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A STUDY ON SUBSTRUCTURES OF R-GROUPS

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Abstract. Throughout this paper, we denote that R is a near-ring and G an R-group. We initiate a study of R-substructures of G, monogenic R-groups, faithful R-groups and faithful D.G. representations of near-rings. Next, we investigate some properties of monogenic R-groups, faithful monogenic R-groups and a generalization of annihilator concepts in R-groups.

1. Introduction

A near-ring R is an algebraic system $(R, +, \cdot)$ with two binary operations $+$ and $\cdot$ such that $(R, +)$ is a group (not necessarily abelian) with neutral element 0, $(R, \cdot)$ is a semigroup and $(a+b)c = ac + bc$ for all a, b, c in R. If R has a unity 1, then R is called *unitary*. A near-ring R with the extra axiom $a0 = 0$ for all $a \in R$ is said to be *zero symmetric*. An element d in R is called *distributive* if $d(a+b) = da + db$ for all a and b in R.

A (two-sided) *ideal* of R is a subset I of R such that (i) $(I, +)$ is a normal subgroup of $(R, +)$, (ii) $a(I+b) - ab \subset I$ for all $a, b \in R$, (iii) $(I+a)b - ab \subset I$ for all $a, b \in R$, equivalently, $IR \subset I$. If I satisfies (i) and (ii) then it is called a *left ideal* of R. If I satisfies (i) and (iii) then it is called a *right ideal* of R.

On the other hand, a *(two-sided) R-subgroup* of R is a subset H of R such that (i) $(H, +)$ is a subgroup of $(R, +)$, (ii) $RH \subset H$ and (iii) $HR \subset H$. If H satisfies (i) and (ii) then it is called a *left R-subgroup* of R. If H satisfies (i) and (iii) then it is called a *right R-subgroup* of R. In case, $(H, +)$ is normal in above, we say that *normal R-subgroup*, *normal left R-subgroup* and *normal right R-subgroup* instead of R-subgroup, left R-subgroup and right R-subgroup, respectively. Note that normal right R-subgroups of R are the same as right ideals of R.

Also, a subset H of R together with (i) $RH \subset H$ and (ii) $HR \subset H$ is called an *R-subset* of R. If this H satisfies (i) then it is called a *left R-subset* of R, and H satisfies (ii) then it is called a *right R-subset* of R.

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We will use the following notations: Given a near-ring R,

$$R_0 = \{a \in R \mid a0 = 0\}$$

which is called the *zero symmetric part* of R,

$$R_c = \{a \in R \mid a0 = a\}$$

which is called the *constant part* of R.

Obviously, we see that R_0 and R_c are subnear-rings of R. Clearly, near-ring R is zero symmetric, in case $R = R_0$ also, in case $R = R_c$, R is called a *constant* near-ring. From the Pierce decomposition theorem, we obtain that

$$R = R_0 \oplus R_c$$

as additive groups. So every element $a \in R$ has a unique representation of the form $a = b + c$, where $b \in R_0$ and $c \in R_c$.

Let $(G, +)$ be a group (not necessarily abelian). In the set

$$M(G) := \{f \mid f : G \longrightarrow G\}$$

of all the self maps of G, if we define the sum $f + g$ of any two mappings f, g in $M(G)$ by the rule $(f + g)x = fx + gx$ for all $x \in G$ and the product $f \cdot g$ by the rule $(f \cdot g)x = f(gx)$ for all $x \in G$, here, for convenience we write the image of f at a variable x, fx instead of $f(x)$, then $(M(G), +, \cdot)$ becomes a near-ring. It is called the *self map near-ring on the group* G. Also, if we define the set

$$M_0(G) := \{f \in M(G) \mid f0 = 0\},$$

then $(M_0(G), +, \cdot)$ is a zero symmetric near-ring.

Let R and S be two near-rings. Then a mapping θ from R to S is called a *near-ring homomorphism* if (i) $\theta(a + b) = \theta a + \theta b$, (ii) $\theta(ab) = \theta a \theta b$. We can replace homomorphism by momomorphism, epimorphism, isomorphism, endomorphism and automorphism, if these terms have their usual meanings as for rings ([1]).

Let R be any near-ring and G an additive group with an identity o. Then G is called an *R-group* if there exists a near-ring homomorphism

$$\theta : (R, +, \cdot) \longrightarrow (M(G), +, \cdot).$$

Such a homomorphism θ is called a *representation of* R on G, we write that rx (left scalar multiplication by R) for $(\theta r)x$ for all $r \in R$ and $x \in G$. If R is unitary and $\theta 1 = 1_G$, then R-group G is called *unitary*. That is, an R-group is an additive group G satisfying (i) $(a + b)x = ax + bx$, (ii) $(ab)x = a(bx)$ and (iii) $1x = x$ (if R has a unity 1), for all $a,\ b \in R$ and $x \in G$.

Evidently, every near-ring R can be given the structure of an R-group (unitary if R is unitary) by left multiplication by R. Moreover, clearly, every group G has an $M(G)$-group structure, from the definition of $M(G)$ on G given by applying the $f \in M(G)$ to the $x \in G$ as a scalar multiplication fx.

A representation θ of R on G is called $faithful$ if $\ker\theta = \{0\}$, that is, $rx = o$ implies that $r = 0$. In this case, also we say that G is a *faithful R-group* or R *acts faithfully on G*.

For an R-group G, a non-empty subset X of G such that $RX \subset X$ is called an *R-subset* of G, a subgroup T of G such that $RT \subset T$ is called an *R-subgroup* of G, a normal subgroup N of G such that $RN \subset N$ is called a *normal R-subgroup* of G and an *R-ideal* of G is a normal subgroup N of G such that

$$a(N + x) - ax \subset N$$

for all $x \in G,\ a \in R$. Also, note that every R-ideal of R is a left ideal of R, but normal R-subgroups of R are not equivalent to R-ideals of R.

Let R be a near-ring and let G be an R-group. If there exists x in G such that $G = Rx$, that is,

$$G = \{rx \mid r \in R\},$$

then G is called a *monogenic R-group* and the element x is called a *generator* of G, more specially, if G is monogenic and for each $x \in G$, $Rx = o$ or $Rx = G$, then G is called a *strongly monogenic R-group.*

For the remainder concepts and results on near-rings, we refer to [6] and [7].

2. Some properties of faithful monogenic R-Groups

A near-ring R is called *distributively generated* (briefly, *d.g.*) by S if

$$(R, +) = gp < S >$$

where S is a semigroup of distributive elements in R (the d.g. concept is motivated by the set of all distributive elements of R which is multiplicatively closed), and $gp < S >$ is an additive group generated by S, we denote it by (R, S). On the other hand, the set of all distributive elements of $M(G)$ are the semigroup $\mathrm{End}(G)$ of all endomorphisms on the group G under composition. We denote that $E(G)$ is the d.g. near-ring generated by $\mathrm{End}(G)$. It is said to be that $E(G)$ is the *endomorphism near-ring* of the group G.

Let (R, S) and (T, U) be d.g. near-rings. Then a near-ring homomorphism

$$\theta : (R, S) \longrightarrow (T, U)$$

is called a *d.g. near-ring homomorphism* if $\theta S \subseteq U$. Note that a semigroup homomorphism $\theta : S \longrightarrow U$ is a d.g. near-ring homomorphism if it is a group homomorphism from $(R, +)$ to $(T, +)$ (C.G. Lyons and J.D.P. Meldrum [3], [4]).

Let (R, S) be a d.g. near-ring. Then an additive group G is called a *d.g. (R, S)-group* if there exists a d.g. near-ring homomorphism

$$\theta : (R, S) \longrightarrow (E(G), \mathrm{End}\,(G))$$

such that $\theta S \subseteq \mathrm{End}(G)$.

If we write rx instead of $(\theta r)x$ for all $x \in G$ and $r \in R$, then a d.g. (R, S)-group is an additive group G satisfying the following conditions:

$$(r+s)x = rx + sx,$$
$$(rs)x = r(sx),$$

for all $x \in G$ and all $r, s \in R$,

$$s(x+y) = sx + sy,$$

for all $x, y \in G$ and all $s \in S$.

Such a homomorphism θ is called a *d.g. representation* of (R, S). This d.g. representation is said to be *faithful* if $\ker \theta = \{0\}$. In this case, G is called a *faithful d.g.* (R, S)*-group.*

Lemma 2.1. *Every distributive near-ring R with $R^2 = R$ is a ring.*

Proof. It is sufficient to show that $(R, +)$ is abelian. Let $x, y \in R$. Then there exist $a, b, c, d \in R$ such that $x = ab$, $y = cd$. If we calculate $(c+a)(b+d)$ by two distributive laws, we have that $(c+a)b + (c+a)d = c(b+d) + a(b+d)$, that is,

$$cb + ab + cd + ad = cb + cd + ab + ad$$

From this equation, we see $x + y = y + x$. Hence $(R, +)$ is abelian. □

Proposition 2.2. If R is a distributive near-ring with unity 1, then R is a ring. Furthermore, if R is a distributive near-ring with unity 1, then every d.g. (R, R)-group is a unitary R-module.

Proof. The first statement is clear by Lemma 2.1, because of $R^2 = R$.

Let G be a d.g. (R, R)-group. Since G is unitary, $x(1+1) = x + x$, for all $x \in G$. Thus we see

$$x + y + x + y = (x+y)(1+1) = x(1+1) + y(1+1) = x + x + y + y,$$

for all $x, y \in G$. This implies that $(G, +)$ is abelian. Since $R = S$, the set of all distributive elements, $r(x+y) = rx + ry$, for all $x, y \in G$ and all $r \in R$. Hence G becomes a unitary R-module. □

Obviously, we get the following statement.

Proposition 2.3. Let (R, S) be a d.g. near-ring. Then all R-subgroups and all R-homomorphic images of a d.g. (R, S)-group are also d.g. (R, S)-groups.

Now, we consider the following special substructures of R and G.

Let G be an R-group and X and Y be non-empty subsets of G. We can define the following.

$$(X : Y) := \{a \in R \mid aX \subset Y\}.$$

We abbreviate that for $x \in G$

$$(\{x\} : Y) =: (x : Y).$$

Similarly for $(Y : x)$.

$(X : o)$ is called the *annihilator* of X, denoted it by $\mathrm{Ann}(X)$. We note that G is a faithful R-group if $(G : o) = \{0\}$, that is, $\mathrm{Ann}(G) = \{0\}$.

In the above notation, note that if Y is a subgroup (normal subgroup, R-subgroup, ideal) of G, then so is $(X : Y)$ in R as an R-group. Moreover, we have the following simple statements.

Proposition 2.4. Let G be an R-group and K_1 and K_2 non-empty subsets of G. Then we have the following conditions:

(1) If K_2 is a normal R-subgroup of G, then $(K_1 : K_2)$ is a normal left R-subgroup of R.

(2) If K_1 is an R-subset of G and K_2 is an R-subgroup of G, then $(K_1 : K_2)$ is a two-sided R-subgroup of R.

(3) If K_1 is an R-subset of G and K_2 is an R-ideal of G, then $(K_1 : K_2)$ is a two-sided ideal of R.

Proof. (1) and (2) are easily proved by simple calculation.

Now, we prove only (3) : Using the condition (1), $(K_1 : K_2)$ is a normal subgroup of R. Let $a \in (K_1 : K_2)$ and $r \in R$. Then

$$(ar)K_1 = a(rK_1) \subset aK_1 \subset K_2,$$

because K_1 is an R-subset of G, so that $ar \in (K_1 : K_2)$. Whence $(K_1 : K_2)$ is a right ideal of R.

Next, let $r_1, r_2 \in R$ and $a \in (K_1 : K_2)$. Then

$$\{r_1(a + r_2) - r_1r_2\}k = r_1(ak + r_2k) - r_1r_2k \in K_2$$

for all $k \in K_1$, since $aK_1 \subset K_2$ and K_2 is an R-ideal of G. Thus $(K_1 : K_2)$ is a left ideal of R. Therefore $(K_1 : K_2)$ is a two-sided ideal of R.

□

Corollary 2.5. *Let R be a near-ring and G an R-group.*

(1) ([6]) For any $x \in G$, $(x : o)$ is a left ideal of R.

(2) ([6]) For any R-subgroup K of G, $(K : o)$ is a two-sided ideal of R.

(3) For any R-subset K of G, $(K : o)$ is a two-sided ideal of R.

(4) For any subset K of G, $(K : o) = \bigcap_{x \in K}(x : o)$.

Lemma 2.6. *([7])* **(Homomorphism theorems)**

(1) Let I be a two-sided ideal of a near-ring R. Then the canonical map $\pi : R \to R/I$ is a near-ring epimorphism. So R/I is a homomorphic image of R, and $\ker \pi = I$.

(2) Let map $\phi : R \to S$ be a near-ring epimorphism. Then $\ker \phi$ is a two-sided ideal of R and $R/\ker \phi \cong S$.

Theorem 2.7. *Let R be a near-ring and G an R-group. Then we have the following conditions:*

(1) $\mathrm{Ann}(G)$ is a two-sided ideal of R. Moreover G is a faithful $R/\mathrm{Ann}(G)$-group.

(2) For any $x \in G$, we get $Rx \cong R/(x:o)$ as R-groups.

Proof. (1) By Corollary 2.5 and Proposition 2.4, $\mathrm{Ann}(G) = (G:o)$ is a two-sided ideal of R.

We now make G an $R/\mathrm{Ann}(G)$-group by defining, for all $r \in R$ *and* $r + \mathrm{Ann}(G) \in R/\mathrm{Ann}(G)$, the action $(r + \mathrm{Ann}(G))x = rx$. If $r + \mathrm{Ann}(G) = r' + \mathrm{Ann}(G)$, then $-r' + r \in \mathrm{Ann}(G)$ hence $(-r' + r)x = o$ for all x in G, that is to say, $rx = r'x$. This tells us that

$$(r + \mathrm{Ann}(G))x = rx = r'x = (r' + \mathrm{Ann}(G))x;$$

thus the action of $R/\mathrm{Ann}(G)$ on G has been shown to be well defined.

The verification of the structure of an $R/\mathrm{Ann}(G)$-group is a routine triviality. Finally, to see that G is a faithful $R/\mathrm{Ann}(G)$-group, we note that if $(r + \mathrm{Ann}(G))x = 0$ for all $x \in G$, then by the definition of $R/\mathrm{Ann}(G)$-group structure, we have $rx = 0$. Hence $r \in \mathrm{Ann}(G)$. This says that only the zero element of $R/\mathrm{Ann}(G)$ annihilates all of G. Thus G is a faithful $R/\mathrm{Ann}(G)$-group.

(2) For any $x \in G$, clearly Rx is an R-subgroup of G. The map $\phi : R \longrightarrow Rx$ defined by $\phi(r) = rx$ is an R-epimorphism, so that from Lemma 2.6, since the kernel of ϕ is $(x:o)$, we deduce that

$$Rx \cong R/(x:o)$$

as R-groups.

□

Corollary 2.8. *Let G be a monogenic R-group with x as a generator. Then we have the following isomorphic relation.*

$$G \cong R/(x:o).$$

Theorem 2.9. *If R is a near-ring and G an R-group, then $R/\mathrm{Ann}(G)$ is embedded in a near-ring $M(G)$.*

Proof. Let $a \in R$. We define $\tau_a : G \longrightarrow G$ by $\tau_a x = ax$ for each $x \in G$. Then τ_a is in $M(G)$. Consider the mapping $\phi : R \longrightarrow M(G)$ defined by $\phi(a) = \tau_a$. Then obviously, we see that from the definition of τ_a

$$\phi(a+b) = \phi(a) + \phi(b) \text{ and } \phi(ab) = \phi(a)\phi(b),$$

that is, ϕ is a near-ring homomorphism from R to $M(G)$.

Next, we must show that $\ker\phi = \mathrm{Ann}(G)$: Indeed, if $a \in \ker\phi$, then $\tau_a = O$, which implies that $aG = \tau_a G = o$, that is, $a \in \mathrm{Ann}(G)$. On the other hand, if $a \in \mathrm{Ann}(G)$, then by the definition of $\mathrm{Ann}(G)$, $aG = o$ hence $O = \tau_a = \phi(a)$, this implies that $a \in \ker\phi$. Therefore from Lemma 2.6 on R-groups, the image of R is a near-ring isomorphic to $R/\mathrm{Ann}(G)$. Consequently, $R/\mathrm{Ann}(G)$ is isomorphic to a subnear-ring of $M(G)$.

□

From Theorem 2.9, we obtain the important statement of the fact that if G is a faithful R-group, then R is embedded in $M(G)$, as in ring theory.

Corollary 2.10. *If (R, S) is a d.g. near-ring, then every monogenic R-group is a d.g. (R, S)-group.*

Proof. Let G be a monogenic R-group with x as a generator. Then the map $\phi : r \mapsto rx$ is an R-epimorphism from R to G as R-groups. We see that by the Corollary 2.8, $G \cong R/\operatorname{Ann}(x)$, where $\operatorname{Ann}(x) = (x : o) = \ker \phi$. From Proposition 2.3, we see that G is a d.g. (R, S)-group. □

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