East Asian Mathematical Journal
Vol. 25 (2009), No. 2, pp. 147–151

AN IMPROVED NEWTON–KANTOROVICH THEOREM AND INTERIOR POINT METHODS

Ioannis K. Argyros

Abstract. We use an improved Newton–Kantorovich theorem introduced in [2] to analyze interior point methods. Our approach requires less number of steps than before [5] to achieve a certain error tolerance for both Newton's and Modified Newton's methods.

1. Introduction

In this study we are concerned with the problem of approximating a locally unique solution x^* of equation

$$F(x) = 0, \tag{1}$$

where F is a differentiable operator defined on a domain D of $\mathbb{R}^i$ (i an integer) with values in $\mathbb{R}^i$.

The famous Newton–Kantorovich theorem [4] has been used extensively to solve equation (1). A survey of such results can be found in [1] and the references there. Recently [2], [3] we improved the Newton–Kantorovich theorem. Here we use this development to show that the results obtained in the elegant work in [5] in connection with interior point methods can be improved if our convergence conditions simply replace the stronger ones given there.

Finally a numerical example is provided to show that fewer iterations than the ones suggested in [5] are needed to achieve the same error tolerance.

2. An improved Newton–Kantorovich theorem

Let $\|\cdot\|$ be a given norm on R^i, and x_0 be a point of D such that the closed ball of radius r centered at x_0,

$$\overline{U}(x_0, r) = \{x \in \mathbb{R}^i : \| x - x_0 \| \leq r\} \tag{2}$$

Received October 9, 2008; Accepted April 8, 2009.
2000 *Mathematics Subject Classification.* 49M15, 65H10, 65K05, 90C33.
Key words and phrases. Newton's method, Modified Newton's method, Newton–Kantorovich theorem, interior point methods.

is included in D, i.e.

$$\overline{U}(x_0, r) \subseteq D. \tag{3}$$

We assume that the Jacobian $F'(x_0)$ is nonsingular and that the following affine-invariant Lipschitz condition is satisfied:

$$\|F'(x_0)^{-1}[F'(x) - F'(y)]\| \leq \omega \ \|x - y\| \tag{4}$$

for all $x, y \in \overline{U}(x_0, r)$.

The famous Newton–Kantorovich Theorem [4] states that if the quantity

$$\alpha := \|F'(x_0)^{-1}F(x_0)\| \tag{5}$$

together with ω satisfy

$$k = \alpha\omega \leq \frac{1}{2}, \tag{6}$$

then there exists $x^* \in \overline{U}(x_0, r)$ with $F(x^*) = 0$. Moreover the sequences produced by Newton's method

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n) \quad (n \geq 0), \tag{7}$$

and by the modified Newton method

$$y_{n+1} = y_n - F'(y_0)^{-1}F(y_n), \quad y_0 = x_0 \quad (n \geq 0) \tag{8}$$

are well defined and converge to x^*.

In [2], [3] we introduced the center–Lipschitz condition

$$\|F'(x_0)^{-1}[F'(x) - F'(x_0)]\| \leq \omega_0\|x - x_0\| \tag{9}$$

for all $x \in \overline{U}(x_0, r)$ and provided a finer local and semilocal convergence analysis of method (7) by using the combination of conditions (4) and (9) given by

$$k^0 = \alpha\overline{\omega} \leq \frac{1}{2}, \tag{10}$$

where,

$$\overline{\omega} = \frac{\omega_0 + \omega}{2}\,. \tag{11}$$

In general

$$\omega_0 \leq \omega \tag{12}$$

holds, and $\frac{\omega}{\omega_0}$ can be arbitrarily large [3]. Note also that

$$k \leq \frac{1}{2} \Rightarrow k^0 \leq \frac{1}{2} \tag{13}$$

but not vice versa unless if $\omega_0 = \omega$. Examples where weaker condition (10) holds but (6) fails have been given in [2], [3].

Similarly by simply replacing ω with ω_0 (since (9) instead of (4) is actually needed in the proof) and condition (6) by the weaker

$$k^1 = \alpha\omega_0 \leq \frac{1}{2} \tag{14}$$

in the proof of Theorem 1 in [5] we show that method (8) also converges to x^* and the improved bounds

$$\|y_n - x^*\| \leq \frac{2\beta_0\lambda_0^2}{1-\lambda_0^2}\xi_0^{n-1} \quad (n \geq 1) \tag{15}$$

where

$$\beta_0 = \frac{\sqrt{1-2k^1}}{\omega_0}, \quad \lambda_0 = \frac{1-\sqrt{1-2k^1}-h^1}{k^1} \text{ and } \xi_0 = 1-\sqrt{1-2k^1}, \tag{16}$$

hold. In case $\omega_0 = \omega$ (15) reduces to (12) in [5]. Otherwise our error bounds are finer. Note also that

$$k \leq \frac{1}{2} \Rightarrow k^1 \leq \frac{1}{2} \tag{17}$$

but not vice versa unless if $\omega_0 = \omega$. Let us provide an example to show that (14) holds but (6) fails.

Example 1. Let $i = 1$, $x_0 = 1$, $D = [p, 2-p]$, $p \in \left[0, \frac{1}{2}\right)$, and define functions F on D by

$$F(x) = x^3 - p. \tag{18}$$

Using (4), (5) and (9) we obtain

$$\alpha = \frac{1}{3}(1-p), \quad \omega = 2(2-p) \text{ and } \omega_0 = 3-p, \tag{19}$$

which imply that

$$k = \frac{2}{3}(1-p)(2-p) > \frac{1}{2} \quad \text{for all } p \in \left[0, \frac{1}{2}\right) \tag{20}$$

whereas condition (14) holds for all $p \in \left[\frac{4-\sqrt{10}}{2}, \frac{1}{2}\right)$.

The above suggest that all results on interior point methods obtained in [5] for Newton's method using (6) can now be rewritten using only (10). The same holds true for the modified Newton's Method where (14) also replaces (6).

For example k_1 and k_2 can be replaced by k_1^0, k_2^0 ($k_2^0 < .5$) in the case of Newton's method (7) and by k_1^1, k_2^1 ($k_2^1 < .5$) in the case of the modified Newton method (8) respectively in all the results in [5] where they appear.

Since $\frac{k_1}{k_1^0}$, $\frac{k_2}{k_2^0}$, $\frac{k_1}{k_1^1}$, $\frac{k_2}{k_2^1}$ can be arbitrarily large [3] for a given triplet α, ω and ω_0, the choices

$$k_1^0 = k_1^1 = .12, \quad k_2^0 = k_2^1 = .24 \text{ when } k_1 = .21 \text{ and } k_2 = .42$$

and

$$k_1^0 = k_1^1 = .24, \quad k_2^0 = k_2^1 = .48 \quad \text{when} \quad k_1 = .245 \text{ and } k_2 = .49$$

are possible. As in [5] denote by N the number of Newton steps, by S the number of the modified Newton steps and by χ the parameter appearing in Corollary 4 in [5]. Then by using formulas (9), (10) and (11) in Corollary 4 and Theorem 2 in [5] we obtain the following tables:

(a) If the HLCP is monotone and only Newton directions are performed, then:

Potra (9)	Argyros (9)
$\chi(.21, .42) > .17$	$\chi(.12, .24) > .1$
$\chi(.245, .49) > .199$	$\chi(.24, .48) > .196$

Potra (10)	Argyros (10)
$N(.21, .42) = 2$	$N(.12, .24) = 1$
$N(.245, .49) = 4$	$N(.24, .48) = 3$

(b) If the HLCP is monotone and Modified Newton directions are performed:

Potra (9)	Argyros (11)
$\chi(.21, .42) > .149$	$\chi(.12, .24) > .098$
$\chi(.245, .49) > .164$	$\chi(.24, .48) > .162$

Potra (11)	Argyros (11)
$S(.21, .42) = 5$	$S(.12, .24) = 1$
$S(.245, .49) = 18$	$S(.24, .48) = 12$

All the above improvements are obtained under weaker hypotheses and the same computational cost (in the case of Newton's method) or less computational cost (in the case of the modified Newton method) since in practice the computation of ω requires that of ω_0 and in general the computation of ω_0 is less expensive than that of ω.

References

[1] I. K. Argyros, *Advances in the Efficiency of Computational Methods and Applications*, World Scientific Publ. Co., River Edge, NJ, 2000.

[2] I. K. Argyros, *On the Newton–Kantorovich hypothesis for solving equations*, J. Comput. Appl. Math., **169** (2004), 315–332.

[3] I. K. Argyros, *A unifying local–semilocal convergence analysis and applications for two–point Newton like methods in Banach space*, J. Math. Anal. Appl., **298** (2004), no. 2, 374–397.

[4] L. V. Kantorovich, and G. P. Akilov, *Functional Analysis in Normed Spaces*, Pergamon Press, Oxford, 1982.

[5] F. A. Potra, *The Kantorovich Theorem and interior point methods*, Math. Program., **102** (2005), no. 1, Ser. A, 47–70.

Cameron university
Department of Mathematics Sciences
Lawton, OK 73505, USA
E-mail address: iargyros@cameron.edu