

ON GENERALIZED NONLINEAR QUASIVARIATIONAL INEQUALITIES

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ABSTRACT. In this paper, we introduce a new generalized nonlinear quasivariational inequality and establish its equivalence with a fixed point problem by using the resolvent operator technique. Utilizing this equivalence, we suggest two iterative schemes, prove two existence theorems of solutions for the generalized nonlinear quasivariational inequality involving generalized cocoercive mapping and establish some convergence results of the sequences generated by the algorithms. Our results include several previously known results as special cases.

1. Introduction

It is well known that variational inequality theory and complementarity theory play important and fundamental roles in mechanics, elasticity, structural analysis, economics, optimization, oceanography, management sciences and other branches of mathematical and engineering sciences. Liu-Kang-Ume [9] investigated a class of variational inclusions using the resolvent operator technique for maximal monotone mappings. Many researchers [1,2,5-15] studied the existence of solutions for several kinds of variational inequalities, quasivariational inequalities and variational inclusions using various fixed point theorems.

Inspired and motivated by the results in [1,2,5-15], in this paper, we introduce a new class of generalized nonlinear quasivariational inequalities, which are more general and include the previously known classes of variational inequalities and quasivariational inequalities as special cases. We also establish its equivalence with a class of fixed point problems by using the resolvent operator technique. Utilizing the equivalence and the Banach fixed-point theorem, we develop two iterative algorithms, give two existence and uniqueness of solution for the generalized nonlinear quasivariational inequality involving

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generalized cocoercive mapping and prove some convergence results of the sequences generated by the algorithms. Our results include several previously known results as special cases.

2. Preliminaries

Let H be a real Hilbert space with a norm $\|\cdot\|$ and inner product $\langle \cdot, \cdot \rangle$, respectively, I stand for the identity mapping on H and 2^H denote the families of all nonempty subsets of H . Let $g, m, A, B : H \rightarrow H$ and $N : H \times H \rightarrow H$ be mappings. Suppose that $W : H \rightarrow 2^H$ is a maximal monotone mapping. For any fixed $f \in H$, we consider the following problem:

Find $u \in H$ such that

$$f \in N(Au, Bu) + W((g - m)u), \quad (2.1)$$

which is known as a *generalized nonlinear quasivariational inequality*.

Remark 2.1. It is easy to see that the generalized nonlinear quasivariational inequality (2.1) includes many classes of variational inequalities and quasivariational inequalities, respectively, in [1,2,5-15] as special cases.

Now we recall the following results and concepts.

Definition 2.1. ([3]) Let $W : H \rightarrow 2^H$ be a maximal monotone mapping. The resolvent operator J_ρ^W associated with W is defined by

$$J_\rho^W x = (I + \rho W)^{-1}x, \quad \forall x \in H,$$

where $\rho > 0$ is a constant.

It is well known that the resolvent operator J_ρ^W is single-valued and nonexpansive.

Definition 2.2. A mapping $g : H \rightarrow H$ is said to be *strongly monotone* and *Lipschitz continuous* if there exist positive constants r, s satisfying

$$\langle gx - gy, x - y \rangle \geq r\|u - v\|^2 \quad \text{and} \quad \|gu - gv\| \leq s\|u - v\|, \quad \forall x, y \in H,$$

respectively.

Definition 2.3. Let $A : H \rightarrow H$ and $N : H \times H \rightarrow H$ be mappings. N is called:

(1) *Lipschitz continuous* with respect to the first argument if there exists a constant $s > 0$ satisfying

$$\|N(x, z) - N(y, z)\| \leq s\|x - y\|, \quad \forall x, y, z \in H;$$

(2) *generalized cocoercive* with respect to A in the first argument if there exist constants $c \geq 0$ and $d \geq 0$ satisfying

$$\begin{aligned} & \langle N(Ax, z) - N(Ay, z), x - y \rangle \\ & \geq -c\|N(Ax, z) - N(Ay, z)\|^2 + d\|x - y\|^2, \quad \forall x, y, z \in H. \end{aligned}$$

In a similar way, we can define the Lipschitz continuity of the mapping N with respect to the second argument.

The following lemmas play a crucial role in the proof of our main results.

Lemma 2.1. ([4]) *Let $\{a_n\}_{n \geq 0}$, $\{b_n\}_{n \geq 0}$ and $\{c_n\}_{n \geq 0}$ be nonnegative sequences satisfying*

$$a_{n+1} \leq (1 - t_n)a_n + b_n t_n + c_n, \quad \forall n \geq 0,$$

where $\sum_{n=0}^{\infty} t_n = \infty$, $\{t_n\}_{n \geq 0} \subset [0, 1]$, $\lim_{n \rightarrow \infty} b_n = 0$ and $\sum_{n=0}^{\infty} c_n < \infty$. Then $\lim_{n \rightarrow \infty} a_n = 0$.

Lemma 2.2. *Let $\lambda \in (0, 1]$ and ρ be a positive constant. Then the following statements are equivalent:*

- (a) the generalized nonlinear quasivariational inequality (2.1) has a solution $u \in H$;
- (b) there exists $u \in H$ satisfying

$$gu = mu + J_{\rho}^W [(g - m)u - \rho N(Au, Bu) + \rho f],$$

where $J_{\rho}^W = (I + \rho W)^{-1}$ is the resolvent operator.

Based on Lemma 2.2 we suggest the following iterative algorithm with error for the generalized nonlinear quasivariational inequality (2.1).

Algorithm 2.1. *Let $g, m, A, B : H \rightarrow H$, $N : H \times H \rightarrow H$, $W : H \rightarrow 2^H$ and $f \in H$. Given $u_0 \in H$, compute $\{u_n\}_{n \geq 0}$ by the following iterative scheme:*

$$u_{n+1} = (1 - a_n)u_n + a_n\{u_n - (g - m)u_n + J_{\rho}^W [(g - m)u_n - \rho N(Au_n, Bu_n) + \rho f]\} + r_n, \quad n \geq 0,$$

where $\{r_n\}_{n \geq 0}$ is arbitrary sequence in H introduced to take into account possible inexact computations and $\{a_n\}_{n \geq 0}$ is arbitrary sequence in $[0, 1]$ satisfying

$$\sum_{n=0}^{\infty} \|r_n\| < +\infty, \quad \sum_{n=0}^{\infty} a_n = +\infty.$$

Algorithm 2.2. *Let $g, m, A, B, : H \rightarrow H$, $N : H \times H \rightarrow H$, $W : H \rightarrow 2^H$ and $f \in H$. Given $u_0 \in H$, compute $\{u_n\}_{n \geq 0}$ by the following iterative scheme:*

$$u_{n+1} = u_n - (g - m)u_n + J_{\rho}^W [(g - m)u_n - \rho N(Au_n, Bu_n) + \rho f], \quad \forall n \geq 0.$$

3. Existence and uniqueness of solution and convergence of algorithms

Now we prove the existence and uniqueness of solution of the generalized nonlinear quasivariational inequality (2.1) and the convergence of Algorithms 2.1 and 2.2.

Theorem 3.1. *Let $g, m, A, B : H \rightarrow H$ be Lipschitz continuous with constants p, q, a and b , respectively, and $g - m$ be strongly monotone with constant r . Assume that $N : H \times H \rightarrow H$ is Lipschitz continuous with respect to the first and second arguments with constants s and t , respectively, and is generalized cocoercive with respect to A in the first argument constants c and d . Assume that $W : H \rightarrow 2^H$ is a maximal monotone mapping. If there exists a positive constant ρ satisfying*

$$\theta = 1 - 2\sqrt{1 - 2r + (p + q)^2} - \sqrt{1 - 2\rho(d - cs^2a^2) + \rho^2s^2a^2} + \rho tb \in (0, 1),$$

then the generalized nonlinear quasivariational inequality (2.1) has a unique solution $u \in H$ and the sequence $\{u_n\}_{n \geq 0}$ defined by Algorithm 2.1 converges strongly to u .

Proof. Let x, y be arbitrary elements in H and define a mapping $G : H \rightarrow H$ by

$$G(x) = (1 - \lambda)x + \lambda\{x - (g - m)x + J_\rho^W[(g - m)x - \rho N(Ax, Bx) + \rho f]\}, \quad \forall x \in H,$$

where $\lambda \in (0, 1]$ is a constant. Since g and m are Lipschitz continuous and $g - m$ is strongly monotone, it follows that

$$\|x - y - (g - m)x + (g - m)y\| \leq [1 - 2r + (p + q)^2]^{\frac{1}{2}} \|x - y\|.$$

Note that A is Lipschitz continuous with constant a and N is Lipschitz continuous with respect to the first and second arguments with constants s and t , respectively, and is generalized cocoercive with respect to A in the first argument constants c and d . It follows that

$$\begin{aligned} & \|x - y - \rho[N(Ax, Bx) - N(Ay, Bx)]\|^2 \\ &= \|x - y\|^2 - 2\rho\langle N(Ax, Bx) - N(Ay, Bx), x - y \rangle \\ & \quad + \rho^2\|N(Ax, Bx) - N(Ay, Bx)\|^2 \\ &\leq \|x - y\|^2 + 2\rho c\|N(Ax, Bx) - N(Ay, Bx)\|^2 \\ & \quad - 2\rho d\|x - y\|^2 + \rho^2s^2a^2\|x - y\|^2 \\ &\leq (1 - 2\rho(d - cs^2a^2) + \rho^2s^2a^2)\|x - y\|^2. \end{aligned}$$

It is easy to verify that

$$\begin{aligned} & \|G(x) - G(y)\| \\ &= \|(1 - \lambda)x + \lambda\{x - (g - m)x + J_\rho^W[(g - m)x - \rho N(Ax, Bx) + \rho f]\} \\ & \quad - (1 - \lambda)y - \lambda\{y - (g - m)y + J_\rho^W[(g - m)y - \rho N(Ay, By) + \rho f]\}\| \\ &\leq (1 - \lambda)\|x - y\| + \lambda\|x - y - (g - m)x + (g - m)y\| \\ & \quad + \lambda\|J_\rho^W[(g - m)x - \rho N(Ax, Bx) + \rho f] \\ & \quad - J_\rho^W[(g - m)y - \rho N(Ay, By) + \rho f]\| \end{aligned}$$

$$\begin{aligned}
 &\leq (1 - \lambda)\|x - y\| + \lambda\|x - y - (g - m)x + (g - m)y\| \\
 &\quad + \lambda\|(g - m)x - (g - m)y - \rho[N(Ax, Bx) - N(Ay, By)]\| \\
 &\leq (1 - \lambda)\|x - y\| + 2\lambda\|x - y - (g - m)x + (g - m)y\| \\
 &\quad + \lambda\|x - y - \rho[N(Ax, Bx) - N(Ay, Bx)]\| \\
 &\quad + \lambda\rho\|N(Ay, Bx) - N(Ay, By)\| \\
 &\leq \{1 - \lambda[1 - 2\sqrt{1 - 2r + (p + q)^2} \\
 &\quad - \sqrt{1 - 2\rho(d - cs^2a^2) + \rho^2s^2a^2 + \rho tb}]\}\|x - y\| \\
 &= [1 - \lambda(1 - \theta)]\|x - y\|,
 \end{aligned}$$

which implies that G is a contraction mapping. Thus G has a unique fixed point $u \in H$, which is also a unique solution of the generalized nonlinear quasivariational inequality (2.1) by Lemma 2.2.

Now we prove that the iterative sequence $\{u_n\}_{n \geq 0}$ defined by Algorithm 2.1 converges to u . It is clear that

$$\begin{aligned}
 u &= (1 - a_n)u + a_n\{u - (g - m)u \\
 &\quad + J_\rho^W[(g - m)u - \rho N(Au, Bu) + \rho f]\}, \quad \forall n \geq 0.
 \end{aligned} \tag{3.1}$$

Using (3.1) and repeating the above arguments, we conclude that

$$\begin{aligned}
 \|u_{n+1} - u\| &\leq (1 - a_n)\|u_n - u\| + a_n\theta\|u_n - u\| + \|r_n\| \\
 &\leq [1 - (1 - \theta)a_n]\|u_n - u\| + \|r_n\|, \quad \forall n \geq 0.
 \end{aligned} \tag{3.2}$$

It follows from Lemma 2.1 and (3.2) that $\lim_{n \rightarrow \infty} u_n = u$. This completes the proof. \square

As a consequence of Theorem 2.1, we have

Theorem 3.2. *Let $g, m, A, B : H \rightarrow H$ be Lipschitz continuous with constants p, q, a and b , respectively, and $g - m$ be strongly monotone with constant r . Assume that $N : H \times H \rightarrow H$ is Lipschitz continuous with respect to the first and second arguments with constants s and t , respectively, and is generalized cocoercive with respect to A in the first argument constants c and d . Assume that $W : H \rightarrow 2^H$ is a maximal monotone mapping. If there exists a positive constant ρ satisfying*

$$\theta = 1 - 2\sqrt{1 - 2r + (p + q)^2} - \sqrt{1 - 2\rho(d - cs^2a^2) + \rho^2s^2a^2 + \rho tb} \in (0, 1),$$

then the generalized nonlinear quasivariational inequality (2.1) has a unique solution $u \in H$ and the sequence $\{u_n\}_{n \geq 0}$ defined by Algorithm 2.2 converges strongly to u .

Remark 3.1. Theorems 3.1-3.2 generalize Theorem 4.1 in [1], Theorems 4.1-4.3 in [2] and Theorem 3.6 in [15].

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