to establish weak and strong convergence results for the iterative scheme (2.3).

In this sequel $\bigcap_{i=1}^{3} F(T_i) = F(T)$ will denote the set of all common fixed points of T_i, $i = 1, 2, 3$.

Lemma 3.1. *Let E be a uniformly convex Banach space and K a nonempty closed convex subset which is also a nonexpansive retract of E. Let $T_i : K \to E$ be asymptotically nonexpansive mapping with sequences $\{k_n^i\} \subset [1, \infty)$ such* *that* $\sum_{n=1}^{\infty}(k_n^i - 1) < \infty$ *for* $i = 1, 2, 3$. *Starting from arbitrary* $x_1 \in K$, *define the sequence* $\{x_n\}$ *by the recursion (2.3) where* $\{a_n\}, \{b_n\}$ *and* $\{c_n\}$ *are sequences in* $[0, 1]$. *If* $F(T) \neq \phi$, *then* $\lim_{n\to\infty} \|x_n - p\|$ *exists, for any* $p \in F(T)$.

Proof. For any given $p \in F(T)$. By (2.3), we have

$$\begin{aligned}\|z_n - p\| &= \|P((1-c_n)x_n + c_nT_3(PT_3)^{n-1}x_n) - p\| \\ &\le (1-c_n)\|x_n - p\| + c_n\|T_3(PT_3)^{n-1}x_n - p\| \\ &\le (1-c_n)\|x_n - p\| + c_n\|T_3(PT_3)^{n-1}x_n - T_3(PT_3)^{n-1}p\| \\ &\le (1-c_n)\|x_n - p\| + c_nk_n^{(3)}\|x_n - p\| \\ &\le k_n^{(3)}\|x_n - p\|.\end{aligned} \tag{3.1}$$

From (2.3) and (3.1), we have

$$\begin{aligned}\|y_n - p\| &= \|P((1-b_n)x_n + b_nT_2(PT_2)^{n-1}z_n) - p\| \\ &\le (1-b_n)\|x_n - p\| + b_n\|T_2(PT_2)^{n-1}z_n - p\| \\ &\le (1-b_n)\|x_n - p\| + b_nk_n^{(2)}\|z_n - p\| \\ &\le (1-b_n)\|x_n - p\| + b_nk_n^{(2)}k_n^{(3)}\|x_n - p\| \\ &\le k_n^{(2)}k_n^{(3)}\|x_n - p\|.\end{aligned} \tag{3.2}$$

Again from (2.3), (3.1) and (3.2), we have

$$\begin{aligned}\|x_{n+1} - p\| &= \|P((1-a_n)x_n + a_nT_1(PT_1)^{n-1}y_n) - p\| \\ &\le (1-a_n)\|x_n - p\| + a_n\|T_1(PT_1)^{n-1}y_n - p\| \\ &\le (1-a_n)\|x_n - p\| + a_nk_n^1\|y_n - p\| \\ &\le [(1-a_n) + a_nk_n^{(1)}k_n^{(2)}k_n^{(3)}]\|x_n - p\| \\ &\le k_n^{(1)}k_n^{(2)}k_n^{(3)}\|x_n - p\|,\end{aligned}$$

that is

$$\begin{aligned}\|x_{n+1} - p\| \le [1 &+ (k_n^{(1)} - 1)(k_n^{(2)} - 1)(k_n^{(3)} - 1) + (k_n^{(1)} - 1)(k_n^{(2)} - 1) \\ &+ (k_n^{(2)} - 1)(k_n^{(3)} - 1) + (k_n^{(3)} - 1)(k_n^{(1)} - 1) + (k_n^{(1)} - 1) \\ &+ (k_n^{(2)} - 1) + (k_n^{(3)} - 1)]\|x_n - p\|.\end{aligned} \tag{3.3}$$

Note that $\sum_{n=1}^{\infty}(k_n^i - 1) < \infty$ for $i = 1, 2, 3$, therefore by Lemma 2.6 ,

$$\lim_{n\to\infty} \|x_n - p\|$$

exists. □

Lemma 3.2. *Let* E *be a uniformly convex Banach space and* K *be a nonempty closed convex subset which is also a nonexpansive retract of* E *and let* $T_i : K \to$

E uniformly L_i- Lipschitzian, for $i = 1, 2, 3$. Starting from arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ by the recursion (2.3) where $\{a_n\}, \{b_n\}$ and $\{c_n\}$ are real sequences in [0,1] and set

$$C_n = \|x_n - T_1(PT_1)^{n-1}x_n\|,\ C_n^{'} = \|x_n - T_2(PT_2)^{n-1}x_n\|$$

and

$$C_n^{''} = \|x_n - T_3(PT_3)^{n-1}x_n\|,\ \forall n \geq 1.$$

If

$$\lim_{n\to\infty} C_n = \lim_{n\to\infty} C_n^{'} = \lim_{n\to\infty} C_n^{''} = 0. \tag{3.4}$$

then

$$\lim_{n\to\infty} \|x_n - T_1x_n\| = \lim_{n\to\infty} \|x_n - T_2x_n\| = \lim_{n\to\infty} \|x_n - T_3x_n\| = 0.$$

Proof. It follows from (2.3) that

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|P((1-a_n)x_n + a_nT_1(PT_1)^{n-1}y_n) - x_n\| \\
&\leq \|T_1(PT_1)^{n-1}y_n - x_n\| \\
&\leq \|T_1(PT_1)^{n-1}y_n - T_1(PT_1)^{n-1}x_n\| + \|T_1(PT_1)^{n-1}x_n - x_n\| \\
&\leq C_n + L_1\|y_n - x_n\| \\
&\leq C_n + L_1\|P((1-b_n)x_n + b_nT_2(PT_2)^{n-1}z_n) - x_n\| \\
&\leq C_n + L_1[\|T_2(PT_2)^{n-1}z_n - x_n\|] \\
&\leq C_n + L_1[\|T_2(PT_2)^{n-1}z_n - T_2(PT_2)^{n-1}x_n\| \\
&\quad + \|T_2(PT_2)^{n-1}x_n - x_n\|] \\
&\leq C_n + C_n^{'}L_1 + L_1L_2\|z_n - x_n\| \\
&\leq C_n + C_n^{'}L_1 + L_1L_2\|T_3(PT_3)^{n-1}x_n - x_n\| \\
&\leq C_n + L_1C_n^{'} + L_1L_2C_n^{''}
\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}
\|y_{n-1} - x_n\| &\leq \|x_{n-1} - x_n\| + \|T_2(PT_2)^{n-2}z_{n-1} - x_n\| \\
&\leq 2\|x_{n-1} - x_n\| + \|T_2(PT_2)^{n-2}z_{n-1} - T_2(PT_2)^{n-2}x_{n-1}\| \\
&\quad + \|T_2(PT_2)^{n-2}x_{n-1} - x_{n-1}\| \\
&\leq 2\|x_{n-1} - x_n\| + C_{n-1}^{'} + L_2C_{n-1}^{''}.
\end{aligned} \tag{3.6}$$

Combine (3.5) with (3.6), we have

$$\|y_{n-1} - x_n\| \leq (1 + 2L_1)\{C_{n-1}^{'} + L_2C_{n-1}^{''}\} + 2C_{n-1}^{'}. \tag{3.7}$$

Combine (3.5) and (3.6) with (3.7), we have

$$\begin{aligned}\|x_n-(PT_1)^{n-1}x_n\| &\le L_1[\|y_{n-1}-x_n\|+\|x_{n-1}-T_1(PT_1)^{n-2}x_{n-1}\| \\ &\quad +\|T_1(PT_1)^{n-2}x_{n-1}-T_1(PT_1)^{n-2}x_n\|] \\ &\le L_1\|y_{n-1}-x_n\|+L_1\|x_n-x_{n-1}\|+C_{n-1} \\ &\le (1+3L_1)\Big[C_{n-1}+L_1C^{'}_{n-1}+L_1L_2C^{''}_{n-1}\Big].\end{aligned} \tag{3.8}$$

It follows from (3.8) that

$$\begin{aligned}\|x_n-T_1x_n\| &\le \|x_n-T_1(PT_1)^{n-1}x_n\|+\|T_1(PT_1)^{n-1}x_n-T_1x_n\| \\ &\le C_n+L\Big[(1+3L_1)[C_{n-1}+L_1C^{'}_{n-1}+L_1L_2C^{''}_{n-1}]\Big].\end{aligned} \tag{3.9}$$

Hence, from (3.4) and (3.8), we have

$$\lim_{n\to\infty}\|x_n-T_1x_n\|=0.$$

Similarly, we can show that

$$\lim_{n\to\infty}\|x_n-T_2x_n\|=\lim_{n\to\infty}\|x_n-T_3x_n\|=0.$$

□

Lemma 3.3. *Let E be a uniformly convex Banach space and K be a nonempty closed convex subset which is also a nonexpansive retract of E and $T_i : K \to E$ be an asymptotically nonexpansive mappings with sequences $\{k_n^{(i)}\} \subset [1,\infty)$, for $i = 1, 2, 3$ such that $\sum(k_n^{(i)}-1) < \infty$ and $F(T) \neq \phi$. Let $\{a_n\}, \{b_n\}$ and $\{c_n\}$ be real sequences in [0,1] such that $\epsilon \le a_n,\ b_n,\ c_n \le 1-\epsilon$ for all $n \in N$ and some $\epsilon > 0$. Starting from arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ by the recursion* (2.3). *Then*

$$\lim_{n\to\infty}\|x_n-T_1x_n\|=\lim_{n\to\infty}\|x_n-T_2x_n\|=0$$

and

$$\lim_{n\to\infty}\|x_n-T_3x_n\|=0.$$

Proof. By Lemma 3.1, we have $\lim_{n\to\infty}\|x_n-p\|$ exists for all $p \in F(T)$. Set $\lim_{n\to\infty}\|x_n-p\|=c$. Now suppose $c > 0$. Taking lim sup on both sides in the inequality (3.1), we have

$$\limsup_{n\to\infty}\|z_n-p\|\le c. \tag{3.10}$$

Similarly, taking lim sup on both sides in the inequality (3.2), we have

$$\limsup_{n\to\infty}\|y_n-p\|\le c. \tag{3.11}$$

Next, we consider

$$\|T_1(PT_1)^{n-1}y_n-p\| \quad\le\quad k_n^{(1)}\|y_n-p\|.$$

Taking lim sup on both sides in the above inequality and using (3.2), we have

$$\limsup_{n\to\infty} \|T_1(PT_1)^{n-1}y_n - p\| \leq c.$$

and

$$\limsup_{n\to\infty} \|x_n - p\| \leq c. \tag{3.12}$$

Again, $\lim_{n\to\infty} \|x_{n+1} - p\| = c$ means that

$$\liminf_{n\to\infty} \|a_n(T_1(PT_1)^{n-1}y_n - p) + (1 - a_n)(x_n - p)\| \geq c. \tag{3.13}$$

On the other hand, by using (2.3) and (3.2), we have

$$\begin{aligned}
&\|a_n(T_1(PT_1)^{n-1}y_n - p) + (1 - a_n)(x_n - p)\| \\
&\qquad \leq a_n\|T_1(PT_1)^{n-1}y_n - p\| + (1 - a_n)\|x_n - p\| \\
&\qquad \leq a_n k_n^{(1)}\|y_n - p\| + (1 - a_n)\|x_n - p\| \\
&\qquad \leq a_n k_n^{(1)} k_n^{(2)} k_n^{(3)}\|x_n - p\| + (1 - a_n)\|x_n - p\| \\
&\qquad \leq k_n^{(1)} k_n^{(2)} k_n^{(3)}\|x_n - p\|.
\end{aligned}$$

Therefore, we have

$$\limsup_{n\to\infty} \|a_n(T_1(PT_1)^{n-1}y_n - p) + (1 - a_n)(x_n - p)\| \leq c. \tag{3.14}$$

Combining (3.13) with (3.14), we obtain that

$$\lim_{n\to\infty} \|a_n(T_1(PT_1)^{n-1}y_n - p) + (1 - a_n)(x_n - p)\| = c.$$

Hence applying Lemma 2.5, we have

$$\lim_{n\to\infty} \|T_1(PT_1)^{n-1}y_n - x_n\| = 0. \tag{3.15}$$

Observe that

$$\begin{aligned}
\|x_n - p\| &\leq \|T_1(PT_1)^{n-1}y_n - x_n\| + \|T1(PT_1)^{n-1}y_n - p\| \\
&\leq \|T_1(PT_1)^{n-1}y_n - x_n\| + k_n^{(1)}\|y_n - p\|,
\end{aligned}$$

which yields that

$$c \leq \liminf_{n\to\infty} \|y_n - p\| \leq \limsup_{n\to\infty} \|y_n - p\| \leq c.$$

That is

$$\lim_{n\to\infty} \|y_n - p\| = c,$$

it gives that

$$\liminf_{n\to\infty} \|b_n(T_2(PT_2)^{n-1}z_n - p) + (1 - b_n)(x_n - p)\| \geq c. \tag{3.16}$$

Similarly, by using (2.3) and (3.10), we have

$$\begin{aligned}\|b_n(T_2(PT_2)^{n-1}z_n - p) + (1-b_n)(x_n - p)\| \\ &\le b_n\|T_2(PT_2)^{n-1}z_n - p\| + (1-b_n)\|x_n - p\| \\ &\le b_n k_n^{(2)}\|z_n - p\| + (1-b_n)\|x_n - p\| \\ &\le b_n k_n^{(2)} k_n^{(3)}\|x_n - p\| + (1-b_n)\|x_n - p\| \\ &\le k_n^{(2)} k_n^{(3)}\|x_n - p\|.\end{aligned}$$

Therefore, we have

$$\limsup_{n\to\infty}\|b_n(T_2(PT_2)^{n-1}z_n - p) + (1-b_n)(x_n - p)\| \le c. \tag{3.17}$$

Combining (3.16) with (3.17), we obtain that

$$\lim_{n\to\infty}\|b_n(T_2(PT_2)^{n-1}z_n - p) + (1-b_n)(x_n - p)\| = c. \tag{3.18}$$

On the other hand, we have

$$\|(T_2(PT_2)^{n-1}z_n - p\| \le k_n^{(2)}\|z_n - p\|.$$

Taking lim sup on both sides in the above inequality and using (3.10), we have

$$\limsup_{n\to\infty}\|T_2(PT_2)^{n-1}z_n - p\| \le c. \tag{3.19}$$

From (3.12),(3.18), (3.19) and applying Lemma 2.5, we have

$$\lim_{n\to\infty}\|T_2(PT_2)^{n-1}z_n - x_n\| = 0. \tag{3.20}$$

Observing that

$$\begin{aligned}\|x_n - p\| &\le \|x_n - T_2(PT_2)^{n-1}z_n\| + \|T2(PT_2)^{n-1}z_n - p\| \\ &\le \|x_n - T_2(PT_2)^{n-1}z_n\| + k_n^{(2)}\|z_n - p\|,\end{aligned}$$

which yields that

$$c \le \liminf_{n\to\infty}\|z_n - p\| \le \limsup_{n\to\infty}\|z_n - p\| \le c,$$

and hence

$$\lim_{n\to\infty}\|z_n - p\| = c.$$

By using the same method, we have

$$\lim_{n\to\infty}\|c_n(T_3(PT_3)^{n-1}x_n - p) + (1-c_n)(x_n - p)\| = c. \tag{3.21}$$

Moreover, we have

$$\|T_3(PT_3)^{n-1}x_n - p)\| \le k_n^{(3)}\|x_n - p\|,$$

which implies that

$$\limsup_{n\to\infty}\|T_3(PT_3)^{n-1}x_n - p\| \le c. \tag{3.22}$$

Combining (3.12),(3.21) with (3.22) and applying Lemma 2.5 , we have

$$\lim_{n\to\infty} \|T_3(PT_3)^{n-1}x_n - x_n)\| = 0. \tag{3.23}$$

Since T_3 is uniformly L_3-Lipschitzian for some $L_3 > 0$, it follows from Lemma 3.1

$$\lim_{n\to\infty} \|x_n - T_3x_n\| = 0.$$

Similarly, we can show that $\lim_{n\to\infty} \|x_n - T_2x_n\| = 0$ and $\|x_n - T_1x_n\| = 0$. □

Lemma 3.4. *Let E be a real uniformly convex Banach space and let K be a nonempty closed convex subset of E. Let $T_i : K \to E$ be asymptotically nonexpansive with sequence $\{k_n^i\} \subset [1,\infty)$ such that $\sum_{n=1}^{\infty}(k_n^i - 1) \leq \infty$ with $F(T) \neq \phi$. From arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ by recursion (2.3) where $\{a_n\},\{b_n\}$ and $\{c_n\}$ be real sequences in $(0,1)$ such that $\epsilon \leq a_n, b_n, c_n \leq 1-\epsilon$ and some $\epsilon \geq 0$. Then for all $p_1, p_2 \in F(T)$,*

$$\lim_{n\to\infty} \|tx_n - (1-t)p_1 - p_2\|$$

exists for all $t \in [0,1]$.

Proof. Let $a_n(t) = \|tx_n + (1-t)p_1 - p_2\|$. Then $\lim_{n\to\infty} a_n(0) = \|p_1 - p_2\|$, from Lemma 3.1 $\lim_{n\to\infty} a_n(1) = \|x_n - p_2\|$ exists. Therefore, it remains to prove this lemma for $t \in (0,1)$. Define $T_i(n) : K \to E$ for $i = 1,2,3$ by

$$\begin{aligned} T_i(n)x = P((1-a_n)x &+ T(PT)^{n-1}((1-b_n)x \\ &+ b_nT(PT)^{n-1}((1-c_n)x + c_nT(PT)^{n-1}x))), \end{aligned}$$

for all $x \in K$. Then

$$\begin{aligned} \|T_i(n)x - T_i(n)y\| &\leq (1+\delta n)\|x-y\| \\ &\leq k_n\|x-y\| \end{aligned}$$

for all $x, y \in k$ where $k_n = (1+\delta_n)$. Since $\sum_{n=1}^{\infty}\delta_n < \infty$. Then $\prod_{n=1}^{\infty} k_n < \infty$. Set

$$S_{n,m} = T_{n+m-1}, T_{n+m-2}...T_{nm} \ m \geq 1.$$

Then

$$\|S_{n,m}\, x - S_{n,m}\, y\| \leq (\prod_{j=1}^{n+m-1} k_j)\|x-y\| \ \forall x, y \in K.$$

Then the rest of the proof follows from the proof of Lemma 3 in [21]. □

Lemma 3.5. *Let E be a real uniformly convex Banach space which has Frechet differentiable norm, K a closed convex nonempty subset of E. Let $T_i : K \to E$ be asymptotically nonexpansive mapping with sequence $\{k_n^i\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n^i - 1) < \infty$ for $i = 1, 2, 3$ and $F(T) \neq \phi$. Let $\{a_n\}, \{b_n\}$ and $\{c_n\}$ be real sequences in $(0, 1)$ such that $\epsilon \leq a_n, b_n, c_n \leq 1 - \epsilon$ and some $\epsilon \geq 0$. From arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ by the recursion (2.3). Then for all $p_1, p_2 \in F(T)$ the*

$$\lim_{n\to\infty} \langle x_n, j(p_1 - p_2)\rangle$$

exits. Furthermore, if $w_w(x_n)$ denotes the set of weak subsequential limits of $\{x_n\}$, then $\langle x^* - y^*, j(p_1 - p_2)\rangle = 0$, for all $p_1, p_2 \in F(T)$ and $x^*, y^* \in w_w(x_n)$.

Proof. This follows basically as in proof of Lemma 3.4 instead of Lemma 3 of [21]. □

4. Weak and strong convergence theorems

In this section, we prove weak and strong convergence theorems.

Theorem 4.1. *Let E be a uniformly convex Banach space and K a nonempty closed convex subset which is also a nonexpansive retract of E. Let $T_i : K \to E$ be an asymptotically nonexpansive mapping with sequence $\{k_n^{(i)}\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n^{(i)} - 1) < \infty, k_n^{(i)} \to 1$, as $n \to \infty$ for $i = 1, 2, 3$. Starting from arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ by the recursion* (2.3), *where $\{a_n\}, \{b_n\}$ and $\{c_n\}$ be real sequences in [0,1] such that $\epsilon \leq a_n, b_n, c_n \leq 1 - \epsilon$ and some $\epsilon > 0$. Assume that one of the following conditions holds:*

1. *E satisfies the Opial property,*
2. *E has Frechet differentiable norm,*
3. *E^* has Kedec -Klee property.*

Then sequence $\{x_n\}$ converges weakly to some $p \in F(T)$.

Proof. For any $p \in F(T)$, it follows from Lemma 3.1 that $\lim_{n\to\infty} \|x_n - p\|$ exists. Since E is reflexive, there exists a subsequence $\{x_{ni}\}$ of $\{x_n\}$ converging weakly to some $p_1 \in K$. Then by Lemma 3.3 and 2.7, $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0$ and $I - T_i$ is demiclosed at 0 for each $i = 1, 2, 3$, respectively. Therefore, we obtain $T_i p_1 = p_1$ for each $i = 1, 2, 3$. That is $p_1 \in F(T)$. In order to show that $\{x_n\}$ converges weakly to p_1, take another subsequence $\{x_{nj}\}$ of $\{x_n\}$ converging weakly to some $p_2 \in K$. Similarly, we can prove that $p_2 \in F(T)$.

Next, we prove that $p_1 = p_2$.

Assume that condition (1) holds, suppose that $p_1 \neq p_2$. Then by Opial property, we obtain

$$\begin{aligned}\lim_{n\to\infty}\|x_n - p_1\| &= \lim_{n_i\to\infty}\|x_{n_i} - p_1\| \\ &< \lim_{n_i\to\infty}\|x_i - p_2\| \\ &= \lim_{n_j\to\infty}\|x_{nj} - p_2\| \\ &< \lim_{n_j\to\infty}\|x_{nj} - p_1\| \\ &= \lim_{n\to\infty}\|x_n - p_1\|.\end{aligned}$$

This is a contradiction of $p_1 \neq p_2$, it implies that $p_1 = p_2$.

Assume that condition (2) holds, from Lemma 3.4, we have

$$\langle p - q, J(p_1 - p_2)\rangle = 0$$

for all $p, q \in w_w(x_n)$. Now

$$\|p_1 - p_2\|^2 = \langle p_1 - p_2, J(p_1 - p_2)\rangle = 0,$$

gives that $p_1 = p_2$.

Finally, let (3) be given. As $\lim_{n\to\infty}\|tx_n + (1-t)p_1 - p_2\|$ exists, therefore by Lemma 2.8, we obtain $p_1 = p_2$. Hence $\{x_n\}$ converges weakly to some point in $F(T)$. □

To prove our strong convergence theorems, we need the following:

Definition 4.2. ([9]) A family $\{T_i : i = 1, 2, \cdots, n\}$ of mappings is said to satisfy Condition $(A^{'})$ if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0, f(r) > 0$ for all $r \in (0, \infty)$ such that

$$\frac{1}{n}\left(\sum_{i=1}^{n}\|x - T_i x\|\right) \geq f(d(x, F(T)))$$

for all $x \in K$, where $d(x, F(T)) = \inf\{\|x - p\| : p \in F(T) = \bigcap_{i=1}^{n} F(T_i) \neq \phi\}$.

It is remarked that the Condition $(A^{'})$ reduces to Condition (A) in ([30]) when $T_i = T$ for all $i = 1, 2, 3 \cdots, n$.

By using Condition $(A^{'})$, for $i = 1, 2, 3$, we obtain the following strong convergence theorem which is a generalization of theorems in [7] and [31].

Theorem 4.3. *Let E be a uniformly convex Banach space and K a nonempty closed convex subset which is also a nonexpansive retract of E. Let $T_i : K \to E$ be an asymptotically nonexpansive mappings with $p \in F(T)$. Let $\{a_n\}, \{b_n\}$ and $\{c_n\}$ be real sequences in [0,1] such that and $\epsilon \leq a_n, b_n, c_n \leq 1 - \epsilon$ for all $n \in N$ and some $\epsilon > 0$. Starting from arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ by the recursion (2.3). Suppose T_i for $i = 1, 2, 3$ satisfy Condition $(A^{'})$. Then $\{x_n\}$ converges strongly to some common fixed point of T_i, for $i = 1, 2, 3$.*

Proof. By Lemma 3.1 $\lim_{n\to\infty}\|x_n-p\|$ exists for all $p\in F(T)$. Let $\lim_{n\to\infty}\|x_n-p\|=c$ for some $c\geq 0$. If $c=0$, there is nothing to prove. Suppose $c>0$. By Lemma 3.3,

$$\lim_{n\to\infty}\|x_n-T_1x_n\|=\lim_{n\to\infty}\|x_n-T_2x_n\|=0$$

and

$$\lim_{n\to\infty}\|x_n-T_3x_n\|=0.$$

By (3.3), we have

$$\begin{aligned}\inf_{p\in f}\|x_{n+1}-p\| \leq \inf_{p\in f}[1&+(k_n^{(1)}-1)(k_n^{(2)}-1)(k_n^{(3)}-1)+(k_n^{(1)}-1)(k_n^{(2)}-1)\\&+(k_n^{(2)}-1)(k_n^{(3)}-1)+(k_n^{(3)}-1)(k_n^{(1)}-1)\\&+(k_n^{(1)}-1)+(k_n^{(2)}-1)+(k_n^{(3)}-1)]\|x_n-p\|.\end{aligned}$$

That is

$$\begin{aligned}d(x_{n+1},F(T))\leq[1&+(k_n^{(1)}-1)(k_n^{(2)}-1)(k_n^{(3)}-1)+(k_n^{(1)}-1)(k_n^{(2)}-1)\\&+(k_n^{(2)}-1)(k_n^{(3)}-1)+(k_n^{(3)}-1)(k_n^{(1)}-1)\\&+(k_n^{(1)}-1)+(k_n^{(2)}-1)+(k_n^{(3)}-1)]d(x_n,F(T)),\end{aligned}$$

gives that $\lim_{n\to\infty}d(x_n,F(T))$ exists by virtue of Lemma 2.6. Now by Condition $(A^{'})$,

$$\lim_{n\to\infty}f(d(x_n,F(T)))=0.$$

Since f is a nondecreasing function and $f(0)=0$, therefore

$$\lim_{n\to\infty}d(x_n,F(T))=0.$$

Now we can take a subsequence $\{x_{nj}\}$ of $\{x_n\}$ and sequence $\{y_j\}\subset F(T)$ such that

$$\|x_{nj}-y_j\|<2^{-j}.$$

Then, by the method of Tan and Xu [36], we know that $\{y_j\}$ is a Cauchy sequence in $F(T)$ and so it converges. Let $y_j\to y$. Since $F(T)$ is closed, therefore $y\in F(T)$ and then $x_{nj}\to y$. As $\lim_{n\to\infty}\|x_n-p\|$ exists, $x_n\to y\in F(T)$, this completes the proof. □

Remark 4.4. Theorems 4.1 and 4.3 are an improvement of Chidume et. al. [7] and Shahzad [31] by setting $c_n=b_n=0$ and $c_n=0$, respectively.

Remark 4.5. Theorems 4.1 and 4.3 are an improvement of Fukhar-ud-din et.al. [10] for three non-self asymptotically maps without error.

Remark 4.6. Theorems 4.1 and 4.3 are also valid for modified Mann iterative scheme [28, 29], and modified Ishikawa iterative scheme [25], by setting P as identity mapping and T is a self mapping.

Remark 4.7. Theorems 4.1 and 4.3 extend the results of Sahu [27] to more general class of non-self maps.

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