East Asian Mathematical Journal
Vol. 25 (2009), No. 2, pp. 127–140

SUBORDINATION RESULTS FOR CERTAIN CLASSES OF MULTIVALENTLY ANALYTIC FUNCTIONS WITH A CONVOLUTION STRUCTURE

J. K. Prajapat and R. K. Raina

Abstract. In this paper a general class of analytic functions involving a convolution structure is introduced. Among the results investigated are the various results depicting useful properties and characteristics of this function class by employing the techniques of differential subordination. Relevances of the main results with some known results are also mentioned briefly.

1. Introduction

Let $\mathcal{A}_p$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \qquad (p \in \mathbb{N} = \{1, 2, ...\}), \tag{1.1}$$

which are analytic and $p-$valent in the open unit disk

$$\mathbb{U} = \{z;\ z \in \mathbb{C} : |z| < 1\}.$$

For the functions f and g in $\mathcal{A}_p$, we say that f is subordinate to g in $\mathbb{U}$, and write $f \prec g$, if there exists a function $w(z)$ in $\mathbb{U}$ such that $|w(z)| < 1$ and $w(0) = 0$ with $f(z) = g(w(z))$ in $\mathbb{U}$. In case, f is univalent in $\mathbb{U}$, then the subordinatation $f \prec g$ is equivalent to $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$.

Let $f \in \mathcal{A}_p$ be given by (1.1) and $g \in \mathcal{A}_p$ be given by

$$g(z) = z^p + \sum_{k=p+1}^{\infty} b_k z^k \qquad (p \in \mathbb{N}), \tag{1.2}$$

then the Hadamard product (or convolution) $f * g$ of f and g is defined (as usual) by

$$(f * g)(z) := z^p + \sum_{k=p+1}^{\infty} a_k b_k z^k := (g * f)(z). \tag{1.3}$$

Received August 16, 2000, Revised October 25, 2000, Accepted November 20, 2000.
2000 *Mathematics Subject Classification.* Primary 30C45; Secondary 26A33, 33C20.
Key words and phrases. Multivalently analytic functions, Hadamard product (or convolution), differential subordination, hypergeometric functions, Dziok-Srivastava linear operator.

For a given function $g(z) \in \mathcal{A}_p$ (defined by (1.2)), we introduce here a new class $\mathcal{M}_p(g; \alpha, m, A, B)$ of functions belonging to the subclasses of $\mathcal{A}_p$ which consists of functions $f(z)$ of the form (1.1) satisfying the following subordination:

$$(1-\alpha)\,\frac{(f*g)(z)}{z^p} + \frac{\alpha}{p}\,\frac{(f*g)'(z)}{z^{p-1}} \prec \left(\frac{1+Az}{1+Bz}\right)^m \tag{1.4}$$

$$(z \in \mathbb{U};\ p \in \mathbb{N};\ 0 < \alpha \le p;\ -1 \le B < A \le 1;\ 0 < m \le 1).$$

For $\alpha = 1$, we put

$$\mathcal{M}_p(g; 1, m, A, B) = \mathcal{N}_p(g; m, A, B),$$

and it may be noted that there exist, as special cases, several new or known interesting subclasses of our function class $\mathcal{M}_p(g; \alpha, m, A, B)$. For example, if the coefficients b_k in (1.2) and the values of m and α in (1.4) are, respectively, choosen as follows:

$$b_k = \frac{(\alpha_1)_{k-p} \dots (\alpha_q)_{k-p}}{(\beta_1)_{k-p} \dots (\beta_s)_{k-p}\ (k-p)!}, \quad m = 1 \quad \text{and} \quad \alpha = \frac{p\lambda}{\alpha_i} \quad (\lambda > 0) \tag{1.5}$$

$$(\alpha_j > 0 (j = 1, ..., q),\ \beta_j > 0 (j = 1, ..., s),\ q \le s + 1;\ q, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}),$$

and also in the process making use of the identity ([7]; see also [8]):

$$z\left(H_s^q[\alpha_i]f\right)'(z) = \alpha_i\left(H_s^q[\alpha_i+1]f(z)\right) - (\alpha_i - p)\left(H_s^q[\alpha_i]f(z)\right) \ (i = 1, ..., q) \tag{1.6}$$

in (1.4), then the class $\mathcal{M}_p(g; \alpha, m, A, B)$ reduces to a known function class studied very recently by Liu [8].

The symbol $(\alpha)_k$ occurring in (1.5) is the well known Pochhammer symbol

$$(\alpha)_0 = 1, \qquad (\alpha)_k = \alpha(\alpha+1)...(\alpha+k-1); k \in \mathbb{N}\ .$$

Furthermore, it may be mentioned here that the operator

$$\left(H_s^q[\alpha_1]f\right)(z) := H_s^q(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s)f(z)$$

involved in the identity (1.6) is the Dziok-Srivastava linear operator (see, for details [7]), which contains such well known operators as the Hohlov linear operator, Saitoh generalized linear operator, the Carlson-Shaffer linear operator, the Ruscheweyh derivative operator as well as its generalized version, the Bernardi-Libera-Livingston operator, and the Srivastave-Owa fractional derivative operator. One may refer to the paper [7] for further details and references of these operators. The Dziok- Srivastava linear operator defined in [7] has further been generalized by Dziok and Raina [5](see also [6]).

On the other hand, if we set the coefficients b_k in (1.2) and the value of the parameter m and α in (1.4), respectively, as follows:

$$b_k = \left(\frac{p+1}{k+1}\right)^\sigma \ (\sigma > 0; k \ge p+1\ (p \in \mathbb{N})), \ \ m = 1 \quad \text{and}\ \alpha = \frac{\lambda}{(p+1)} \ (\lambda > 0) \tag{1.7}$$

and apply the following identity:

$$z\left(I^{\sigma}f(z)\right)' = (p+1)I^{\sigma-1}f(z) - I^{\sigma}f(z) \qquad (p \in \mathbb{N};\ \sigma > 0) \tag{1.8}$$

in (1.4), then the class $\mathcal{M}_p(g;\alpha,m,A,B)$ reduces to the class $\Omega_p^{\sigma}(A,B,\lambda)$ studied very recently by Sham *et al.* [15], where the operator $I^{\sigma}f(z)$ in (1.8) is defined by ([15, p.1, Eqn. (1.1)])

$$\begin{aligned} I^{\sigma}f(z) &= \frac{(p+1)^{\sigma}}{z\,\Gamma(\sigma)}\int_0^z \left(log\,\frac{z}{t}\right)^{\sigma-1} f(t)dt \\ &= z^p + \sum_{k=p+1}^{\infty}\left(\frac{p+1}{k+1}\right)^{\sigma} a_k z^k \qquad (f \in \mathcal{A}_p;\ p \in \mathbb{N};\ \sigma > 0). \end{aligned}$$

Moreover, if we choose the coefficients b_k in (1.2) and the value of the parameter m and α in (1.4), respectively, as follows:

$$b_k = \frac{k}{p}\left(\frac{k+\mu}{p+\mu}\right)^r \quad (p \in \mathbb{N};\ r \in \mathbb{N}_0;\ \mu \geq 0), \qquad m = 1 \quad \text{and} \quad \alpha = 1, \tag{1.9}$$

then the class $\mathcal{M}_p(g;\alpha,m,A,B)$ transforms into a (presumably) new class $\mathcal{R}_p^r(\mu,A,B)$ defined by

$$\mathcal{R}_p^r(\mu,A,B) := \left\{ f:\ \ f \in \mathcal{A}_p \ \text{ and } \ \frac{1}{p}\frac{[I_p(r,\mu)f]'(z)}{z^{p-1}} \prec \frac{1+Az}{1+Bz} \right\} \tag{1.10}$$

$$(z \in \mathbb{U};\ r \in \mathbb{N}_0;\ \mu \geq 0;\ -1 \leq B < A \leq 1),$$

involving the operator $I_p(r,\mu)$ which is defined by ([18])

$$I_p(r,\mu)f(z) = z^p + \sum_{k=p+1}^{\infty}\left(\frac{k+\mu}{p+\mu}\right)^r a_k z^k \tag{1.11}$$

$$(z \in \mathbb{U};\ p \in \mathbb{N};\ r \in \mathbb{N}_0;\ \mu \geq 0).$$

The class $\mathcal{R}_p^r(1,A,B)$ was studied by Srivastava *et al.* [16]. Furthermore, on specializing suitably the coefficients b_k in (1.2) and the parameters m and α in (1.4) , one may obtain the function classes investigated very recently by Dinggong and Liu [4], Ozkan [13] and Srivastava *et al.* [17].

In the present paper we derive various useful and interesting properties and characteristics of the function classes $\mathcal{M}_p(g;\alpha,m,A,B)$ and $\mathcal{N}_p(g;m,A,B)$(defined above) by using the techniques of differential subordination. Relevances of the main results and their connections with known results are also briefly pointed out.

2. Preliminaries and key lemmas

We require the following lemmas in the sequel to investigate the function classes $\mathcal{M}_p(g;\alpha,m,A,B)$ and $\mathcal{N}_p(g;m,A,B)$ (defined above).

Lemma 2.1. *(Miller and Mocanu [11]) Let $h(z)$ be a convex (univalent) function in $\mathbb{U}$ with $h(0)=1$, and let the function $\phi(z)=1+p_1z+p_2z^2+...$ be analytic in $\mathbb{U}$. If*

$$\phi(z)+\frac{z\phi'(z)}{\gamma}\prec h(z) \tag{2.1}$$

for $\gamma\neq 0$ and $\Re(\gamma)\geq 0$, then

$$\phi(z)\prec\psi(z):=\frac{\gamma}{z^{\gamma}}\int_0^z t^{\gamma-1}\,h(t)\,dt\prec h(z) \tag{2.2}$$

and $\psi(z)$ is the best dominant.

Lemma 2.2. *([10, p. 132]) Let $q(z)$ be analytic and univalent in $\mathbb{U}$ and let $\theta(\omega)$ and $\phi(\omega)$ be analytic in domain $\mathbb{D}$ containing $q(\mathbb{U})$ with $\phi(\omega)\neq 0$ when $\omega\in q(\mathbb{U})$. Set*

$$Q(z)=zq'(z)\phi[q(z)],\qquad h(z)=\theta[q(z)]+Q(z),$$

and suppose that

(i) $Q(z)$ *is univalent and starlike in* $\mathbb{U}$,

(ii)

$$\Re\left(\frac{zh'(z)}{Q(z)}\right)=\Re\left(\frac{\theta'[q(z)]}{\phi[q(z)]}+\frac{zQ'(z)}{Q(z)}\right)>0\quad(z\in\mathbb{U}).$$

If $p(z)$ is analytic in $\mathbb{U}$ with $p(0)=q(0)$, $p(\mathbb{U})\subset\mathbb{D}$ and

$$\theta[p(z)]+zp'(z)\phi[p(z)]\prec\theta[q(z)]+zq'(z)\phi[q(z)]=h(z).$$

Then $p(z)\prec q(z)$ and $q(z)$ is the best dominant.

The generalized hypergeometric function ${}_pF_q$ is defined by

$$\begin{aligned} {}_qF_s(z) &\equiv {}_qF_s(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s;\ z)\\ &=:\sum_{n=0}^{\infty}\frac{(\alpha_1)_n\ ...\ (\alpha_q)_n}{(\beta_1)_n\ ...\ (\beta_s)_n}.\ \frac{z^n}{n!} \end{aligned} \tag{2.3}$$

$(z\in\mathbb{U};\ \alpha_j\in\mathbb{C}(j=1,...,q),\ \ \beta_j\in\mathbb{C}\backslash\{0,-1,-2,...\}(j=1,...,s),\ q\leq s+1$ $q,s\in\mathbb{N}_0)$.

The following identities are well known [1, pp. 556-558].

Lemma 2.3. *For real or complex numbers a, b and c $(c\neq 0,-1,-2,...)$:*

$$(i)\quad \int_0^1 t^{b-1}\,(1-t)^{c-b-1}\,(1-zt)^{-a}\,dt=\frac{\Gamma(a)\,\Gamma(c-b)}{\Gamma(c)}\ {}_2F_1\,(a,b;\ c;z) \tag{2.4}$$

$$(ii)\qquad F_1\,(a,b;\ c;z)=(1-z)^{-a}\ {}_2F_1\,\left(a,c-b;c;\frac{z}{z-1}\right) \tag{2.5}$$

$$(iii) \qquad {}_2F_1\left(1,1;\ 3; \frac{z}{z-1}\right) = \frac{2(z-1)}{z}\left[1 + \frac{ln(1-z)}{z}\right] \tag{2.6}$$

For a function $f \in \mathcal{A}_p$ given by (1.1), the generalized Bernardi-Libera-Livingston integral operator [3] (see also [9]) is defined by

$$\begin{aligned}\mathcal{K}_{p,\gamma}(f)(z) &:= \frac{\gamma+p}{z^\gamma}\int_0^z t^{\gamma-1} f(t)\,dt \\ &= z^p + \sum_{k=p+1}^{\infty} \frac{\gamma+p}{\gamma+k}\, a_k\, z^k \qquad (\gamma > -p;\ p \in \mathbb{N}).\end{aligned} \tag{2.7}$$

It readily follows from (2.7) that the operator $\mathcal{K}_{p,\gamma}(f)(z)$ is a self-preserving operator on $\mathcal{A}_p$, so that

$$f(z) \in \mathcal{A}_p \ \Rightarrow\ \mathcal{K}_{p,\gamma}(f)(z) \in \mathcal{A}_p \qquad (\gamma > -p;\ p \in \mathbb{N}).$$

3. Main results

Our first main result based on differential subordination is given by the following:

Theorem 3.1. *If* $\mathcal{M}_p(g;\alpha,m,A,B)$*, then*

$$\frac{(f*g)(z)}{z^p} \prec \mathcal{X}(z) \prec \left(\frac{1+Az}{1+Bz}\right)^m \qquad (z \in \mathbb{U}), \tag{3.1}$$

where

$$\mathcal{X}(z) = \left(\frac{A}{B}\right)^m \sum_{i=0}^{m} \frac{(-m)_i}{i!}\left(\frac{A-B}{A}\right)^i (1+Bz)^{-i}\ {}_2F_1\left(i,1;1+\frac{p}{\alpha};\frac{Bz}{1+Bz}\right)$$

and $\mathcal{X}(z)$ *is the best dominant of (3.1).*
Also

$$\Re\left(\frac{(f*g)(z)}{z^p}\right) > \mathcal{X}(-1) \tag{3.2}$$

and the result (3.2) is sharp.

Proof. Let $f(z) \in \mathcal{M}_p(g;\alpha,m,A,B)$, and assume that

$$\frac{(f*g)(z)}{z^p} = \theta(z). \tag{3.3}$$

We may express the function $\theta(z)$ as

$$\theta(z) = 1 + c_1 z + c_2 z^2 + \ \ldots \tag{3.4}$$

which is analytic in $\mathbb{U}$ with $\theta(0) = 1$. From (3.3), we obtain

$$\begin{aligned}(1-\alpha)\,\frac{(f*g)(z)}{z^p} + \frac{\alpha}{p}\,\frac{(f*g)'(z)}{z^{p-1}} &= \theta(z) + \frac{\alpha}{p}\, z\,\theta'(z) \\ &\prec \left(\frac{1+Az}{1+Bz}\right)^m = h(z) \qquad (z \in \mathbb{U}).\end{aligned} \tag{3.5}$$

We observe that the function $h(z)$ involved in (3.5) is analytic and convex in $\mathbb{U}$ because

$$\Re\left(1+\frac{zh''(z)}{h'(z)}\right) = -1+(1-m)\,\Re\left(\frac{1}{1+Az}\right)+(1+m)\,\Re\left(\frac{1}{1+Bz}\right)$$
$$> -1+\frac{1-m}{1+|A|}+\frac{1+m}{1+|B|} \;\geq\; 0 \qquad (z\in\mathbb{U}).$$

Therefore, applying Lemma 1, we get

$$\theta(z) \prec \frac{p}{\alpha}\, z^{-\frac{p}{\alpha}} \int_0^1 t^{\frac{p}{\alpha}-1}\left(\frac{1+At}{1+Bt}\right)^m dt.$$

Upon expanding the binomial expressions in the integrand, changing the order of integration and summation (justified on account of the conditions mentioned in (1.4)), and carrying out elementary calculations, we find that

$$\begin{aligned}\theta(z) &\prec \left(\frac{A}{B}\right)^m \sum_{i=0}^{m}\frac{(-m)_i}{i!}\left(\frac{A-B}{A}\right)^i {}_2F_1\left(\frac{p}{\alpha}, i; 1+\frac{p}{\alpha}; -Bz\right)\\ &= \mathcal{X}(z).\end{aligned} \tag{3.6}$$

Now using the transformation (2.5) of Lemma 3, we finally get

$$\mathcal{X}(z) = \left(\frac{A}{B}\right)^m \sum_{i=0}^{m}\frac{(-m)_i}{i!}\left(\frac{A-B}{A}\right)^i (1+Bz)^{-i}\, {}_2F_1\left(i,1;1+\frac{p}{\alpha};\frac{Bz}{1+Bz}\right).$$

Next to prove (3.2), we observe that (3.1) is equivalent to

$$\frac{(f*g)(z)}{z^p} = \frac{p}{\alpha}\int_0^1 u^{\frac{p}{\alpha}-1}\left(\frac{1+A\,u\,w(z)}{1+B\,u\,w(z)}\right)^m du,$$

where $w(z)$ is analytic in $\mathbb{U}$ with $w(0)=1$ and $|w(z)|<1$ in $\mathbb{U}$.
Hence

$$\Re\left(\frac{(f*g)(z)}{z^p}\right) = \frac{p}{\alpha}\int_0^1 u^{\frac{p}{\alpha}-1}\,\Re\left(\frac{1+A\,u\,w(z)}{1+B\,u\,w(z)}\right)^m du$$
$$> \frac{p}{\alpha}\int_0^1 u^{\frac{p}{\alpha}-1}\left(\frac{1-A\,u}{1-B\,u}\right)^m du.$$

To establish sharpness of the result (3.2), it is sufficient to show that

$$\inf_{|z|<1}\{\Re(\mathcal{X}(z)\} = \mathcal{X}(-1). \tag{3.7}$$

We observe from (3.1) that for $|z|\leq r\ \ (0<r<1)$:

$$\begin{aligned}\Re\{\mathcal{X}(z)\} &\geq \frac{p}{\alpha}\int_0^1 u^{\frac{p}{\alpha}-1}\,\Re\left(\frac{1+Aur}{1+Bur}\right)^m du = \mathcal{X}(-r) \qquad (|z|\leq r\ \ (0<r<1))\\ &\to \mathcal{X}(-1) \quad \text{as} \quad r\to 1-,\end{aligned}$$

which establishes (3.7) and this completes the proof of Theorem 1. □

Corollary 3.1. *If $f(z) \in \mathcal{M}_p(g; \alpha, m, A, B)$, then*

$$\Re\left(\frac{(f*g)(z)}{z^p}\right)^{\frac{1}{n}} > \left[\left(\frac{A}{B}\right)^m \sum_{i=0}^{m} \frac{(-m)_i}{i!}\left(\frac{A-B}{A}\right)^i (1-B)^{-i}\ {}_2F_1\left(i,1;1+\frac{p}{\alpha};\frac{B}{B-1}\right)\right]^{\frac{1}{n}} \quad (B \neq 0;\ n \geq 1).$$

Proof. Using the elementary inequality

$$\Re(w^{\frac{1}{n}}) \geq (\Re(\,w))^{\frac{1}{n}} \quad \text{for} \quad \Re(w) > 0 \quad \text{and} \quad n \geq 1$$

in Theorem 1, we get the desired result. □

Remark 1. We note that if we use the parametric substitutions given by (1.5) and apply the identity (1.6), then Theorem 1 and Corollary 1 correspond to the results given recently by Liu [8, p. 3]. Also, making use of the parametric substitutions given by (1.7), and applying the identity (1.8), Theorem 1 yields the recently established results due to Sham *et al.* [15, p. 2, Theorem 2.1 and Theorem 2.2]. Another known result due to Srivastava *et al.* [16, p. 4, Theorem 1] is obtainable from Theorem 1 when the parameters involved in it are specialized by means of (1.9).

Theorem 3.2. *Let the function $\psi(z)$ defined by*

$$\psi(z) = z^p\ {}_{r+1}F_{s+1}\ (\alpha_1, ..., \alpha_r, 1+\lambda^{-1}; \beta_1, ..., \beta_s, \lambda^{-1};\ z) \quad (r \leq s+1;\ \lambda > 0;\ z \in \mathbb{U}) \tag{3.8}$$

be in the class $\mathcal{N}_p(g; m, A, B)$. Then the function

$$\theta(z) \ = \ z^p\ {}_rF_s\ (\alpha_1, ..., \alpha_r; \beta_1, ..., \beta_s;\ z)$$

satisfies the condition

$$\frac{(\theta*g)'(z)}{p\ z^{p-1}} \prec \sigma(z) \prec \left(\frac{1+Az}{1+Bz}\right)^m \qquad (z \in \mathbb{U}), \tag{3.9}$$

where

$$\sigma(z) = \left(\frac{A}{B}\right)^m \sum_{i=0}^{m} \frac{(-m)_i}{i!}\left(\frac{A-B}{A}\right)^i (1+Bz)^{-i}\ {}_2F_1\left(i,1,1+\frac{1}{\lambda};\frac{Bz}{1+Bz}\right).$$

The function $\sigma(z)$ is the best dominant of (3.9).
Also

$$\Re\left(\frac{(\theta*g)'(z)}{p\ z^{p-1}}\right) > \sigma(-1) \tag{3.10}$$

and the result (3.10) is best possible.

Proof. From (1.3) and (3.8), we get

$$\frac{(\psi * g)'(z)}{p\, z^{p-1}} = 1 + \sum_{k=p+1}^{\infty} [1+\lambda(k-p)]\, b_k\, \frac{(\alpha_1)_{k-p} \cdots (\alpha_r)_{k-p}\, k}{(\beta_1)_{k-p} \cdots (\beta_s)_{k-p}\, p} \frac{z^{k-p}}{(k-p)!}$$
$$= w(z) + \lambda\, z\, w'(z),$$

where

$$w(z) = 1 + \sum_{k=p+1}^{\infty} b_k\, \frac{(\alpha_1)_{k-p} \cdots (\alpha_r)_{k-p}\, k}{(\beta_1)_{k-p} \cdots (\beta_s)_{k-p}\, p} \frac{z^{k-p}}{(k-p)!}$$
$$= \frac{(\theta * g)'(z)}{p\, z^{p-1}} \qquad (z \in \mathbb{U}).$$

By hypothesis $\psi(z) \in \mathcal{N}_p(g; m, A, B)$, then the assertions (3.9) and (3.10) follow directly by applying the same procedure as adopted in the proof of Theorem 1. □

Theorem 3.3. *If $f(z) \in \mathcal{A}_p$ such that*

$$(1-\alpha)\, \frac{(\mathcal{K}_{p,\gamma}\,(f) * g)(z)}{z^p} + \alpha\, \frac{(f * g)(z)}{z^p} \prec \left(\frac{1+Az}{1+Bz}\right)^m \qquad (\alpha > 0,\ z \in \mathbb{U}), \tag{3.11}$$

where $\mathcal{K}_{p,\gamma}$ is defined by (2.7), then

$$\frac{(\mathcal{K}_{p,\gamma}\,(f) * g)(z)}{z^p} \prec \rho(z) \prec \left(\frac{1+Az}{1+Bz}\right)^m \qquad (z \in \mathbb{U}), \tag{3.12}$$

with $\rho(z)$ given by

$$\rho(z) = \left(\frac{A}{B}\right)^m \sum_{i=0}^{m} \frac{(-m)_i}{i!} \left(\frac{A-B}{A}\right)^i (1+Bz)^{-i}\; {}_2F_1\left(i, 1; 1 + \frac{p+\gamma}{\alpha}; \frac{Bz}{1+Bz}\right)$$

and $\rho(z)$ is the best dominant of (3.12). Also

$$\Re\left(\frac{(\mathcal{K}_{p,\gamma}\,(f) * g)(z)}{z^p}\right) > \rho(-1). \tag{3.13}$$

The result (3.13) is sharp.

Proof. It follows from (2.7) that

$$z\,(\mathcal{K}_{p,\gamma}\,(f) * g)'(z) = (\gamma + p)\,(f * g)(z) - \gamma\,(\mathcal{K}_{p,\gamma}\,(f) * g)(z), \tag{3.14}$$

and if we assume that

$$\frac{(\mathcal{K}_{p,\gamma}\,(f) * g)(z)}{z^p} = q(z),$$

then following the same process as in the proof of Theorem 1, we arrive at the desired results (3.12) and (3.13). □

Putting $m = \alpha = 1$ in Theorem 3 and observing that

$$(\mathcal{K}_{p,\gamma}(f) * g)(z) \;=\; \frac{\gamma+p}{z^{\gamma}} \int_0^z t^{\gamma-1}(f*g)(t)\,dt \qquad (f \in \mathcal{A}_p,\; z \in \mathbb{U}),$$

we get the following:

Corollary 3.2. *If $f(z) \in \mathcal{A}_p$ such that*

$$\frac{(f*g)(z)}{z^p} \;\prec\; \frac{1+Az}{1+Bz} \qquad (z \in \mathbb{U}),$$

then

$$\Re\left(\frac{\gamma+p}{z^{\gamma+p}} \int_0^z t^{\gamma-1}(f*g)(t)\,dt\right) \;>\; \xi \qquad (z \in \mathbb{U}),$$

where

$$\xi = \begin{cases} \frac{A}{B} + \left(1 - \frac{A}{B}\right)(1-B)^{-1}\,{}_2F_1\left(1,1;\; p+\gamma+1;\, \frac{B}{B-1}\right) & (B \neq 0); \\ \quad 1 - \frac{p+\gamma}{p+\gamma+1}\, A & (B = 0), \end{cases}$$

and the result is best possible.

A special case of Corollary 2 when

$$A = 1-2\beta \;(0 \le \beta < 1), \quad B = -1, \quad p = 1 \quad \text{and} \quad g(z) = z/(1-z)$$

would immediately yield the following result.

Corollary 3.3. *If $f(z) \in \mathcal{A}$ and*

$$\Re\left(\frac{f(z)}{z}\right) \;>\; \beta \qquad (0 \le \beta < 1;\; z \in \mathbb{U}),$$

then

$$\Re\left(\frac{\gamma+1}{z^{\gamma+1}} \int_0^z t^{\gamma-1} f(t)dt\right) \;> \xi^*,$$

where

$$\xi^* \;=\; \beta + (1-\beta)\left({}_2F_1\left(1,1;\gamma+2;\frac{1}{2}\right) - 1\right).$$

Remark 2. Corollary 3 provides an improvement of the result due to Obradovic [12].

Theorem 3.4. *If $f(z) \in \mathcal{M}_p(g;\alpha,m,A,B)$, then*

$$(1-\alpha)\,\frac{((\mathcal{K}_{p,\gamma}\, f)*g)(z)}{z^p} + \frac{\alpha}{p}\,\frac{((\mathcal{K}_{p,\gamma}\, f)*g)'(z)}{z^{p-1}} \prec \tau(z) \prec \left(\frac{1+Az}{1+Bz}\right)^m \quad (z \in \mathbb{U}), \tag{3.15}$$

where $\mathcal{K}_{p,\gamma}$ is defined by (2.7) and the function $\tau(z)$ is given by

$$\tau(z) = \left(\frac{A}{B}\right)^m \sum_{i=0}^{m} \frac{(-m)_i}{i!} \left(\frac{A-B}{A}\right)^i (1+Bz)^{-i}\,{}_2F_1\left(i,1;1+p+\gamma;\frac{Bz}{1+Bz}\right),$$

which is the best dominant of (3.15).
Also

$$\Re\left((1-\alpha)\,\frac{((\mathcal{K}_{p,\gamma}\,f)*g)(z)}{z^p}+\frac{\alpha}{p}\,\frac{((\mathcal{K}_{p,\gamma}\,f)*g)'(z)}{z^{p-1}}\right)>\tau(-1), \tag{3.16}$$

and the result (3.16) is the best possible.

Proof. Let $f(z)\in\mathcal{M}_p(g;\alpha,m,A,B)$, and assume that

$$(1-\alpha)\,\frac{((\mathcal{K}_{p,\gamma}\,f)*g)(z)}{z^p}+\frac{\alpha}{p}\,\frac{((\mathcal{K}_{p,\gamma}\,f)*g)'(z)}{z^{p-1}}\;=\;q(z). \tag{3.17}$$

Now using the identity (3.14) in (3.17) and differentiating the resulting equation with respect to z, we obtain

$$\begin{aligned}(1-\alpha)\,\frac{(f*g)(z)}{z^p}+\frac{\alpha}{p}\,\frac{(f*g)'(z)}{z^{p-1}}&=q(z)+\frac{z\,q'(z)}{\gamma+p}\\ &\prec\left(\frac{1+Az}{1+Bz}\right)^m\quad(z\in\mathbb{U}),\end{aligned}$$

which on using Lemma 1 (when γ is replaced by $\gamma+p$) yields

$$q(z)\prec(\gamma+p)\,z^{-(\gamma+p)}\int_0^z t^{(\gamma+p)-1}\left(\frac{1+At}{1+Bt}\right)^m dt.$$

The assertion (3.15) and the estimate (3.16) can now be deduced on the same lines as given in the proof of Theorem 1. This evidently completes the proof of Theorem 4. □

Remark 3. It may be worthwhile to point out here that an extended form of the above result (Theorem 4) can be established if one uses a generalized form of the fractional calculus operator $D_{0,z}^{(\alpha,\alpha',\beta,\beta',\gamma)}$ (see [14]) instead of the operator $\mathcal{K}_{p,\gamma}(f)(z)$ defined here (in the corrected form) by

$$D_{0,z}^{\alpha,\alpha',\beta,\beta',\gamma}f(z)=\frac{d^n}{dz^n}I_{0,z}^{\alpha,\alpha',\beta+n,\beta',n-\gamma}f(z)\quad(n-1\leq\gamma<n;n\in\mathbb{N}),$$

where $I_{0,z}^{\alpha,\alpha',\beta,\beta',\gamma}f(z)$ denotes the generalized fractional integral operator [14 , p. 2]. Incidentally, the image formula [14, p. 4, Eqn. (2.2)] for the function z^k under the operator $D_{0,z}^{\alpha,\alpha',\beta,\beta',\gamma}f(z)$ needs to be corrected to the form

$$D_{0,z}^{\alpha,\alpha',\beta,\beta',\gamma}z^k=\tfrac{d^m}{dz^m}I_{0,z}^{\alpha,\alpha',\beta+m,\beta',m-\gamma}z^k$$

$$=\frac{\Gamma(1+k)(\Gamma(1+k-\alpha'+\beta')\Gamma(1+k-\alpha-\alpha'-\beta-\gamma)}{\Gamma(1+k+\beta')\Gamma(1+k-\alpha'-\beta-\gamma)\Gamma(1+k-\alpha-\alpha'-\gamma)}z^{k-\alpha-\alpha'-\gamma}.$$

$$(\gamma\geq0;\alpha,\alpha',\beta,\beta',\gamma\in\mathbb{R};k>\max(0,\alpha'-\beta',\alpha+\alpha'+\beta+\gamma)-1)$$

Theorem 3.5. *Let*

$$f \in \mathcal{A}_p; \quad 0 \leq \alpha < p; \quad 0 < m \leq 1; \quad -1 \leq a < \frac{1}{m} - 1 \quad \textit{and} \quad -1 < A < 1.$$

Suppose also that

$$\frac{(f * g)(z)}{z^p} \neq 0 \qquad (z \in \mathbb{U})$$

and

$$\left(\frac{(f * g)(z)}{z^p}\right)^a \left((1-\alpha)\,\frac{(f * g)(z)}{z^p} + \frac{\alpha}{p}\,\frac{(f * g)'(z)}{z^{p-1}}\right) \prec h(z)\ (z \in \mathbb{U}), \quad (3.18)$$

where

$$h(z) = \left(\frac{1+Az}{1-z}\right)^{m(a+1)} \left(1 + \frac{\alpha\, m\, (A+1)z}{p\,(1-z)(1+Az)}\right),$$

then

$$\frac{(f * g)(z)}{z^p} \prec \left(\frac{1+Az}{1-z}\right)^m.$$

Proof. Define a function $\Lambda(z)$ by

$$\Lambda(z) = \frac{(f * g)(z)}{z^p} \qquad (z \in \mathbb{U}). \quad (3.19)$$

We note that the function $\Lambda(z)$ is of the form (3.4) and analytic in $\mathbb{U}$ with $\Lambda(0) = 1$. Differentiating (3.19) with respect to z, then the subordination (3.18) becomes

$$[\Lambda(z)]^{a+1} + \frac{\alpha}{p}[\Lambda(z)]^a\, z\Lambda'(z) \prec h(z). \quad (3.20)$$

By setting

$$q(z) = \left(\frac{1+Az}{1-z}\right)^m, \qquad \theta(w) = w^{a+1} \qquad \text{and} \qquad \phi(w) = \frac{\alpha}{p}\, w^a \quad (3.21)$$

and noting that $q(z)$ is analytic and univalent in $\mathbb{U}$ with $q(0) = 1$, and also both $\theta(w)$ and $\phi(w)$ are analytic with $\phi(w) \neq 0$ in $C\backslash\{0\}$. We observe further that

$$Q(z) = zq'(z)\,\phi[q(z)] = \frac{\alpha\, m\,(1+A)z(1+Az)^{m(a+1)-1}}{p\,(1-z)^{a(m+1)+1}} \quad (3.22)$$

is univalent and starlike in $\mathbb{U}$, since

$$\begin{aligned} \Re\left(\frac{zQ'(z)}{Q(z)}\right) &= \Re\left[1 + (am+m-1)\frac{Az}{1+Az} + (am+m+1)\frac{z}{1-z}\right] \\ &> 1 - \frac{(am+m-1)|A|}{1+|A|} - \frac{am+m+1}{2} \\ &= \frac{(1-m(a+1))(1+3|A|)}{2(1+|A|)} \ > \ 0. \end{aligned}$$

Also, we infer that

$$h(z) = \theta[q(z)] + Q(z) = \left(\frac{1+Az}{1-z}\right)^{m(a+1)} \left(1 + \frac{\alpha\, m\, (A+1)z}{p\,(1-z)(1+Az)}\right) \;(z \in \mathbb{U}),$$

and

$$\begin{aligned}\Re\left(\frac{zh'(z)}{Q(z)}\right) &= \Re\left(\frac{\theta'[q(z)]}{\phi[q(z)]} + \frac{zQ'(z)}{Q(z)}\right)\\ &= \frac{p(a+1)}{\alpha} + \Re\left(\frac{zQ'(z)}{Q(z)}\right) \;\geq\; 0 \qquad (z \in \mathbb{U};\; \alpha \neq 0).\end{aligned} \tag{3.23}$$

for $z \in \mathbb{U}$. The inequality (3.23) thus shows that the function $h(z)$ is close-to-convex and univalent in $\mathbb{U}$, and it follows from (3.20)-(3.23) that

$$\theta[\Lambda(z)] + z\,\Lambda'(z)\,\phi[\Lambda(z)] \prec \theta[q(z)] + zq'(z)\,\phi[q(z)] = h(z).$$

Therefore, by virtue of Lemma 2, we conclude that $\Lambda(z) \prec q(z)$, which completes the proof of Theorem 5. □

Since

$$h(z) = 1 + \frac{1}{1-z} - \frac{1}{1+Az}$$

takes real values of z with $h(0) = 1$, and $h(\mathbb{U})$ is symmetric with respect to the real axis and

$$\Re\{h(z)\} > \frac{3}{2} - \frac{1}{1-|A|} \qquad (z \in \mathbb{U}),$$

therefore, by letting

$$m = \alpha = p = 1 \qquad \text{and} \qquad a = -1,$$

in Theorem 5, we obtain the following result.

Corollary 3.4. *Let*

$$-1 < A < 1 \qquad \textit{and} \qquad \frac{(f*g)(z)}{z} \neq 0 \qquad (z \in \mathbb{U}).$$

If $f(z) \in \mathcal{A}_p$ *satisfies*

$$\Re\left(\frac{z\,(f*g)'(z)}{(f*g)(z)}\right) > \frac{3}{2} - \frac{1}{1-|A|},$$

then

$$\frac{(f*g)(z)}{z} \prec \frac{1+Az}{1-z}.$$

Further, on setting

$$A = 0 \qquad \text{and} \qquad g(z) = \frac{z}{1-z}$$

in Corollary 4, we get

Corollary 3.5. *Let*

$$\frac{f(z)}{z} \neq 0 \qquad (z \in \mathbb{U}).$$

If $f(z) \in \mathcal{A}$ *satisfies*

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \frac{1}{2},$$

then

$$\frac{f(z)}{z} \prec \frac{1}{1-z}.$$

Remark 4. We observe that if we use the parametric substitutions given by (1.5) (with $p = 1$), apply the identity (1.6) for $p = 1$, then Theorem 5 corresponds to the result given in [2, p. 535, Theorem 2.2].

References

[1] Abramowitz, M. and Stegun, I. A.(Editors), *Handbook of Mathematical Functions and Formulas, Graphs and Mathematical Tables*, Dover Publications, New York, 1971.

[2] Aghalary, R., Joshi, S. B., Mohapatra, R. N. and Ravichandran, V., S*ubordinations for analytic functions defined by Dziok-Srivastava linear operator*, Appl. Math. Comput., **37** (2006), 533–542.

[3] Bernardi, S. D., *Convex and starlike univalent functions*, Trans. Amer. Math. Soc., **135** (1969), 429–446.

[4] Dinggong, Y. and Liu, J. L., *On a class of analytic functions involving Ruscheweyh derivatives*, Bull. Korean Math. Soc., **39(1)** (2002), 123–131.

[5] Dziok, J. and Raina, R. K., *Families of analytic functions associated with the Wright generalized hypergeometric functions*, Demonstratio Math., **37** (2004), 533–542.

[6] Dziok, J., Raina, R. K. and Srivastava, H. M., *Some classes of analytic functions associated with operator on Hilbert space involving Wright's generalized hypergeometric functions*, Proc. Janggeon Math. Soc., **7** (2004), 43–55.

[7] Dziok, J. and Srivastava, H. M., *Classes of analytic functions associated with the generalized hypergeometric functions*, Appl. Math. Comput., **103** (1999), 1–13.

[8] Liu, J. L., *On subordination for certain multivalent analytic functions associated with the generalized hypergeometric function*, J. Inequal. Pure Appl. Math., **7(4)** (Article 131)(2006), 1–6 (electronic).

[9] Livingston, A. E., *On the radius of the univalence of certain analytic functions*, Proc. Amer. Math. Soc., **17** (1966), 352–357.

[10] Miller S. S. and Mocanu P. T., *Differential Subordinates: Theory and Applications, Series in Pure and Applied Mathematics*, No. 225, Marcel Dekker, New York, (2000).

[11] Miller, S. S. and Mocanu, P. T., *Differential subordinations and univalent functions*, Michigan Math. J., **28** (1981), 157–171.

[12] Obradovic, M., *On certain inequalities for some regular functions in* $|z|$, Int. J. Math. Sci., **8** (1985), 671–678.

[13] Ozkan, O., *Some subordination results on multivalent functions defined by integral operator*, J. Inequal. Appl., **Volume 2007**(2007), Article ID 71616, 1–8 (electronic).

[14] R. K. Raina *Some results associated with fractional calculus operators involving Appell hypergeometric function*, J. Inequal. Pure Appl. Math., **10** (2009), 1–7 (electronic).

[15] Sham, S., Kulkurni, S. R. and Jahangiri, J. M., *Subordination properties of p-valent functions defined by integral operators*, Internat. J. Math. Math. Sci., **Volume 2006**(2006), Article ID 94572, 1–3 (electronic).

[16] Srivastava, H. M., Patel, J. and Mohapatra, G. P., *Some applications of differential subordination to a general class of multivalent functions*, Adv. Stud. Contemp. Math., **3(1)** (2001), 1–15.

[17] Srivastava, H. M., Patel, J. and Mohapatra, G. P., *A certain class of p-valently analytic functions*, Math. Comput. Modelling, **41** (2005), 321–324.

[18] Srivastava, H. M., Suchithra, K., Stephen, B. A. and Sivasubramanian, S., *Inclusion and neighborhood properties of certain subclasses of analytic and multivalent functions of complex order*, J. Inequal. Pure Appl. Math., **7(5)** (Article 191)(2006), 1–8 (electronic).

J. K. Prajapat
Department of Mathematics
Sobhasaria Engineering College
NH-11 Gokulpura, Sikar 332001, Rajasthan, India
E-mail address: jkp_0007@rediffmail.com

R. K. Raina
10/11 Ganpati Vihar, Opposite Sector 5
Udaipur 313002, Rajasthan, India
E-mail address: rkraina_7@hotmail.com