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# ON NONLINEAR VARIATIONAL INCLUSIONS WITH $(A,\eta)$-MONOTONE MAPPINGS

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Abstract. In this paper, we introduce a generalized system of nonlinear relaxed co-coercive variational inclusions involving $(A,\eta)$-monotone mappings in the framework of Hilbert spaces. Based on the generalized resolvent operator technique associated with $(A,\eta)$-monotonicity, we consider the approximation solvability of solutions to the generalized system. Since $(A,\eta)$-monotonicity generalizes $A$-monotonicity and $H$-monotonicity, The results presented this paper improve and extend the corresponding results announced by many others.

## 1. Introduction

Variational inclusions problems are among the most interesting and intensively studied classes of mathematical problems and have wide applications in the fields of optimization and control, economics and transportation equilibrium and engineering sciences. Variational inclusions problems have been generalized and extended in different directions using the novel and innovative techniques. Various kinds of iterative algorithms to solve the variational inequalities and variational inclusions have been developed by many authors. There exists a vast literature [1-26] on the approximation solvability of nonlinear variational inequalities as well as nonlinear variational inclusions using projection type methods, resolvent operator type methods or averaging techniques. In most of the resolvent operator methods, the maximal monotonicity has played a key role, but more recently introduced notions of $A$-monotonicity [22] and $H$-monotonicity [6,7] have not only generalized the maximal monotonicity, but gave a new edge to resolvent operator methods. Recently, Verma [20] generalized the recently introduced and studied notion of $A$-monotonicity to the case of $(A,\eta)$-monotonicity, while examining the sensitivity analysis for a class of nonlinear variational inclusion problems based on the generalized resolvent operator technique. Resolvent operator techniques have been in use for a while in literature, especially with the general framework involving

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set-valued maximal monotone mappings, but it got a new empowerment by the recent developments of $A$-monotonicity and $H$-monotonicity. Furthermore, these developments added a new dimension to the existing notion of the maximal monotonicity and its applications to several other fields such as convex programming and variational inclusions. In this paper, inspired and motivated by the recent research going on in this area, we explore the approximation solvability of a generalized system of nonlinear variational inclusion problems in the framework Hilbert spaces.

## 2. Preliminaries

In this section, we explore some basic properties derived from the notion of $(A, \eta)$-monotonicity. Let $X$ be a real Hilbert space with the norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$, respectively. Let $\eta : X \times X \to X$ be a single-valued mapping. The mapping $\eta$ is said to be $\tau$-Lipschitz continuous if there is a constant $\tau > 0$ such that

$$\|\eta(u, v)\| \leq \tau\|y - v\|, \quad \forall u, v \in X.$$

Let $M$ be a multi-valued mapping from a Hilbert space $X$ to $2^X$, the power set of $X$. We recall following:

(i) The set $D(M)$ defined by

$$D(M) = \{u \in X : M(u) \neq \emptyset\},$$

is called the effective domain of $M$.

(ii) The set $R(M)$ defined by

$$R(M) = \bigcup_{u \in X} M(u),$$

is called the range of $M$.

(iii) The set $G(M)$ defined by

$$G(M) = \{(u, v) \in X \times X : u \in D(M), v \in M(u)\},$$

is the graph of $M$.

**Definition 1.** Let $\eta : X \times X \to X$ be a single-valued mapping and let $M : X \to 2^X$ be a multi-valued mapping on $X$.

(i) The mapping $M$ is said to be $(r, \eta)$-strongly monotone if

$$\langle u^* - v^*, \eta(u, v)\rangle \geq r\|u - v\|, \quad \forall (u, u^*), (v, v^*) \in G(M).$$

(ii) The mapping $M$ is said to be $\eta$-pseudo-monotone if $\langle v^*, \eta(u, v)\rangle \geq 0$ implies

$$\langle u^*, \eta(u, v)\rangle \geq 0, \quad \forall (u, u^*), (v, v^*) \in G(M).$$

(iii) The mapping $M$ is said to be $(m, \eta)$-relaxed monotone if there exists a positive constant $m$ such that

$$\langle u^* - v^*, \eta(u, v)\rangle \geq -m\|u - v\|^2, \quad \forall (u, u^*), (v, v^*) \in G(M).$$

**Definition 2.** (see [6,7]). Let $H : X \to X$ be a nonlinear mapping on a Hilbert space $X$ and let $M : X \to 2^X$ be a multi-valued mapping on $X$. The mapping $M$ is said to be $H$-monotone if $(H + \rho M)X = X$ for all $\rho > 0$.

**Definition 3.** (see [22]). Let $A : X \to X$ be a nonlinear mapping on a Hilbert space $X$ and let $M : X \to 2^X$ be a multi-valued mapping on $X$. The mapping $M$ is said to be $A$-monotone if

(i) $M$ is $m$-relaxed monotone;
(ii) $A + \rho M$ is maximal monotone for all $\rho > 0$.

*Remark* 1. $A$-monotonicity which was introduce by Verma [22] generalizes the notion of $H$-monotonicity introduced by Fang and Huang [6,7].

**Definition 4.** (see [20]). A mapping $M : X \to 2^X$ is said to be maximal $(m, \eta)$-relaxed monotone if

(i) $M$ is $(m, \eta)$-relaxed monotone;
(ii) for $(u, u^*) \in X \times X$ and

$$\langle u^* - v^*, \eta(u, v)\rangle \geq -m\|u - v\|^2, \quad (v, v^*) \in G(M),$$

we have $u^* \in M(u)$.

**Definition 5.** (see [20]). Let $A : X \to X$ and $\eta : X \times X \to X$ be two single-valued mappings. The mapping $M : X \to 2^X$ is said to be $(A, \eta)$-monotone if

(i) $M$ is $(m, \eta)$-relaxed monotone;
(ii) $R(A + \rho M) = X$ for all $\rho > 0$.

Note that alternatively, the map $M : X \to 2^X$ is said to be $(A, \eta)$-monotone if

(i) $M$ is $(m, \eta)$-relaxed monotone;
(ii) $A + \rho M$ is $\eta$-pseudo-monotone for $\rho > 0$.

*Remark* 2. $(A, \eta)$-monotonicity which was introduced by Verma [20] generalizes the notion of $A$-monotonicity.

**Definition 6.** (see [20]). Let $A : X \to X$ be an $(r, \eta)$-strong monotone mapping and let $M : X \to X$ be an $(A, \eta)$-monotone mapping. Then the generalized resolvent operator $J_{M,\rho}^{A,\eta} : X \to X$ is defined by

$$J_{M,\rho}^{A,\eta}(u) = (A + \rho M)^{-1}(u), \quad \forall u \in X,$$

where $\rho > 0$ is a constant.

**Definition 7.** (see [18]). The mapping $T : X \times X \to X$ is said to be relaxed $(\beta, \gamma)$-co-coercive with respect to $A$ in the first argument if there exists two positive constants $\alpha, \beta$ such that

$$\langle T(x, u) - T(y, u), Ax - Ay\rangle \geq (-\beta)\|T(x, u) - T(y, u)\|^2 + \gamma\|x - y\|^2,$$

for all $(x, y, u) \in X \times X \times X$.

**Proposition 2.1.** *(see [20]). Let $A : X \to X$ be an $r$-strongly monotone mapping and let $M : X \to 2^X$ be an $A$-monotone mapping. Then the operator $(A + \rho M)^{-1}$ is single-valued.*

**Proposition 2.2.** *(see [20]). Let $\eta : X \times \to X$ be a single-valued mapping, $A : X \to X$ be $(r, \eta)$-strongly monotone mapping and $M : X \to 2^X$ be an $(A, \eta)$-monotone mapping. Then the mapping $(A + \rho M)^{-1}$ is single-valued.*

## 3. Results on algorithmic convergence analysis

Let $N_1, N_2 : X \times X \to X$, $\eta_1, \eta_2 : X \times X \to X$ $g_1, g_2 : X \to X$ be nonlinear mappings. Let $M_1 : X \to 2^X$ be an $(A_1, \eta_1)$-monotone mapping and $M_2 : X \to 2^X$ an $(A_2, \eta_2)$-monotone mapping, respectively. Consider the the following nonlinear system of variational inclusions (NSVI) problem: determine elements $(u, v) \in X \times X$ such that

$$0 \in A_1 g_1(u) - A_1 g_1(v) + \rho_1[N_1(v, u) + M_1 g_1(u)], \tag{3.1}$$

$$0 \in A_2 g_2(v) - A_2 g_2(u) + \rho_2[N_2(u, v) + M_2 g_2(v)]. \tag{3.2}$$

Next, we consider some special cases of NSVI problem (3.1)-(3.2).

(I) If $A_1 = A_2 = A$, $M_1 = M_2 = M$, $g_1 = g_2 = g$ and $N_1 = N_2 = N$, then NSVI problem (3.1)-(3.2) reduces to the following NSVI problem: find $(u, v) \in X \times X$ such that

$$0 \in Ag(u) - Ag(v) + \rho_1[N(v, u) + Mg(u)], \tag{3.3}$$

$$0 \in Ag(v) - Ag(u) + \rho_2[N(u, v) + Mg(v)]. \tag{3.4}$$

(II) If $A_1 = A_2 = A$, $M_1 = M_2 = M$, $g_1 = g_2 = I$ and $N_1 = N_2 = N$, then NSVI problem (3.1)-(3.2) reduces to the following NSVI problem: find $(u, v) \in X \times X$ such that

$$0 \in Au - Av + \rho_1[N(v, u) + Mu], \tag{3.5}$$

$$0 \in Av - Au + \rho_2[N(u, v) + Mv]. \tag{3.6}$$

(III) If $M_1 = M_2 = M$, $N_1 = N_2 = N$, $u = v$, $g_1 = g_2 = I$ and $\rho_1 = \rho_2 = \rho$ in NSVI (3.1)-(3.2), we have the following NVI problem: find an element $u \in X$ such that

$$0 \in N(u, u) + Mu, \tag{3.7}$$

In order to prove our main results, we need the following lemmas.

**Lemma 3.1.** *Let $X$ be a real Hilbert space and $\eta : X \times X \to X$ a $\tau$-Lipschitz continuous mapping. Let $A : X \to X$ be a $(r, \eta)$-strongly monotone mapping and $M : X \to 2^X$ a $(A, \eta)$-monotone mapping. Then the generalized resolvent operator $J^{A,\eta}_{M,\rho} : H \to H$ is $\tau/(r - \rho m)$, that is,*

$$\|J^{A,\eta}_{M,\rho}(x) - J^{A,\eta}_{M,\rho}(y)\| \le \frac{\tau}{r - \rho m}\|x - y\|, \quad \forall x, y \in X.$$

**Lemma 3.2.** *Let $X$ be a real Hilbert space, $A_i : H \to H$ a $(r_i, \eta_i)$-strongly monotone mapping and $M_i : H \to 2^H$ a $(A_i, \eta_i)$-monotone mapping. Let $\eta_i : H \times H \to H$ be a $\tau_i$-Lipschitz continuous mapping for each $i = 1, 2$. Then $(u, v)$ is the solution of NSVI (3.1)-(3.2) if and only if it satisfies*

$$g_1(u) = J^{A_1,\eta_1}_{M_1,\rho_1}[A_1 g_1(v) - \rho_1 N_1(v, u)], \tag{3.9}$$

$$g_2(v) = J^{A_2,\eta_2}_{M_2,\rho_2}[A_2 g_2(u) - \rho_2 N_2(u, v)]. \tag{3.10}$$

Next, we give the iterative algorithms in this work.

**Algorithm 3.1.** For any $(u_0, v_0) \in X \times X$, compute the sequences $\{u_n\}$ and $\{v_n\}$ by the iterative process:

$$\begin{cases} u_{n+1} = u_n - g_1(u_n) + J^{A_1,\eta_1}_{M_1,\rho_1}[A_1 g_1(v_n) - \rho_1 N_1(v_n, u_n)], \\ g_2(v_n) = J^{A_2,\eta_2}_{M_2,\rho_2}[A_2 g_2(u_n) - \rho_2 N_2(u_n, v_n)], \quad n \geq 0. \end{cases}$$

(I) If $A_1 = A_2 = A$, $M_1 = M_2 = M$, $\eta_1 = \eta_2$, $g_1 = g_2 = g$ and $N_1 = N_2 = N$ in Algorithm 3.1, then we have the following algorithm:

**Algorithm 3.2.** For any $(u_0, v_0) \in X \times X$, compute the sequences $\{u_n\}$ and $\{v_n\}$ by the iterative process:

$$\begin{cases} u_{n+1} = u_n - g(u_n) + J^{A,\eta}_{M,\rho_1}[Ag(v_n) - \rho_1 N(v_n, u_n)], \\ g(v_n) = J^{A,\eta}_{M,\rho_2}[Ag(u_n) - \rho_2 N(u_n, v_n)], \quad n \geq 0. \end{cases}$$

(II) If $A_1 = A_2 = A$, $M_1 = M_2 = M$, $\eta_1 = \eta_2$, $g_1 = g_2 = I$ and $N_1 = N_2 = N$ in Algorithm 3.1, then we have the following algorithm:

**Algorithm 3.3.** For any $(u_0, v_0) \in X \times X$, compute the sequences $\{u_n\}$ and $\{v_n\}$ by the iterative processes:

$$\begin{cases} u_{n+1} = J^{A,\eta}_{M,\rho_1}[Av_n - \rho_1 N(v_n, u_n)], \\ v_n = J^{A,\eta}_{M,\rho_2}[Au_n - \rho_2 N(u_n, v_n)], \quad n \geq 0. \end{cases}$$

(III) If $M_1 = M_2 = M$, $N_1 = N_2 = N$, $\eta_1 = \eta_2$, $u = v$ and $\rho_1 = \rho_2 = \rho$ in Algorithm 3.1, then we have the following algorithm:

**Algorithm 3.4.** For any $u_0 \in X$, compute the sequence $\{u_n\}$ by the iterative processes:

$$u_{n+1} = J^{A,\eta}_{M,\rho}[Au_n - \rho N(u_n, u_n)], \quad n \geq 0.$$

Now, we are in a position to prove our main results.

**Theorem 3.3.** *Let $X$ be a real Hilbert space, $A_i : X \times X$ a $(r_i, \eta_i)$-strongly monotone and $s_i$-Lipschitz continuous mapping and $M_i : X \to 2^X$ a $(A_i, \eta_i)$-monotone mapping for each $i = 1, 2$, respectively. Let $\eta_i : X \times X \to X$ be a $\tau_i$-Lipschitz continuous mapping. Let $N_i : X \times X \to X$ be relaxed $(\alpha_i, \beta_i)$-co-coercive (with respect to $A_i g_i$) and $\mu_i$-Lipschitz continuous in the first variable. Let $N_i$ be a $\nu_i$-Lipschitz continuous mapping in the second variable and $g_i : X \to X$ a relaxed $(\gamma_i, \delta_i)$-co-coercive and $\sigma_i$-Lipschitz mapping for each $i =$*

*1, 2. Assume that $\Omega_1 \neq \emptyset$, where $\Omega_1$ denotes the set of solutions to the NSVI problem (3.1)-(3.2). Let $(u^*, v^*) \in \Omega_1$, $\{u_n\}$ and $\{v_n\}$ be sequences generated by Algorithm 3.1. Suppose that the following conditions are satisfied:*

$$\frac{\tau_1\tau_2\theta_1\theta_2}{(r_1-\rho_1 m_1)[(1-\theta_3)(r_2-\rho_2 m_2)-\tau_2\rho_2\nu_2]}+\frac{\tau_1\rho_1\nu_1}{r_1-\rho_1 m_1}<1-\theta_4,$$

*where*

$$\theta_1=\sqrt{\sigma_1^2 s_1^2-2\rho_1\beta_1+2\rho_1\alpha_1\mu_1^2+\rho_1^2\mu_1^2},$$

$$\theta_2=\sqrt{\sigma_2^2 s_2^2-2\rho_2\beta_2+2\rho_2\alpha_2\mu_2^2+\rho_2^2\mu_2^2},$$

$$\theta_3=\sqrt{1+2\sigma_2^2\gamma_2-2\delta_2+\sigma_2^2}$$

*and*

$$\theta_4=\sqrt{1+2\sigma_1^2\gamma_1-2\delta_1+\sigma_1^2}.$$

*Then the sequences $\{u_n\}$ and $\{v_n\}$ converge strongly to $u^*, v^*$, respectively.*

*Proof.* By the assumption, we have

$$\begin{cases} u^* = u^* - g_1(u^*) + J_{M_1,\rho_1}^{A_1,\eta_1}[A_1 g_1(v^*) - \rho_1 N_1(v^*, u^*)], \\ g_2(v^*) = J_{M_2,\rho_2}^{A_2,\eta_2}[A_2 g_2(u^*) - \rho_2 N_2(u^*, v^*)]. \end{cases}$$

It follows that

$$\begin{aligned}
&\|u_{n+1}-u^*\| \\
&= \|u_n - g_1(u_n) + J_{M_1,\rho_1}^{A_1,\eta_1}[A_1 g_1(v_n) - \rho_1 N_1(v_n,u_n)] - u^*\| \\
&= \|u_n - g_1(u_n) + J_{M_1,\rho_1}^{A_1,\eta_1}[A_1 g_1(v_n) - \rho_1 N_1(v_n,u_n)] - u^* + g_1(u^*) \\
&\quad - J_{M_1,\rho_1}^{A_1,\eta_1}[A_1 g_1(v^*) - \rho_1 N_1(v^*,u^*)]\| \\
&\le \|u_n - u^* - [g_1(u_n) - g_1(u^*)]\| \\
&\quad + \|J_{M_1,\rho_1}^{A_1,\eta_1}[A_1 g_1(v_n) - \rho_1 N_1(v_n,u_n)] - J_{M_1,\rho_1}^{A_1,\eta_1}[A_1 g_1(v^*) - \rho_1 N_1(v^*,u^*)]\| \\
&\le \|u_n - u^* - [g_1(u_n) - g_1(u^*)]\| \\
&\quad + \frac{\tau_1}{r_1-\rho_1 m_1}\|A_1 g_1(v_n) - A_1 g_1(v^*) - \rho_1[N_1(v_n,u_n) - N_1(v^*,u_n)] \\
&\quad - \rho_1[N_1(v^*,u_n) - N_1(v^*,u^*)]\|.
\end{aligned} \tag{3.11}$$

It follows from relaxed $(\alpha_1, \beta_1)$-cocoercive monotonicity and $\mu_1$-Lipschitz continuity of $N_1$ in the first variable, $A_1$ is $s_1$-Lipschitz continuous and $g_1$ is $\sigma_1$-Lipschitz continuous that

$$\begin{aligned}
&\|A_1g_1(v_n) - A_1g_1(v^*) - \rho_1(N_1(v_n, u_n) - N_1(v^*, u_n))\|^2 \\
&= \|A_1g_1(v_n) - A_1g_1(v^*)\|^2 \\
&\quad - 2\rho_1\langle N_1(v_n, u_n) - N_1(v^*, u_n), A_1g_1(v_n) - A_1g_1(v^*)\rangle \\
&\quad + \rho_1^2\|N_1(v_n, u_n) - N_1(v^*, u_n)\|^2 \\
&\le \theta_1^2\|v_n - v^*\|^2,
\end{aligned} \tag{3.12}$$

where $\theta_1 = \sqrt{\sigma_1^2 s_1^2 - 2\rho_1\beta_1 + 2\rho_1\alpha_1\mu_1^2 + \rho_1^2\mu_1^2}$. Observe that the $\nu_1$-Lipschitz continuity of $N_1$ in the second argument yields that

$$\|N_1(v^*, u_n) - N_1(v^*, u^*)\| \le \nu_1\|u_n - u^*\|. \tag{3.13}$$

On the other hand, we have

$$\begin{aligned}
&\|g_2(v_n) - g_2(v^*)\| \\
&= \|J_{M_2,\rho_2}^{A_2,\eta_2}[A_2g(u_n) - \rho_2N_2(u_n, v_n)] - J_{M_2,\rho_2}^{A_2,\eta_2}[A_2g_2(u^*) - \rho_2N_2(u^*, v^*)]\| \\
&\le \frac{\tau_2}{r_2 - \rho_2m_2}\|A_2g(u_n) - A_2g_2(u^*) - \rho_2[N_2(u_n, v_n) - N_2(u^*, v^*)]\| \\
&\le \frac{\tau_2}{r_2 - \rho_2m_2}\|A_2g(u_n) - A_2g_2(u^*) - \rho_2[N_2(u_n, v_n) - N_2(u^*, v_n)] \\
&\quad - \rho_2[N_2(u^*, v_n) - N_2(u^*, v^*)]\|.
\end{aligned} \tag{3.14}$$

It follows from relaxed $(\alpha_2, \beta_2)$-cocoercive monotonicity and $\mu_2$-Lipschitz continuity of $N_2$ in the first variable, $A_2$ is $s_2$-Lipschitz continuous and $g_2$ is $\sigma_2$-Lipschitz continuous that

$$\begin{aligned}
&\|A_2g_2(u_n) - A_2g_2(u^*) - \rho(N_2(u_n, v_n) - N_2(u^*, v_n))\|^2 \\
&= \|A_2g_2(u_n) - A_2g_2(u^*)\|^2 \\
&\quad - 2\rho_2\langle N_2(u_n, v_n) - N_2(u^*, v_n), A_2g_2(u_n) - A_2g_2(u^*)\rangle \\
&\quad + \rho_2^2\|N_2(u_n, v_n) - N_2(u^*, v_n)\|^2 \\
&\le \theta_2^2\|u_n - u^*\|^2,
\end{aligned} \tag{3.15}$$

where $\theta_2 = \sqrt{\sigma_2^2 s_2^2 - 2\rho_2\beta_2 + 2\rho_2\alpha_2\mu_2^2 + \rho_2^2\mu_2^2}$. Observe that the $\nu_2$-Lipschitz continuity of $N_2$ in the second argument yields that

$$\|N_2(u^*, v_n) - N_2(u^*, v^*)\| \le \nu_2\|v_n - v^*\|. \tag{3.16}$$

Substituting (3.15) and (3.16) into (3.14), we have

$$\|g_2(v_n) - g_2(v^*)\| \le \frac{\tau_2\theta_2}{r_2 - \rho_2m_2}\|u_n - u^*\| + \frac{\tau_2\rho_2\nu_2}{r_2 - \rho_2m_2}\|v_n - v^*\|. \tag{3.17}$$

Observe that

$$\|v_n - v^*\| \leq \|v_n - v^* - [g_2(v_n) - g_2(v^*)]\| + \|g_2(v_n) - g_2(v^*)\|. \tag{3.18}$$

Since the relaxed $(\gamma_2, \delta_2)$-cocoercive monotonicity and $\sigma_2$-Lipschitz continuity of $g_2$ that

$$\begin{aligned}
&\|v_n - v^* - g_2(v_n) - g_2(v^*)\|^2 \\
&= \|v_n - v^*\|^2 - 2\langle g_2(v_n) - g_2(v^*), v_n - v^*\rangle + \|g_2(v_n) - g_2(v^*)\|^2 \\
&\leq \|v_n - v^*\|^2 - 2[-\gamma_2\|g_2(v_n) - g_2(v^*)\|^2 + \delta_2\|v_n - v^*\|^2] \\
&\quad + \|g_2(v_n) - g_2(v^*)\|^2 \\
&\leq \|v_n - v^*\|^2 + 2\sigma_2^2\gamma_2\|v_n - v^*\|^2 - 2\delta_2\|v_n - v^*\|^2 + \sigma_2^2\|v_n - v^*\|^2 \\
&= \theta_3^2\|v_n - v^*\|^2,
\end{aligned} \tag{3.19}$$

where $\theta_3 = \sqrt{1 + 2\sigma_2^2\gamma_2 - 2\delta_2 + \sigma_2^2}$. Substitute (3.17) and (3.19) into (3.18) yields that

$$\|v_n - v^*\| \leq \theta_3|v_n - v^*\| + \frac{\tau_2\theta_2}{r_2 - \rho_2 m_2}\|u_n - u^*\| + \frac{\tau_2\rho_2\nu_2}{r_2 - \rho_2 m_2}\|v_n - v^*\|,$$

which implies that

$$\|v_n - v^*\| \leq \frac{\tau_2\theta_2}{(1-\theta_3)(r_2 - \rho_2 m_2) - \tau_2\rho_2\nu_2}\|u_n - u^*\|. \tag{3.20}$$

Substitute (3.20) into (3.12) yields that

$$\begin{aligned}
&\|A_1 g_1(v_n) - A_1 g_1(v^*) - \rho(N_1(v_n, u_n) - N_1(v^*, u_n))\| \\
&\leq \frac{\tau_2\theta_1\theta_2}{(1-\theta_3)(r_2 - \rho_2 m_2) - \tau_2\rho_2\nu_2}\|u_n - u^*\|,
\end{aligned} \tag{3.21}$$

On the other hand, it follows from relaxed $(\gamma_1, \delta_1)$-cocoercive monotonicity and $\sigma_1$-Lipschitz continuity of $g_1$ that

$$\begin{aligned}
&\|u_n - u^* - g_1(u_n) - g_1(u^*)\|^2 \\
&= \|u_n - u^*\|^2 - 2\langle g_1(u_n) - g_1(u^*), u_n - u^*\rangle + \|g_1(u_n) - g_1(u^*)\|^2 \\
&\leq \|u_n - u^*\|^2 - 2[-\gamma_1\|g_1(u_n) - g_1(u^*)\|^2 + \delta_1\|u_n - u^*\|^2] \\
&\quad + \|g_1(u_n) - g_1(u^*)\|^2 \\
&\leq \|u_n - u^*\|^2 + 2\sigma_1^2\gamma_1\|u_n - u^*\|^2 - 2\delta_1\|u_n - u^*\|^2 + \sigma_1^2\|u_n - u^*\|^2 \\
&= \theta_4^2\|u_n - u^*\|^2,
\end{aligned} \tag{3.22}$$

where $\theta_4 = \sqrt{1 + 2\sigma_1^2\gamma_1 - 2\delta_1 + \sigma_1^2}$. Substituting (3.13), (3.21) and (3.22) into (3.11), we arrive at

$$\begin{aligned}
&\|u_{n+1} - u^*\| \\
&\leq \theta_4\|u_n - u^*\| + \frac{\tau_1\tau_2\theta_1\theta_2}{(r_1 - \rho_1 m_1)[(1-\theta_3)(r_2 - \rho_2 m_2) - \tau_2\rho_2\nu_2]}\|u_n - u^*\| \\
&\quad + \frac{\tau_1\rho_1\nu_1}{r_1 - \rho_1 m_1}\|u_n - u^*\| \\
&= (\theta_4 + \frac{\tau_1\tau_2\theta_1\theta_2}{(r_1 - \rho_1 m_1)[(1-\theta_3)(r_2 - \rho_2 m_2) - \tau_2\rho_2\nu_2]} + \frac{\tau_1\rho_1\nu_1}{r_1 - \rho_1 m_1})\|u_n - u^*\|.
\end{aligned} \tag{3.23}$$

From the assumption, we see

$$\theta_4 + \frac{\tau_1\tau_2\theta_1\theta_2}{(r_1 - \rho_1 m_1)[(1-\theta_3)(r_2 - \rho_2 m_2) - \tau_2\rho_2\nu_2]} + \frac{\tau_1\rho_1\nu_1}{r_1 - \rho_1 m_1} < 1.$$

It follows that the conclusion holds. This completes the proof. □

As some applications of Theorem 3.3, we have the following results immediately.

**Corollary 3.4.** *Let $X$ be a real Hilbert space, $A : H \times H$ a $(r, \eta)$-strongly monotone and $s$-Lipschitz continuous mapping and $M_i : X \to 2^X$ a $(A, \eta)$-monotone mapping, respectively. Let $\eta : X \times X \to X$ be a $\tau$-Lipschitz continuous mapping. Let $N : X \times X \to X$ be relaxed $(\alpha, \beta)$-co-coercive (with respect to $Ag$) and $\mu$-Lipschitz continuous in the first variable. Let $N$ be a $\nu$-Lipschitz continuous mapping in the second variable and $g : X \to X$ a relaxed $(\gamma, \delta)$-co-coercive and $\sigma_i$-Lipschitz mapping. Assume that $\Omega_2 \neq \emptyset$, where $\Omega_2$ denotes the set of solutions to the NSVI problem (3.3)-(3.3). Let $(u^*, v^*) \in \Omega_2$, $\{u_n\}$ and $\{v_n\}$ be sequences generated by Algorithm 3.2. Suppose that the following conditions are satisfied:*

$$\frac{\tau^2\theta_1\theta_2}{(r - \rho_1 m)[(1-\theta_3)(r - \rho_2 m) - \tau\rho_2\nu]} + \frac{\tau\rho_1\nu}{r - \rho_1 m} < 1 - \theta_3,$$

*where*

$$\theta_1 = \sqrt{\sigma^2 s^2 - 2\rho_1\beta + 2\rho_1\alpha\mu^2 + \rho_1^2\mu^2},$$
$$\theta_2 = \sqrt{\sigma^2 s^2 - 2\rho_2\beta + 2\rho_2\alpha\mu^2 + \rho^2\mu^2}$$

*and*

$$\theta_3 = \sqrt{1 + 2\sigma^2\gamma - 2\delta + \sigma^2}.$$

*Then the sequences $\{u_n\}$ and $\{v_n\}$ converge strongly to $u^*, v^*$, respectively.*

**Corollary 3.5.** *Let $X$ be a real Hilbert space, $A : H \times H$ a $(r, \eta)$-strongly monotone and $s$-Lipschitz continuous mapping and $M_i : X \to 2^X$ a $(A, \eta)$-monotone mapping, respectively. Let $\eta : X \times X \to X$ be a $\tau$-Lipschitz continuous mapping. Let $N : X \times X \to X$ be relaxed $(\alpha, \beta)$-co-coercive (with respect to $A$) and $\mu$-Lipschitz continuous in the first variable. Let $N$ be a $\nu$-Lipschitz continuous* *mapping in the second variable. Assume that $\Omega_3 \neq \emptyset$, where $\Omega_3$ denotes the set of solutions to the NSVI problem (3.5)-(3.6). Let $(u^*, v^*) \in \Omega_3$, $\{u_n\}$ and $\{v_n\}$ be sequences generated by Algorithm 3.3. Suppose that the following conditions are satisfied:*

$$\frac{\tau^2\theta_1\theta_2}{(r-\rho_1 m)[(r-\rho_2 m)-\tau\rho_2\nu]} + \frac{\tau\rho_1\nu}{r-\rho_1 m} < 1,$$

*where*

$$\theta_1 = \sqrt{\sigma^2 s^2 - 2\rho_1\beta + 2\rho_1\alpha\mu^2 + \rho_1^2\mu^2},$$

*and*

$$\theta_2 = \sqrt{\sigma^2 s^2 - 2\rho_2\beta + 2\rho_2\alpha\mu^2 + \rho^2\mu^2}.$$

*Then the sequences $\{u_n\}$ and $\{v_n\}$ converge strongly to $u^*, v^*$, respectively.*

**Corollary 3.6.** *Let $X$ be a real Hilbert space, $A : H \times H$ a $(r, \eta)$-strongly monotone and $s$-Lipschitz continuous mapping and $M_i : X \to 2^X$ a $(A, \eta)$-monotone mapping, respectively. Let $\eta : X \times X \to X$ be a $\tau$-Lipschitz continuous mapping. Let $N : X \times X \to X$ be relaxed $(\alpha, \beta)$-co-coercive (with respect to $A$) and $\mu$-Lipschitz continuous in the first variable. Let $N$ be a $\nu$-Lipschitz continuous mapping in the second variable. Assume that $\Omega_4 \neq \emptyset$, where $\Omega_4$ denotes the set of solutions to the NSVI problem (3.7). Let $u^* \in \Omega_4$, $\{u_n\}$ be a sequence generated by Algorithm 3.4. Suppose that the following conditions are satisfied:*

$$\tau\sqrt{s^2 - 2\rho\beta + 2\rho\alpha\mu^2 + \rho^2\mu^2} + \tau\rho\nu < r - \rho m.$$

*Then the sequences $\{u_n\}$ converges strongly to $u^*$.*

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