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COMBINATORIAL WEBS OF QUANTUM LIE SUPERALGEBRA $\mathfrak{sl}(1|1)$

Dongseok Kim

Abstract. Temperley-Lieb algebras had been generalized to web spaces for rank 2 simple Lie algebras which led us to link invariants for these Lie algebras as a generalization of Jones polynomial. Recently, Geer found a new generalization of Jones polynomial for some Lie superalgebras. In this paper, we study the quantum $\mathfrak{sl}(1|1)$ representation theory using the web space and find a finite presentation of the representation category (for generic q) of the quantum $\mathfrak{sl}(1|1)$.

1. Introduction

After the historical invention of Jones polynomial [7, 8], it brought a Renaissance of the study of knots and links [3, 10, 22]. To generalize Jones polynomial there are three important ways to produce the quantum link invariants. First one can use the quantum Yang-Baxter equation: using a solution of the Quantum Yang-Baxter equation, $\mathcal{R}$-matrix, one can obtain the link invariant as a trace of a representation of the braid into a tensor power from a presentation of a link by a closed braid [2, 3]. From the original work of N. Reshetikhin and V. Turaev one can construct of a functor from a monoidal category $\mathcal{C}$-colored framed tangles to $\mathcal{C}$ where $\mathcal{C}$ is a ribbon category, the image of framed links lands in the coefficient ring $\mathcal{C}[q^{1/2}, q^{-1/2}]$ [9, 17, 18]. The last is the main topic of the present article that we find link invariants using skein expansion of crossing together with a geometric counterpart for the invariants vectors which is called webs and their presentation using the representation theory of quantum algebras [1, 11–16, 19].

Recently, Geer found a new generalization of Jones polynomial for some Lie superalgebras [4–6]. The Lie superalgebra $\mathfrak{sl}(1|1)$ is the simplest special linear Lie superalgebra. As a start of the research on the presentation of the representation category of quantum superalgebras, we study the quantum $\mathfrak{sl}(1|1)$

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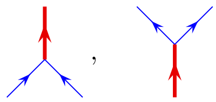
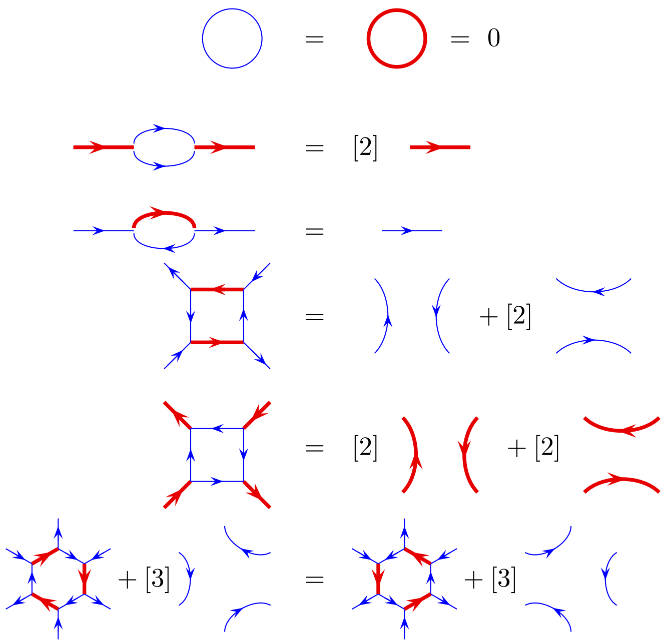
representation theory using the web space and find a finite presentation of the representation category $\mathrm{Mod}(\mathfrak{sl}(1|1))_q$ for generic q in the following theorem.

Theorem 1.1. *The representation category generated by vertices*



that satisfy the relations



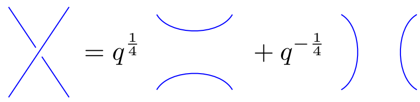
(1)
(2)
(3)
(4)
(5)
(6)

is isomorphic to $\mathrm{Mod}(\mathfrak{sl}(1|1))_q$ *after completion with projectors.*

The outline of this paper is as follows. In section 2, we review the representation theory of the quantum Lie algebras. In section 3, we develop the representation theory of the quantum $\mathfrak{sl}(1|1)$. In section 4 we prove Theorem 1.1. In section 5, we discuss further research problems on the representation theory of the quantum $\mathfrak{sl}(1|1)$.

2. Preliminaries

The Kauffman bracket polynomial can be defined by these two skein relations



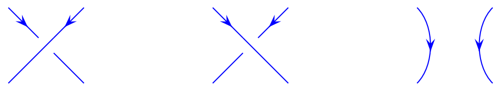
together with

$$\bigcirc = -[2] = -\frac{q^{\frac{2}{2}} - q^{-\frac{2}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} = -q^{\frac{1}{2}} - q^{-\frac{1}{2}}.$$

The specialized HOMFLY polynomials P_n can be calculated as

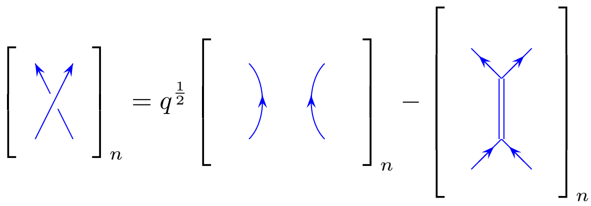
$$P_n(\emptyset) = 1,$$
$$P_n(\bigcirc \cup D) = (\frac{q^{\frac{n}{2}} - q^{-\frac{n}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}})P_n(D),$$
$$q^{\frac{n}{2}} P_n(L_+) - q^{-\frac{n}{2}} P_n(L_-) = (q^{\frac{1}{2}} - q^{-\frac{1}{2}})P_n(L_0),$$

where $\emptyset$ is the empty diagram, $\bigcirc$ is the trivial knot and L_+, L_- and L_0 are skein triple.



L_+ L_- L_0

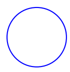
The HOMFLY polynomial of links can be recovered from the representation theory of the quantum $\mathfrak{sl}(n)$. For $n = 1$ and for any link, $P_1(q) = 1$. For $n = 2$, $P_2(q)$ is the Jones polynomial [8, 17, 18, 22]. The polynomial $P_n(q)$ can be computed by linearly expanding each crossing into a sum of diagrams of planar trivalent graphs where the edges of these planar graphs are oriented and colored by 1 or 2 [15].



2.1. The quantum representation category of Lie algebras

Let $\mathfrak{g}$ be a complex simple Lie algebra. Let $\mathrm{Mod}(\mathfrak{g})_q$ be the category of finite-dimensional representations over $\mathbb{Q}(q)$ of the quantum group $U_q(\mathfrak{g})$, which deform representations of $U(\mathfrak{g})$. Then $\mathrm{Mod}(\mathfrak{g})_q$ is well-known to be a ribbon category. In particular, it is a pivotal category. This means that any planar graph has a well-defined scalar value, provided that each edge is labeled with a representation of $U_q(\mathfrak{g})$ and each vertex is labeled with an invariant vector.

For $\mathfrak{g} = \mathfrak{sl}(2)$, there is a well-known presentation of its representation category as a pivotal category, the Temperley-Lieb category. In this presentation, there is a single unoriented edge type V_1, the fundamental representation, and at first there are no vertices. Therefore the only possible "word" in this presentation is a set of curves properly embedded in a disk. We assume the sole relation



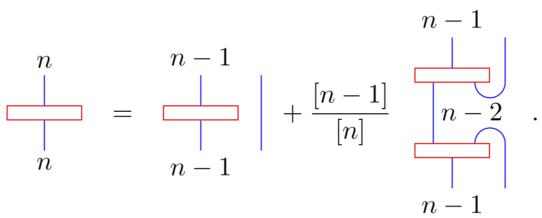
$$= -[2] = -\frac{q^{\frac{2}{2}} - q^{-\frac{2}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} = -q^{\frac{1}{2}} - q^{-\frac{1}{2}}.$$

This relation implies that the invariant space $\mathrm{Inv}(V_1^{\otimes n})$, or the skein space of a disk with n marked boundary points, has a basis, the set of crossingless matchings, or chord diagrams. Strictly speaking, this is not all of $\mathrm{Mod}(\mathfrak{g})_q$, but only the subcategory supported on the objects $V_1^{\otimes n}$. However, we can model any other irreducible representation V_n by the highest-weight projection

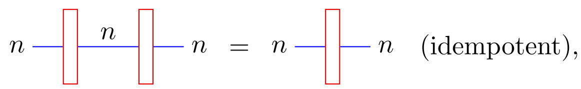
$$c : V_1^{\otimes n} \to V_1^{\otimes n}$$

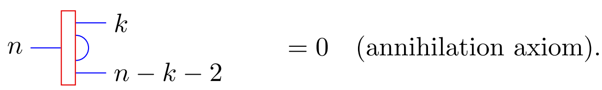
whose image is V_n. Expressed combinatorially in the Temperley-Lieb category, this projection is called a *Jones-Wenzl projector*, or more simply a *clasp*.

The Temperley-Lieb category can be completed to all of $\mathrm{Mod}(\mathfrak{sl}(2))_q$ using clasps. A recursive formula for a clasp of weight n is

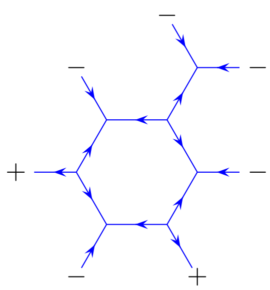


Its properties are





Kuperberg found generalizations of this presentation to rank 2 Lie algebras, $\mathfrak{sl}(3), \mathfrak{sp}(4)$ and G_2, called "spiders" [13]. Words in the generating edges and vertices are then called *webs*. For $\mathfrak{sl}(3)$, an example of the webs with a boundary $(+-+----)$ is



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The main property of a spider presentation is that the relations are *confluent*. Confluence means that the relations have the following two properties:

(1) Each relation has a single leading term which is the most complicated with respect to a natural partial ordering.

(2) If the relations are applied to simplify webs, they can be made to commute by applying further simplifying relations.

2.2. Lie superalgebras

Formally, a *Lie superalgebra* is a (nonassociative) $\mathbb{Z}_2$-graded algebra, or superalgebra, over a commutative ring (typically $\mathbb{R}$ or $\mathbb{C}$) whose product $[\cdot,\cdot]_s$, called the *Lie superbracket* or *supercommutator*, satisfies the two conditions (analogs of the usual Lie algebra axioms, with grading):

(1) Super skew-symmetry:

$$[x, y]_s = -(-1)^{|x||y|}[y, x]_s,$$

(2) The super Jacobi identity:

$$(-1)^{|z||x|}[x, [y, z]_s]_s + (-1)^{|x||y|}[y, [z, x]_s]_s + (-1)^{|y||z|}[z, [x, y]_s]_s = 0$$

where x, y, and z are pure in the $\mathbb{Z}_2$-grading. Here, $|x|$ denotes the degree of x (either 0 or 1).

3. The representation theory of the quantum Lie superalgebra $\mathfrak{sl}(1|1)$

$\mathfrak{sl}(1|1)$ is a Lie superalgebra with basis E, N, X, Y where $deg(E) = deg(N) = 0, deg(X) = deg(Y) = 1$ and relations

$$[N, X]_s = X,\ [N, Y]_s = -Y,\ [X, Y]_s = E,$$
$$[E, N]_s = [E, X]_s = [E, Y]_s = 0.$$

We consider one dimensional representation 1_n spanned by v^n, $deg(v^n) = 0$ with $\mathfrak{sl}(1|1)$ acting by

$$Ev^n = Xv^n = Yv^n = 0,\ Nv = nv.$$

We also consider two dimensional representation $2_{e,n}$ spanned by $\upsilon_0^{e,n}, \upsilon_1^{e,n}$ and the action of $\mathfrak{sl}(1|1)$ by

$$\begin{aligned}
&E\upsilon_0^{e,n} = e\upsilon_0^{e,n},\ N\upsilon_0^{e,n} = n\upsilon_0^{e,n},\ X\upsilon_0^{e,n} = 0,\ Y\upsilon_0^{e,n} = e\upsilon_1^{e,n},\\
&E\upsilon_1^{e,n} = e\upsilon_1^{e,n},\ N\upsilon_1^{e,n} = (n-1)\upsilon_0^{e,n},\ X\upsilon_1^{e,n} = \upsilon_0^{e,n},\ Y\upsilon_1^{e,n} = 0.
\end{aligned}$$

The R-matrix is an intertwiner for any pair of representations V, W:

$$R : V \otimes W \longrightarrow W \otimes V,$$

which is a signed permutation

$$R(x \otimes y) = (-1)^{deg(x)deg(y)} y \otimes x.$$

Then it clearly holds the Yang-Baxter equation

$$R^2 = Id,\ R_1R_2R_1 = R_2R_1R_2.$$

The tensor product $2_{e_1,n_1} \otimes 2_{e_2,n_2}$ contains two dimensional subrepresentations $2_{e_1+e_2,n_1+n_2}$ and $\overline{2_{e_1+e_2,n_1+n_2}}$ spanned by $\upsilon_0^{e_1,n_1} \otimes \upsilon_0^{e_2,n_2}$, $e_2\upsilon_0^{e_1,n_1} \otimes \upsilon_1^{e_2,n_2} + e_1\upsilon_1^{e_1,n_1} \otimes \upsilon_0^{e_2,n_2}$, and $\upsilon_0^{e_1,n_1} \otimes \upsilon_1^{e_2,n_2} - \upsilon_1^{e_1,n_1} \otimes \upsilon_0^{e_2,n_2}$, $\upsilon_1^{e_1,n_1} \otimes \upsilon_1^{e_2,n_2}$.

If $e_1 + e_2 \neq 0$, these representations $2_{e_1+e_2,n_1+n_2}$, $\overline{2_{e_1+e_2,n_1+n_2}}$ intersect trivially. Consequently, as a $\mathfrak{sl}(1|1)$ module, $2_{e_1,n_1} \otimes 2_{e_2,n_2}$ can be decomposed into

$$2_{e_1,n_1} \otimes 2_{e_2,n_2} \cong 2_{e_1+e_2,n_1+n_2} \oplus \overline{2_{e_1+e_2,n_1+n_2}}.$$

Let π_+ be the intertwiner

$$\pi_+ : 2_{e,n} \otimes 2_{e,n} \longrightarrow 2_{2e,2n}$$

define by

$$\begin{aligned}
&\pi_+(\upsilon_0^{e,n} \otimes \upsilon_0^{e,n}) = \upsilon_0^{2e,2n},\\
&\pi_+(\upsilon_1^{e,n} \otimes \upsilon_0^{e,n}) = \upsilon_1^{2e,2n},\\
&\pi_+(\upsilon_0^{e,n} \otimes \upsilon_1^{e,n}) = \upsilon_1^{2e,2n},\ \pi_+(\upsilon_1^{e,n} \otimes \upsilon_1^{e,n}) \qquad = 0.
\end{aligned}$$

Let ι_+ be the intertwiner

$$\iota_+ : 2_{2e,2n} \longrightarrow 2_{e,n} \otimes 2_{e,n}$$

define by

$$\begin{aligned}
&\iota_+(\upsilon_0^{2e,2n}) = 2\upsilon_0^{e,n} \otimes \upsilon_0^{e,n},\\
&\iota_+(\upsilon_1^{2e,2n}) = \upsilon_0^{e,n} \otimes \upsilon_1^{e,n} + \upsilon_1^{e,n} \otimes \upsilon_0^{e,n}.
\end{aligned}$$

Then, the composition $\pi_+ \circ \iota_+$ is the zero intertwiner.

Let us specialize to the tensor product of representations $2_{e,n}$ and $\overline{2_{-e,1-n}}$ where $e \neq 0$.

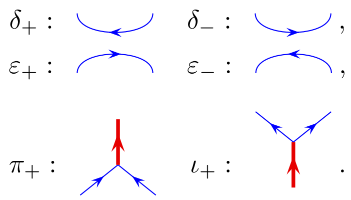
Define intertwiners

$$\begin{aligned}\delta_+ &: 1_0 \longrightarrow 2_{e,n} \otimes \overline{2_{-e,1-n}},\\ \varepsilon_+ &: 2_{e,n} \otimes \overline{2_{-e,1-n}} \longrightarrow 1_0,\\ \delta_- &: 1_0 \longrightarrow \overline{2_{-e,1-n}} \otimes 2_{e,n},\\ \varepsilon_- &: \overline{2_{-e,1-n}} \otimes 2_{e,n} \longrightarrow 1_0,\end{aligned}$$

by

$$\begin{aligned}\delta_+(v^0) &= v_1^{e,n} \otimes \overline{v_0^{-e,1-n}} - v_0^{e,n} \otimes \overline{v_1^{-e,1-n}},\\ \delta_-(v^0) &= \overline{v_1^{e,n}} \otimes v_0^{-e,1-n} - \overline{v_0^{e,n}} \otimes v_1^{-e,1-n},\\ \varepsilon_+(v_0^{e,n} \otimes \overline{v_0^{-e,1-n}}) &= 0,\\ \varepsilon_+(v_1^{e,n} \otimes \overline{v_0^{-e,1-n}}) &= v^0,\\ \varepsilon_+(v_0^{e,n} \otimes \overline{v_1^{-e,1-n}}) &= v^0,\\ \varepsilon_+(v_1^{e,n} \otimes \overline{v_1^{-e,1-n}}) &= 0,\\ \varepsilon_-(\overline{v_0^{e,n}} \otimes v_0^{-e,1-n}) &= 0,\\ \varepsilon_-(\overline{v_1^{e,n}} \otimes v_0^{-e,1-n}) &= -v^0,\\ \varepsilon_-(\overline{v_0^{e,n}} \otimes v_1^{-e,1-n}) &= v^0,\\ \varepsilon_-(\overline{v_1^{e,n}} \otimes v_1^{-e,1-n}) &= 0.\end{aligned}$$

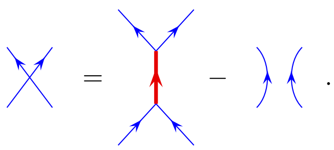
For a diagrammatic description, we use a directed blue line for $2_{e,n}$, a directed blue line of opposite direction for $\overline{2_{-e,1-n}}$, a directed think red line for $2_{2e,2n}$ and a directed thick red line of opposite direction for $\overline{2_{-2e,1-2n}}$. Then



The following equation about the R-matrix,

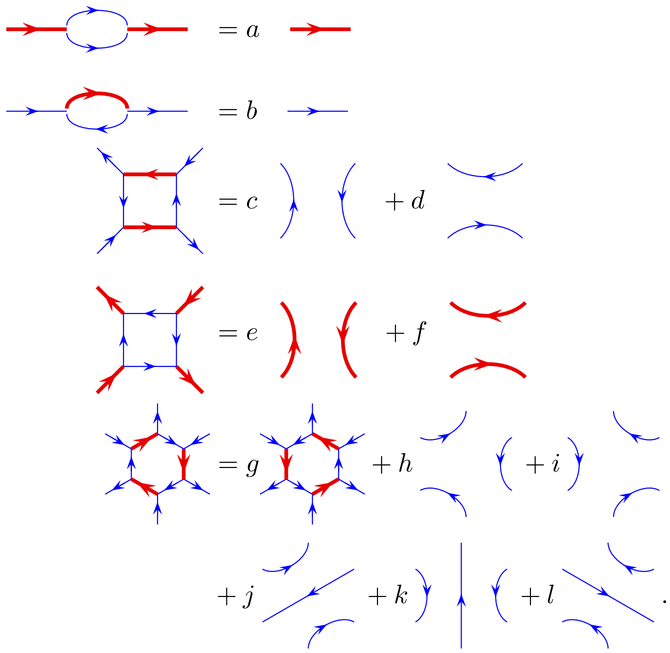
$$R = \pi_+ \circ \iota_+ - Id,$$

can be depicted as

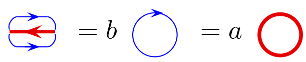


4. Presentation

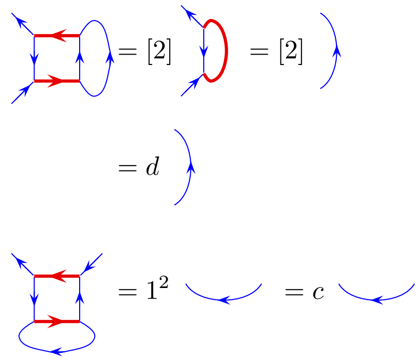
Now, we are set to prove Theorem 1.1. First, the equation (1) follows from the quantum trace of each representation. To prove other equations, we set



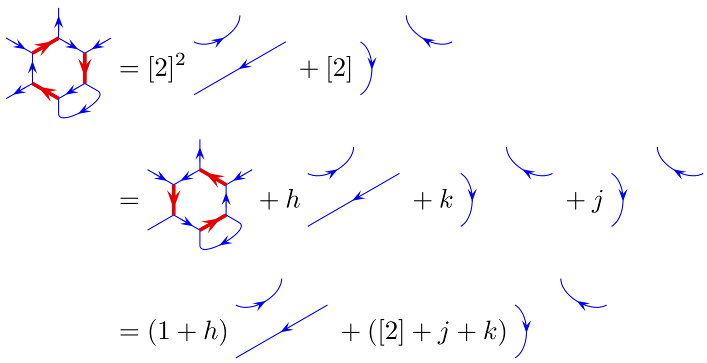
For the presentation of the representation category of the quantum Lie algebras, we can easily solve a and b using the following consistency relation.



However, due to the equation (1) we can not find a, b from these equations. Thus we set $a = [2]$ and $b = 1$ by scaling generating vertices which are comparable to Viro's [21]. We consider the following consistency relation.



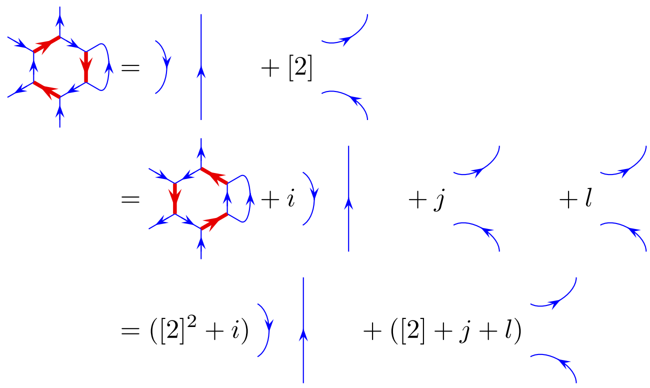
One can be solved for $c = 1$ and $d = [2]$. By symmetry, one can easily see $e = f$. Similarly we can solve $e = f = [2]$. For the last equation which is known as Kekule relations [14], we first find $g = 1$ and $j = l$ from the symmetry. The following consistency relation



leads to the equations

$$h = [3], \quad j + k = 0.$$

The other consistency relation



leads to the equations

$$i = [3], \quad j + l = 0.$$

Thus, we find $g = 1, h = [3], i = -[3]$ and $j = k = l = 0$. Therefore, it completes the proof of Theorem 1.1. We further conjecture that the presentation in Theorem 1.1 is confluent thus a spider presentation.

5. Further research problems

The key ingredient to find the quantum representation category of Lie algebras is to use the quantum dimension of fundamental representation and other irreducible representations. However, an analogue for quantum superalgebras does not work, the super dimension of a finite dimensional modules over a Lie superalgebra is zero. This was reflected on the relation (1). However, this will lead us to trivial link invariants for all links. Geer and Patureau-Mirand suggested that we can make a nontrivial trace for the special linear Lie superalgebras $\mathfrak{sl}(n|m), n \neq m$ [5]. Therefore, we need to come up with a different and new way to make our link invariant to be nontrivial.

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