East Asian Mathematical Journal
Vol. 25 (2009), No. 4, pp. 495–504

LIGHTLIKE SUBMANIFOLDS OF AN INDEFINITE QUATERNION KAEHLERIAN MANIFOLD

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Abstract. We introduce three classes of quaternion lightlike submanifolds, screen real lightlike submanifolds and CR lightlike submanifolds of an indefinite quaternion Kaehlerian manifold and study the geometry of leaves of their distributions.

1. Introduction

It is well known that on an indefinite quaternion Kaehlerian manifold $\bar{M}$ with an indefinite Riemannian metric $\bar{g}$ there exists a 3-dimensional vector bundle V of tensors of type $(1,1)$ with local cross section of almost Hermitian structures $\{\Phi_\alpha; \alpha = 1,2,3\}$ satisfying certain conditions(cf. [7]). In case M is nondegenerate submanifold in an indefinite quaternion Kaehlerian manifold $\bar{M}$, the theory of quaternion CR submanifolds of quaternion Kaehlerian manifolds was introduced in [1]. In case of lightlike(or degenerate) submanifold M of indefinite quaternion Kaehlerian manifold $\bar{M}$, several vector bundles involved, for example, $Rad(TM)$, $S(TM)$, $ltr(TM)$ and $S(TM^\perp)$, etc. (see section 2). Therefore there would be several classes of lightlike submanifolds according to the action of almost Hermitian structures $\{\Phi_\alpha; \alpha = 1,2,3\}$ on subbundles of the tangent bundle TM. In this article we introduce the general notion of quaternion CR lightlike submanifolds, which is similar to the concept of lightlike CR submanifolds of indefinite Kaehler manifolds([4]). We study the integrability conditions of their distributions, and investigate the geometry of leaves of the distributions involved in the induced quaternion CR structure on M. Contrary to the nondegenerate case, but similar to the lightlike CR submanifolds of indefinite Kaehler manifolds, they do not include the quaternion lightlike and the screen real lightlike submanifolds.

Received May 6, 2009; Accepted December 1, 2009.
2000 *Mathematics Subject Classification.* 53C25, 53C25.
Key words and phrases. quaternion lightlike submanifold, quaternion CR lightlike submanifold, screen real submanifold, irrotational, indefinite quaternion Kaehlerian manifold.

2. Preliminaries

In this section, we recall briefly some results from the general theory of lightlike submanifolds(cf. [4]). An m-dimensional submanifold M immersed in an indefinite Riemannian manifold $(\bar{M}, \bar{g})$ of dimension $n+m$ is called a *lightlike submanifold* if it admits a degenerate metric g induced from $\bar{g}$ whose radical distribution $Rad(TM) := TM \cap TM^{\perp}$ is of rank $r(1 \leq r \leq m)$, where

$$TM^{\perp} := \cup_{x \in M}\{u \in T_x\bar{M} \ ; \ \bar{g}(u,v) = 0, \forall v \in T_xM\}.$$

Let $S(TM)$ be a *screen distribution* which is an indefinite Riemannian complementary distribution of $Rad(TM)$ in TM, i.e., $TM = Rad(TM) \perp S(TM)$, where the symbol $\perp$ denotes the orthogonal direct sum. We consider a *screen transversal vector bundle* $S(TM^{\perp})$, which is an indefinite Riemannian complementary vector bundle of $Rad(TM)$ in $TM^{\perp}$. Let $(M, g, S(TM))$ be a lightlike submanifold of an indefinite Riemannian manifold $(\bar{M}, \bar{g})$. For any local basis $\{\xi_i; i = 1, \cdots, r\}$ of $Rad(TM)$, there exists a local frame $\{N_i; i = 1, \cdots, r\}$ of sections with values in the orthogonal compelment of $S(TM^{\perp})$ in $[S(TM)]^{\perp}$ such that $\bar{g}(\xi_i, N_j) = \delta_{ij}$ and $\bar{g}(N_i, N_j) = 0$. It follows that there exists a *lightlike transversal vector bundle* $ltr(TM)$ locally spanned by $\{N_i\}$. Let $tr(TM)$ (called a *tranaversal vector bundle*) be complementary (but not orthogonal) vector bundle to TM in $T\bar{M}\mid_M$. Then we have the following decompositions :

$$\begin{aligned}
tr(TM) &= ltr(TM) \perp S(TM^{\perp}), \\
T\bar{M}\mid_M &= S(TM) \perp \{Rad(TM) \oplus ltr(TM)\} \perp S(TM^{\perp}) \\
&= TM \oplus tr(TM).
\end{aligned}$$

We say that a submanifold $(M, g, S(TM), S(TM^{\perp}))$ of $\bar{M}$ is

case 1 : r-*lightlike* if $r < min\{m, n\}$;
case 2 : *co-isotropic* if $r = n < m$, $S(TM^{\perp}) = \{0\}$;
case 3 : *isotropic* if $r = m < n$, $S(TM) = \{0\}$;
case 4 : *totally lightlike* if $r = m = n$, $S(TM) = \{0\}$ and $S(TM^{\perp}) = \{0\}$.

Throughout the paper $\Gamma(\bullet)$ denotes the smooth sections of the vector bundle $\bullet$. As TM and $tr(TM)$ are complementary vector subbundles of $T\bar{M}\mid_M$ we put

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y), \ \ \forall X, Y \in \Gamma(TM), \tag{1}$$
$$\bar{\nabla}_X U = -A_U X + \nabla^t_X U, \ \ \forall X \in \Gamma(TM), \ U \in \Gamma(tr(TM)), \tag{2}$$

where $\{\nabla_X Y, A_U X\}$ and $\{h(X,Y), \nabla^t_X U\}$ belong to $\Gamma(TM)$ and $\Gamma(tr(TM))$, respectively. ∇ and ∇^t are linear connections on M and on the vector bundle $tr(TM)$, respectively.

We note that the lightlike second fundamental forms of a lightlike submanifold M do not depend on $S(TM), S(TM^{\perp})$ and $ltr(TM)$. According to the decomposition $tr(TM) = ltr(TM) \perp S(TM^{\perp})$, (1) and (2) become in the form

$$\begin{aligned}\bar{\nabla}_X Y &= \nabla_X Y + h^l(X,Y) + h^s(X,Y), \ \ \forall X,Y \in \Gamma(TM), & (3)\\ \bar{\nabla}_X N &= -A_N X + \nabla^l_X N + D^s(X,N), \ \ N \in \Gamma(ltr(TM)), & (4)\\ \bar{\nabla}_X W &= -A_W X + \nabla^s_X W + D^l(X,W), \ \ W \in \Gamma(S(TM^{\perp})), & (5)\end{aligned}$$

where $h^l(X,Y), \nabla^l_X N, D^l(X,W) \in \Gamma(ltr(TM))$, and $h^s(X,Y), D^s(X,N), \nabla^s_X W \in \Gamma(S(TM^{\perp}))$.

Then, by using (1),(3),(4),(5) and the fact that $\bar{\nabla}$ is a metric connection, we obtain

$$\bar{g}(h^s(X,Y),W) + \bar{g}(Y, D^l(X,W)) = g(A_W X, Y), \tag{6}$$

$$\bar{g}(D^s(X,N),W) = \bar{g}(N, A_W X). \tag{7}$$

Denote the projections of TM on $S(TM)$ and $Rad(TM)$ by $\mathbf{S}$ and $\mathbf{R}$, respectively. From the decomposition $TM = S(TM) \perp Rad(TM)$, we have

$$\begin{aligned}\nabla_X \mathbf{S}Y &= \nabla^*_X \mathbf{S}Y + h^*(X, \mathbf{S}Y), & (8)\\ \nabla_X \xi &= -A^*_\xi X + \nabla^{*t}_X \xi & (9)\end{aligned}$$

for $X,Y \in \Gamma(TM)$ and $\xi \in \Gamma(Rad(TM))$, where $\{\nabla^*_X \mathbf{S}Y, A^*_\xi X\}$ and $\{h^*(X,\mathbf{S}Y), \nabla^{*t}_X \xi\}$ belong to $\Gamma(S(TM))$ and $\Gamma(Rad(TM))$, respectively. It follows that ∇^* and ∇^{*t} are linear connections on distributions $S(TM)$ and $Rad(TM)$, respectively. By using the above equations we obtain

$$\begin{aligned}\bar{g}(h^l(X,\mathbf{S}Y),E) &= g(A^*_E X, \mathbf{S}Y), & (10)\\ \bar{g}(h^*(X,\mathbf{S}Y),N) &= g(A_N X, \mathbf{S}Y), & (11)\\ \bar{g}(h^l(X,\xi),\xi) = 0, & \qquad A^*_\xi \xi = 0. & (12)\end{aligned}$$

In general, the induced connection ∇ on M is not metric connection, since

$$(\nabla_X g)(Y,Z) = \bar{g}(h^l(X,Y),Z) + \bar{g}(h^l(X,Z),Y), \tag{13}$$

which is easily obtained by using (3) and the fact that $\bar{\nabla}$ is a metric connection. However, it is important to note that ∇^* is a metric connection on $S(TM)$.

3. Quaternion Lightlike Submanifolds

Let $(\bar{M}, V)$ be an almost quaternion manifold (cf. [8])with a canonical local basis $\{\Phi_\alpha; \alpha = 1,2,3\}$ of a 3-dimensional vector bundle V consisting of tensors of type $(1,1)$ over $\bar{M}$. Consider an indefinite Riemannian metric $\bar{g}$ on $\bar{M}$ satisfying

$$\bar{g}(\Phi_\alpha X, \Phi_\alpha Y) = \bar{g}(X,Y), \quad \alpha = 1,2,3 \tag{14}$$

for any vector fields X,Y on $\bar{M}$. Then $(\bar{M}, \bar{g}, V)$ is called an *indefinite almost quaternion Kaehlerian manifold.*

If the Levi-Civita connection $\bar{\nabla}$ of $(\bar{M}, \bar{g}, V)$ satisfies

$$\begin{aligned}\bar{\nabla}_X \Phi_1 &= a_3(X)\Phi_2 - a_2(X)\Phi_3,\\ \bar{\nabla}_X \Phi_2 &= -a_3(X)\Phi_1 + a_1(X)\Phi_3,\\ \bar{\nabla}_X \Phi_3 &= a_2(X)\Phi_1 - a_1(X)\Phi_2\end{aligned} \tag{15}$$

for any vector field X on $\bar{M}$, a_1, a_2, a_3 being local 1-forms over the open for which $\{\Phi_\alpha; \alpha = 1,2,3\}$ is a local basis of V, we shall say that $(\bar{M}, \bar{g}, V)$, or simply $\bar{M}$ is an *indefinite quaternion Kaehlerian manifold.*

Let $(M, g, S(TM), S(TM^\perp))$ be a lightlike submanifold of an indefinite quaternion Kaehlerian manifold $(\bar{M}, \bar{g}, V)$. For any vector field X tangent to M, we put

$$\Phi_\alpha X = \phi_\alpha X + \psi_\alpha X, \quad \alpha = 1,2,3 \tag{16}$$

where $\phi_\alpha X$ and $\psi_\alpha X$ are the tangential and transversal parts of $\Phi_\alpha X$, repectively. Moreover, each ϕ_α is skew symmetric on $S(TM)$.

A lightlike submanifold M of $(\bar{M}, \bar{g}, V)$ is called to be *quaternion* if for any $\alpha = 1,2,3$

$$\Phi_\alpha X = \phi_\alpha X, \quad i.e., \quad \Phi_\alpha X \in \Gamma(TM), \quad \forall X \in \Gamma(TM). \tag{17}$$

Remark 1. Let $(M, g, S(TM), S(TM^\perp))$ be a quaternion lightlike submanifold of an indefinite quaternion Kaehlerian manifold $\bar{M}$. Then the distributions $Rad(TM)$ and $S(TM)$ are Φ_α-invariant $(\alpha = 1,2,3)$. It follows that distributions $S(TM^\perp), ltr(TM)$ and $tr(TM)$ are also Φ_α-invariant$(\alpha = 1,2,3)$.

Differentiating the equations (17) and comparing the tangential and transversal parts, we get from Remark 3.1

$$\begin{aligned}(\nabla_X \phi_1)Y &= a_3(X)\phi_2 Y - a_2(X)\phi_3 Y,\\ (\nabla_X \phi_2)Y &= -a_3(X)\phi_1 Y + a_1(X)\phi_3 Y,\\ (\nabla_X \phi_3)Y &= a_2(X)\phi_1 Y - a_1(X)\phi_2 Y,\end{aligned} \tag{18}$$

and

$$h(X, \phi_\alpha Y) = \Phi_\alpha h(X,Y) = h(\phi_\alpha X, Y), \quad \alpha = 1,2,3 \tag{19}$$

for any vector fields X, Y on M.

Due to (18), the aggreate $\{M, \phi_\alpha(\alpha = 1,2,3), g\}$ is a singular quaternion Kaehlerian manifold in a sense that the induced metric tensor g on M is degenerate.

From (19) we have the following, which is the same result as in case of nondegenerate submanifolds (cf. [3]).

Proposition 3.1. *Any quaternion lightlike submanifold of an indefinite quaternion Kaehlerian manifold is totally geodesic.*

Proposition 3.2. *Let $(M, g, S(TM), S(TM^{\perp}))$ be a quaternion lightlike submanifold of an indefinite quaternion Kaehlerian manifold $(\bar{M}, V, \bar{g})$. Then $S(TM)$ is integrable if and only if*

$$h^*(X, \phi_\alpha Y) = h^*(\phi_\alpha X, Y),\ \forall X, Y \in \Gamma(S(TM)),$$

where α is 1,2 or 3.

Proof. Differentiating the equation (17) and comparing radical parts, we get for any vector fields X, Y tangent to M

$$\mathbf{R}(\phi_\alpha \nabla_Y X) = h^*(Y, \phi_\alpha X),$$

which implies that

$$\mathbf{R}(\phi_\alpha [X, Y]) = h^*(X, \phi_\alpha Y) - h^*(Y, \phi_\alpha X).$$

Therefore $h^*(X, \phi_\alpha Y) - h^*(Y, \phi_\alpha X) = 0$ if and only if $\mathbf{R}(\phi_\alpha[X, Y]) = 0$ if and only if $\phi_\alpha[X, Y] \in S(TM)$, i.e., $[X, Y] \in S(TM)$, which completes the proof. □

Proposition 3.3. *Let $(M, g, S(TM), S(TM^{\perp}))$ be a quaternion lightlike submanifold of of an indefinite quaternion Kaehlerian manifold $\bar{M}$. Then $Rad(TM)$ defines a totally geodesic foliation on M.*

Proof. For any $X, Y \in \Gamma(Rad(TM))$ and $Z \in \Gamma(TM)$ we get

$$\begin{aligned} \bar{g}(\nabla_X Y, \mathbf{S}Z) &= \bar{g}(\bar{\nabla}_X Y, \mathbf{S}Z) \\ &= X(\bar{g}(Y, \mathbf{S}Z)) - \bar{g}(Y, \bar{\nabla}_X \mathbf{S}Z) \\ &= 0, \end{aligned}$$

since M is totally geodesic. Hence $\mathbf{S}\nabla_X Y = 0$, which means that $Rad(TM)$ defines a totally geodesic foliation. □

4. Quaternion CR lightlike Submanifolds.

Let $(M, g, S(TM), S(TM^{\perp}))$ be a lightlike submanifold of an indefinite quaternion Kaehlerian manifold $\bar{M}$. We say that M is a *quaternion CR lightlike submanifold* of $\bar{M}$ if the following conditions (A) and (B) are satisfied :
(A) $Rad(TM)$ is a distribution on M such that

$$Rad(TM) \cap \Phi_\alpha(Rad(TM)) = \{0\}, \forall \alpha = 1, 2, 3;$$

(B) There exist vector bundles D_0 and D' over M such that

$$\begin{aligned} S(TM) &= \{\oplus_{\alpha=1}^3 \Phi_\alpha(Rad(TM)) \oplus D'\} \perp D_0, \\ \Phi_\alpha(D_0) &= D_0,\ \ \Phi_\alpha(D') = \mu_\alpha \perp \nu_\alpha, \forall \alpha = 1, 2, 3, \end{aligned}$$

where D_0 is non-degenerate, μ_α and ν_α are vector subbundles of $S(TM^{\perp})$ and $ltr(TM)$, respectively.

Thus, we have the following decomposition

$$TM = D \oplus D',\ \ D = Rad(TM) \perp \oplus_{\alpha=1}^3 \Phi_\alpha(Rad(TM)) \perp D_0. \tag{20}$$

Proposition 4.1. *Let M be a quaternion CR lightlike submanifold of an indefinite quaternion Kaehlerian manifold $\bar{M}$. Then we have for $\alpha = 1, 2, 3$*
(i) ν_α is a non-zero totally lightlike vector subbundle of $T\bar{M}\mid_M$,
(ii) μ_α is a non-degenerate vector subbundle of $S(TM^\perp)$,
(iii) D' is a degenerate distribution on M.

Proof. ν_α is totally lightlike as a vector subbundle of the totally lightlike vector bundle of $ltr(TM)$. Suppose that $\nu_\alpha = \{0\}$. Then $D' = \Phi_\alpha\mu_\alpha$, which showes that $S(TM)$ is degenerate. This is a contradiction. Next, we suppose that μ_α is degenerate. Then the condition (B) leads to a contradiction that $S(TM)$ is degenerate too. Finally, (iii) is a consequence of (i). □

Proposition 4.2. *There exist no isotropic or totally lightlike quaternion CR lightlike submanifolds on $\bar{M}$.*

Proof. If M is isotropic or totally lightlike, then $S(TM) = \{0\}$. Hence conditions (A) and (B) of the definition are not satisfied. □

Denote the projections of TM on D and D' by $\mathbf{P}$ and $\mathbf{Q}$, respectively. For a quaternion CR lightlike submanifold M we put

$$\Phi_\alpha X = f_\alpha X + \omega_\alpha X,\ \alpha = 1, 2, 3\ \ \forall X \in \Gamma(TM), \tag{21}$$

where $f_\alpha X = \Phi_\alpha \mathbf{P} X \in \Gamma(D)$ and $\omega_\alpha X = \Phi_\alpha \mathbf{Q} X \in \Gamma(\mu_\alpha \perp \nu_\alpha)$. Then f_α is a tensor field of type $(1,1)$ on M and ω_α is $\Gamma(\mu_\alpha \perp \nu_\alpha)$-valued 1-form on M.

On the other hand, we set

$$\Phi_\alpha U = B_\alpha U + C_\alpha U,\ \alpha = 1, 2, 3\ \ \forall U \in \Gamma(tr(TM)), \tag{22}$$

where $B_\alpha U \in \Gamma(TM)$ and $C_\alpha U \in \Gamma(tr(TM))$.

Differentiating (21) and (22) with (15), we obtain

Lemma 4.3. *Let M be a quaternion CR lightlike submanifold of an indefinite quaternion Kaehlerian manifold $\bar{M}$. Then we have for $\forall X, Y \in \Gamma(TM)$ and*

$U \in \Gamma(tr(TM))$

$$(\nabla_X f_1)Y = a_3(X)f_2Y - a_2(X)f_3Y + A_{\omega_1 Y}X + B_1h(X,Y), \quad (23)$$
$$(\nabla_X f_2)Y = -a_3(X)f_1Y + a_1(X)f_3Y + A_{\omega_2 Y}X + B_2h(X,Y), \quad (24)$$
$$(\nabla_X f_3)Y = a_2(X)f_1Y - a_1(X)f_2Y + A_{\omega_3 Y}X + B_3h(X,Y), \quad (25)$$
$$(\nabla^t_X \omega_1)Y = a_3(X)\omega_2Y - a_2(X)\omega_3Y + C_1h(X,Y) - h(X,f_1Y), \quad (26)$$
$$(\nabla^t_X \omega_2)Y = -a_3(X)\omega_1Y + a_1(X)\omega_3Y + C_2h(X,Y) - h(X,f_2Y), \quad (27)$$
$$(\nabla^t_X \omega_3)Y = a_2(X)\omega_1Y - a_1(X)\omega_2Y + C_3h(X,Y) - h(X,f_3Y), \quad (28)$$
$$(\nabla_X B_1)U = a_3(X)B_2U - a_2(X)B_3U - f_1(A_UX) + A_{C_1U}X, \quad (29)$$
$$(\nabla_X B_2)U = -a_3(X)B_1U + a_1(X)B_3U - f_2(A_UX) + A_{C_2U}X, \quad (30)$$
$$(\nabla_X B_3)U = a_2(X)B_1U - a_2(X)B_2U - f_3(A_UX) + A_{C_3U}X, \quad (31)$$
$$(\nabla^t_X C_1)U = a_3(X)C_2U - a_2(X)C_3U - \omega_1(A_UX) - h(X,B_1U), \quad (32)$$
$$(\nabla^t_X C_2)U = -a_3(X)C_1U + a_1(X)C_3U - \omega_2(A_UX) - h(X,B_2U), \quad (33)$$
$$(\nabla^t_X C_3)U = a_2(X)C_1U - a_2(X)C_2U - \omega_3(A_UX) - h(X,B_3U), \quad (34)$$

where $(\nabla_X f_\alpha)Y := \nabla_X f_\alpha Y - f_\alpha(\nabla_X Y)$, $(\nabla^t_X \omega_\alpha)Y := \nabla^t_X \omega_\alpha Y - \omega_\alpha(\nabla_X Y)$, $(\nabla_X B_\alpha)U := \nabla_X B_\alpha U - B_\alpha(\nabla^t_X U)$, $(\nabla^t_X C_\alpha)U := \nabla^t_X C_\alpha U - C_\alpha(\nabla^t_X U)$ *for* $\alpha = 1, 2, 3$.

Theorem 4.4. *Let M be a quaternion CR lightlike submanifold of an indefinite quaternion Kaehlerian manifold $\bar{M}$. Then we have the followings :*
(i) The quaternion distribution D is integrable if and only if the second fundamental form of M satisfies

$$h(X, \Phi_\alpha Y) = h(\Phi_\alpha X, Y), \quad \forall X, Y \in \Gamma(D);$$

(ii) The totally real distribution D' is integrable if and only if the shape operator of M satisfies

$$A_{\Phi_\alpha X}Y = A_{\Phi_\alpha Y}X, \quad \forall X, Y \in \Gamma(D'),$$

where α is $1, 2$ *or* 3.

Proof. From (26) $\sim$ (28) it follows that

$$h(X, \Phi_\alpha Y) = C_\alpha h(X,Y) + \omega_\alpha(\nabla_X Y), \ \forall X, Y \in \Gamma(D).$$

Then taking into account that h is symmetric and ∇ is torsion free, we deduce

$$\omega_\alpha([X,Y]) = h(X, \Phi_\alpha Y) - h(Y, \Phi_\alpha X).$$

It is clear that $\omega_\alpha[X,Y] = 0$ for any $X, Y \in \Gamma(D)$ if and only if D is integrable. Hence we prove assertion (i). Next, in a similar way, from (23) $\sim$ (25) we get

$$f_\alpha(\nabla_X Y) = A_{\Phi_\alpha X}Y - A_{\Phi_\alpha Y}X \ \ \forall X, Y \in \Gamma(D'),$$

which proves assertion (ii). $\square$

Next, let $\{N_i; i = 1, \cdots, r\}$ be a basis of $\Gamma(ltr(TM))$ with respect to the basis $\{\xi_i; i = 1, \cdots, r\}$ of $\Gamma(Rad(TM))$ such that $\{N_{\alpha i}; i = 1, \cdots, p\}$ is basis of ν_α. Also consider an orthonormal basis $\{W_{\alpha a}; a = 1, \cdots, q\}$ of μ_α. Then we can restate (i) of Theorem 4.4 in the following form.

Corollary 4.5. *Let M be a quaternion CR lightlike submanifold of an indefinite quaternion Kaehlerian manifold $\bar{M}$. Then the quaternion distribution D is integrable if and only if the second fundamental form of M satisfies*

$$\bar{g}(h(X, \Phi_\alpha Y) - h(Y, \Phi_\alpha X), \xi_{\alpha i}) = 0, \quad i = 1, \cdots, p,$$

$$\bar{g}(h(X, \Phi_\alpha Y) - h(Y, \Phi_\alpha X), W_{\alpha a}) = 0, \quad a = 1, \cdots, q,$$

where α is $1, 2, 3$.

Theorem 4.6. *Let M be a quaternion CR lightlike submanifold of an indefinite quaternion Kaehlerian manifold $\bar{M}$. Then D defines a totally geodesic foliation on M if and only if for any $X, Y \in \Gamma(TM)$*

$$h(X, Y) \ \textit{has no components in} \ \Gamma(\mu_\alpha \perp \nu_\alpha), \alpha = 1, 2 \ \textit{or}\ 3. \tag{35}$$

Proof. Clearly, D defines a totally geodesic foliation on M, if and only if

$$\nabla_X Y \in \Gamma(D), \ \forall X, Y \in \Gamma(D). \tag{36}$$

Using the first decomposition in (20) and $D' = \Phi_\alpha(\mu_\alpha \perp \nu_\alpha)$, we know that (36) holds if and only if

$$\bar{g}(\nabla_X Y, \Phi_\alpha \xi_{\alpha i}) = 0, \ \forall i \in \{1, \cdots, p\}, \tag{37}$$

$$\bar{g}(\nabla_X Y, \Phi_\alpha W_{\alpha a}) = 0, \ \forall a \in \{1, \cdots, q\}. \tag{38}$$

By using (15), we get

$$\bar{g}(\nabla_X Y, \Phi_\alpha \xi_{\alpha i}) = -\bar{g}(h(X, \Phi_\alpha Y), \xi_{\alpha i}), \tag{39}$$

and

$$\bar{g}(\nabla_X Y, \Phi_\alpha W_{\alpha a}) = -\bar{g}(h(X, \Phi_\alpha Y), W_{\alpha a}). \tag{40}$$

Since D is a quaternion distribution, our assertion follows from (37) $\sim$ (40). $\square$

Theorem 4.7. *Let M be a quaternion CR lightlike submanifold of an indefinite quaternion Kaehlerian manifold $\bar{M}$. Then a totally real distribution D' defines a totally geodesic foliation on M if and only if for any $X, Y \in \Gamma(D')$ we have*

$$h^*(X, Y) = 0, \tag{41}$$

$$A_{\Phi_\alpha Y} X \ \textit{has no component in}\ \Gamma(D_0), \tag{42}$$

$$A_{\Phi_\alpha Y} X + B_\alpha h(X, Y) \ \textit{has no component in}\ \Gamma(Rad(TM)), \tag{43}$$

where α is 1,2,3.

Proof. It is clear from (20) that D' defines a totally geodesic foliation if and only if

$$\bar{g}(\nabla_X Y, N) = \bar{g}(\nabla_X Y, Z) = \bar{g}(\nabla_X Y, \Phi_\alpha N) = 0,\ \forall \alpha = 1, 2, 3 \tag{44}$$

for any $X, Y \in \Gamma(D')$ and $Z \in \Gamma(D_0)$, $N \in \Gamma(ltr(TM))$.

By using (1) and (2), we obtain from (44)

$$\bar{g}(\nabla_X Y, N) = \bar{g}(h^*(X, Y), N), \tag{45}$$

$$\bar{g}(\nabla_X Y, Z) = \bar{g}(A_{\Phi_\alpha Y} X, Z'), Z = \Phi_\alpha Z', \tag{46}$$

and

$$\bar{g}(\nabla_X Y, \Phi_\alpha N) = \bar{g}(A_{\Phi_\alpha Y} X + B_\alpha h(X, Y), N), \tag{47}$$

where we have used (15) and the fact that D_0 is a quaternion distribution. Our assertion follows from (44) $\sim$ (47). □

Let M be a quaternion CR lightlike submanifold of an indefinite quaternion Kaehlerian manifold $\bar{M}$ such that both D and D' define totally geodesic foliations on M. Then M is locally represented as a product of $S \times S'$ where S and S' are leaves of D and D' respectively and both are totally geodesic immersed in M. In this case we call M a *QR lightlike product.*

Theorem 4.8. *Let M be a totally geodesic quaternion CR lightlike submanifold of an indefinite quaternion Kaehlerian manifold $\bar{M}$. Suppose that there exists a transversal vector bundle of M which is parallel along D' with respect to the Levi-Civita connection on $\bar{M}$, i.e.,*

$$\bar{\nabla}_X U \in \Gamma(tr(TM)),\ \forall U \in \Gamma(tr(TM)),\ X \in \Gamma(D'). \tag{48}$$

Then M is a QR lightlike product.

Proof. Since M is totally geodesic, it follows from Theorem 4.6 that D is a totally geodesic foliation on M. The condition (48) and (2) imply that $A_U X = 0$ for any $X \in \Gamma(D')$ and $U \in \Gamma(tr(TM))$. Hence both (42) and (43) hold. Moreover, by using (11), we have (41). Thus, by theorem 4.7, D' is a totally geodesic foliation. Therefore M is a QR lightlike product.

□

A lightlike submanifold M of an indefinite Riemannian manifold is *irrotational*([6]) if $\bar{\nabla}_X \xi \in \Gamma(TM)$ for any $X \in \Gamma(TM)$ and $\xi \in \Gamma(Rad(TM))$.

From (3) we conclude that M is an irrotational lightlike submanifold if and only if the followings hold :

$$h^s(X, \xi) = 0,\ \ h^l(X, \xi) = 0,\ \forall X \in \Gamma(TM),\ \xi \in \Gamma(Rad(TM)).$$

We say that a lightlike submanifold M of an indefinite quaternion Kaehlerian manifold $\bar{M}$, is *screen real* if $Rad(TM)$ and $S(TM)$ are a quaternion distribution and a totally real distribution with respect to $\Phi_\alpha, \alpha = 1, 2, 3$, respectively.

Proposition 4.9. *There exists a Levi-Civita connection on an irrotational screen real lightlike submanifold of an indefinite quaternion Kaehlerian manifold.*

Proof. From (3) we have $\bar{g}(h^l(X,Y),\xi) = \bar{g}(\bar{\nabla}_X Y,\xi), \quad \forall X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(Rad(TM))$. By using (14) and (15) we get

$$\bar{g}(\bar{\nabla}_X Y,\xi) = \bar{g}(\Phi_\alpha \bar{\nabla}_X Y, \Phi_\alpha \xi) = \bar{g}(\bar{\nabla}_X \Phi_\alpha Y, \Phi_\alpha \xi),$$

where α is $1,2,3$. Since $\bar{\nabla}$ is a metric connection we have

$$0 = \bar{\nabla}_X \bar{g}(\Phi_\alpha Y, \Phi_\alpha \xi) = -\bar{g}(\bar{\nabla}_X \Phi_\alpha Y, \Phi_\alpha \xi) - \bar{g}(\Phi_\alpha Y, \bar{\nabla}_X \Phi_\alpha \xi).$$

It follows from the above equations and M being irrotational that

$$\bar{g}(h^l(X,Y),\xi) = -\bar{g}(\Phi_\alpha Y, \bar{\nabla}_X \Phi_\alpha \xi) = -\bar{g}(\Phi_\alpha Y, h^s(X, \Phi_\alpha \xi)) = 0.$$

Hence $h^l = 0$. Then the proof follows from (13). □

Proposition 4.10. *A quaternion CR lightlike submanifolds are nontrivial.*

Proof. Suppose M is a quaternion lightlike submanifold of an indefinite quaternion Kaehlerian manifold. Then we see from Remark 3.2 that radical distribution is invariant, which is not consistent with condition (A) of the definition. Similarly, the case of screen real can be also argued. □

Acknowledgement
This work was supported by 2008 Research Fund of University of Ulsan.

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