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ON INTERVAL VALUED FUZZY QUASI-IDEALS OF SEMIGROUPS

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Abstract. In this paper we shall introduce the notion of an i-v fuzzy interior ideal, an i-v fuzzy quasi-ideal and an i-v fuzzy bi-ideal in a semigroup. We study some properties of i-v fuzzy subsets and using their properties we characterize regular semigroups.

1. Introduction

In 1975, Zadeh ([11]) introduced a new notion of fuzzy subsets viz., interval valued fuzzy subsets (in short, i-v fuzzy subsets) where the values of the membership functions are closed intervals of numbers instead of a number. In ([3]), Biswas defined interval valued fuzzy subgroups and investigated some elementary properties. Subsequently, Jun and Kim ([7]) and Davvaz ([4]) applied a few concept of i-v fuzzy subsets in near-rings. In this paper we introduce the notion of an i-v fuzzy interior ideal, an i-v fuzzy quasi-ideal and an i-v fuzzy bi-ideal in a semigroup. We investigate some of their properties. We give examples which are i-v fuzzy interior ideal and i-v fuzzy bi-ideal but not i-v fuzzy ideal and i-v fuzzy quasi-ideal respectively. We find the equivalent conditions on which these i-v fuzzy subsets coincide. Finally we characterize regular semigroups through their i-v fuzzy subsets. We also find equivalent conditions on regular semigroups through i-v fuzzy subsets.

2. Basic definitions and preliminary results

Let S be a semi group. Let A and B be subsets of S, the *multiplication* of A and B is defined as $AB = \{ ab \in S \mid a \in A \text{ and } b \in B \}$. A nonempty subset A of S is called a *subsemigroup* of S if $AA \subseteq A$. A nonempty subset A of S is called a *left* (*right*)*ideal* of S if $SA \subseteq A$ ($AS \subseteq A$). A is called a *two-sided ideal* (simply *ideal*) of S if it is both a left and a right ideal of S. A nonempty subset A of S is called an *interior ideal* of S if $SAS \subseteq A$, and a *quasi-ideal* of S if $AS \cap SA \subseteq A$. A subsemigroup A of S is called a *bi-ideal* of S if $ASA \subseteq A$.

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A semigroup S is called *regular* if for each element $a \in S$ there exists $x \in S$ such that $a = axa$. A function f from a nonempty set A to the unit interval $[0,\ 1]$ is called a *fuzzy subset* of A.

Definition 2.1. An *interval number* $\overline{a}$ on $[0, 1]$ is a closed subinterval of $[0,\ 1]$, that is, $\overline{a} = [a^-,\ a^+]$ such that $0 \leq a^- \leq a^+ \leq 1$ where a^- and a^+ are the lower and upper end points of $\overline{a}$ respectively.

In this notation $\overline{0} = [0,\ 0]$ and $\overline{1} = [1,\ 1]$. For any interval numbers $\overline{a} = [a^-,\ a^+]$ and $\overline{b} = [b^-,\ b^+]$ on $[0,\ 1]$, define

(i) $\overline{a} \leq \overline{b}$ if and only if $a^- \leq b^-$ and $a^+ \leq b^+$.

(ii) $\overline{a} = \overline{b}$ if and only if $a^- = b^-$ and $a^+ = b^+$.

Definition 2.2. Let X be any set. A mapping $\overline{A} : X \to D[0,\ 1]$ is called an *interval-valued fuzzy subset* (briefly, *i-v fuzzy subset*) of X, where $D[0,\ 1]$ denotes the family of all closed subintervals of $[0, 1]$ and $\overline{A}(x) = [A^-(x),\ A^+(x)]$ for all $x \in X$, where A^- and A^+ are fuzzy sets of X such that $A^-(x) \leq A^+(x)$ for all $x \in X$.

Thus $\overline{A}(x)$ is an interval (a closed subset of $[0,\ 1]$) and not a number from the interval $[0,\ 1]$ as in the case of fuzzy set.

Definition 2.3. A mapping $\min^i : D[0,\ 1] \times D[0,\ 1] \to D[0,\ 1]$ defined by

$\min^i(\overline{a}, \overline{b}) = [\min\{a^-,\ b^-\}, \min\{a^+,\ b^+\}]$ for all $\overline{a}, \overline{b} \in D[0,\ 1]$ is called an *interval min-norm*.

A mapping $\max^i : D[0,\ 1] \times D[0,\ 1] \to D[0,\ 1]$ defined by

$\max^i(\overline{a}, \overline{b}) = [\max\{a^-,\ b^-\}, \max\{a^+,\ b^+\}]$ for all $\overline{a},\ \overline{b} \in D[0,\ 1]$ is called an *interval max-norm*.

Let $\min^i$ and $\max^i$ be the interval $min - norm$ and $max - norm$ on $D[0,\ 1]$ respectively. Then the following are true.

(i) $\min^i\{\overline{a},\ \overline{a}\} = \overline{a}$ and $\max^i\{\overline{a}, \overline{a}\} = \overline{a}$ for all $\overline{a} \in D[0,\ 1]$

(ii) $\min^i\{\overline{a},\ \overline{b}\} = \min^i\{\overline{b},\ \overline{a}\}$ and $\max^i\{\overline{a},\ \overline{b}\} = \max^i\{\overline{b},\ \overline{a}\}$ for all $\overline{a},\ \overline{b},\ \in D[0,\ 1]$

(iii) If $\overline{a} \geq \overline{b}\ \in D[0,\ 1]$, then
$\min^i\{\overline{a},\ \overline{c}\} \geq \min^i\{\overline{b},\ \overline{c}\}$ and
$\max^i\{\overline{a},\ \overline{c}\} \geq \max^i\{\overline{b},\ \overline{c}\}$ for all $\overline{c}\ \in D[0,\ 1]$.

Definition 2.4. Let $\overline{A}$ be an i-v fuzzy set of a set X and $[t_1,\ t_2] \in D[0,\ 1]$. Then the set $\overline{U}(\overline{A} : [t_1,\ t_2]) = \{\, x \in X \mid \overline{A}(x) \geq [t_1,\ t_2]$, is called the *upper level set* of $\overline{A}$.

Note that

$$\begin{aligned}\overline{U}(\overline{A} : [t_1,\ t_2]) &= \{x \in X \mid [A^-(x), A^+(x)] \geq [t_1, t_2]\}\\ &= \{x \in X \mid A^-(x) \geq t_1\} \cap \{x \in X \mid A^+(x) \geq t_2\}\\ &= (\cup(A^-; t_1)) \cap (\cup(A^+; t_2)).\end{aligned}$$

Definition 2.5. Let $\overline{A}$, $\overline{B}$, $\overline{A}_i$ $(i \in \Omega)$ be interval valued fuzzy subsets of X. The following are defined by
(i) $\overline{A} \leq \overline{B}$ if and only if $\overline{A}(x) \leq \overline{B}(x)$.
(ii) $\overline{A} = \overline{B}$ if and only if $\overline{A}(x) = \overline{B}(x)$.
(iii) $(\overline{A} \cup \overline{B})(x) = \max^i\{\overline{A}(x), \overline{B}(x)\}$.
(iv) $(\overline{A} \cap \overline{B})(x) = \min^i\{\overline{A}(x), \overline{B}(x)\}$.
(v) $(\bigcap_{i\in\Omega} \overline{A}_i)(x) = \inf^i\{\overline{A}_i(x) \mid i \in \Omega\}$.
(vi) $(\bigcup_{i\in\Omega} \overline{A}_i)(x) = \sup^i\{\overline{A}_i(x) \mid i \in \Omega\}$.

where $\inf^i\{\overline{A}_i(x) \mid i \in \Omega\} = [\inf_{i\in\Omega}\{A_i^-(x)\}, \inf_{i\in\Omega}\{A_i^+(x)\}]$ is the interval valued infemum norm and $\sup^i\{\overline{A}_i(x) \mid i \in \Omega\} = [\sup_{i\in\Omega}\{A_i^-(x)\}, \sup_{i\in\Omega}\{A_i^+(x)\}]$ is the interval valued supremum norm.

Definition 2.6. Let $'.'$ be a binary composition in a set S. The *product* $\overline{A} \circ \overline{B}$ of any two i-v fuzzy subsets $\overline{A}$, $\overline{B}$ of S is defined by

$$(\overline{A} \circ \overline{B})(x) = \begin{cases} \sup\limits_{x=a.b}{}^i\{\min^i\{\overline{A}(a),\ \overline{B}(b)\}\}, & \text{if } x \text{ is expressed as } x = a.b \\ \overline{0} & \text{otherwise.} \end{cases}$$

Since semigroup S is associative, the operation $\circ$ is associative. We denote xy instead of $x.y$ and $\overline{A}\,\overline{B}$ for $\overline{A} \circ \overline{B}$.

Definition 2.7. Let I be a subset of a semigroup S. Define a function $\overline{\chi}_I : S \to D[0,\ 1]$ by

$$\overline{\chi}_I(x) = \begin{cases} \overline{1} & \text{if } x \in I \\ \overline{0} & \text{otherwise} \end{cases}$$

for all $x \in S$. Clearly $\overline{\chi}_I$ is an i-v fuzzy subset of S. Throughout this paper $\overline{\chi}_S$ is denoted by $\overline{S}$ and S will denote a semigroup unless otherwise mentioned.

Definition 2.8. An i-v fuzzy subset $\overline{\lambda}$ of S is called an *i-v fuzzy subsemigroup* of S if $\overline{\lambda}(ab) \geq \min^i\{\overline{\lambda}(a),\ \overline{\lambda}(b)\}$, for all $a,\ b \in S$.

Definition 2.9. An i-v fuzzy subset $\overline{\lambda}$ of S is called an *i-v fuzzy left* (*right*)*ideal* of S if $\overline{\lambda}(ab) \geq \overline{\lambda}(b)(\overline{\lambda}(ab) \geq \overline{\lambda}(a))$, for all $a,\ b \in S$.

An i-v fuzzy subset $\overline{\lambda}$ of S is called an *i-v fuzzy two-sided ideal (simply i-v fuzzy ideal)* of S if it is both an i-v fuzzy left ideal and an i-v fuzzy right ideal of S.

Every i-v fuzzy right(left, two-sided) ideal of S is an i-v fuzzy subsemigroup of S. However the converse is not true in general as shown in the following example.

Example 2.10. Let $S = \{0, 1, 2, 3\}$ be a semigroup with the multiplication table given below:

$\bullet$	0	1	2	3
0	0	0	0	0
1	0	1	2	0
2	0	0	0	0
3	0	3	0	0

Define $\overline{\lambda} : S \to D[0,\, 1]$ by $\overline{\lambda}(0) = [0.8,\, 0.9], \overline{\lambda}(1) = [0.6,\, 0.7], \overline{\lambda}(2) = [0.1,\, 0.2]$ and $\overline{\lambda}(3) = [0.3,\, 0.4]$. Then $\overline{\lambda}$ is an i-v fuzzy subsemigroup of S. $\overline{\lambda}$ is not an i-v fuzzy left (right, two-sided) ideal of S. For, $(\overline{\lambda}\,\overline{\lambda})(x) \leq \overline{\lambda}(x)$ for all $x \in S$.

$$\begin{aligned}(\overline{\lambda S})(2) &= \sup_{2=ab}{}^{i}\{\min{}^{i}\{\overline{\lambda}(a), \overline{S}(b)\}\} \\ &= \min{}^{i}\{\overline{\lambda}(1), \overline{S}(2)\} \text{ as } 2{=}1.2 \\ &= \min{}^{i}\{[0.6, 0.7], [1, 1]\} \\ &= [0.6, 0.7] \nleq \overline{\lambda}(2) = [0.1, 0.2] \text{ and} \\ (\overline{S}\,\overline{\lambda})(3) &= \min{}^{i}\{\overline{\lambda}(1), \overline{\lambda}(3)\} \text{ as } 3{=}1.3 \\ &= [0.6, 0.7] \nleq \overline{\lambda}(3) = [0.3, 0.4].\end{aligned}$$

Thus $\overline{\lambda}$ is neither an i-v fuzzy right ideal nor an i-v fuzzy left ideal of S. That is $\overline{\lambda}$ is not an i-v fuzzy ideal of S.

Lemma 2.11. *Let $\overline{\lambda}, \overline{\mu}$ and $\overline{\nu}$ be i-v fuzzy subsets of S, then*
(i) $\overline{\lambda} \cup (\overline{\mu} \cap \overline{\nu}) = (\overline{\lambda} \cup \overline{\mu}) \cap (\overline{\lambda} \cup \overline{\nu})$ *and*
(ii) $\overline{\lambda} \cap (\overline{\mu} \cup \overline{\nu}) = (\overline{\lambda} \cap \overline{\mu}) \cup (\overline{\lambda} \cap \overline{\nu})$

Proof. Straight forward. □

Lemma 2.12. *Let $\overline{\lambda}, \overline{\mu}$ and $\overline{\nu}$ be i-v fuzzy subsets of S. Then,*
(i) $\overline{\lambda}(\overline{\mu} \cup \overline{\nu}) = (\overline{\lambda}\,\overline{\mu}) \cup (\overline{\lambda}\,\overline{\nu})$; $(\overline{\mu} \cup \overline{\nu})\overline{\lambda} = (\overline{\mu}\,\overline{\lambda}) \cup (\overline{\nu}\,\overline{\lambda})$
(ii) $\overline{\lambda}(\overline{\mu} \cap \overline{\nu}) \leq (\overline{\lambda}\,\overline{\mu}) \cap (\overline{\lambda}\,\overline{\nu})$; $(\overline{\mu} \cap \overline{\nu})\overline{\lambda} \leq (\overline{\mu}\,\overline{\lambda}) \cap (\overline{\nu}\,\overline{\lambda})$

Proof. Straight forward. □

Lemma 2.13. *Let $\overline{\lambda}$, $\overline{\mu}$ and $\overline{\nu}$ be i-v fuzzy subsets of S. If $\overline{\lambda} \leq \overline{\mu}$ then $\overline{\lambda}\,\overline{\nu} \leq \overline{\mu}\,\overline{\nu}$ and $\overline{\nu}\,\overline{\lambda} \leq \overline{\nu}\,\overline{\mu}$.*

Proof. Omitted as it is straight forward. □

Proposition 2.14. *Let A be a nonempty subset of a semigroup S. A is a subsemigroup* (*resp. left ideal, right ideal, two-sided ideal*) *of S if and only if $\overline{\chi}_A$ is an i-v fuzzy subsemigroup* (*resp. left ideal, right ideal, two-sided ideal*) *of S.*

Proof. Let A be a subsemigroup of S.

$$\overline{\chi}_A(x) = \begin{cases} \overline{1} & \text{if } x \in A \\ \overline{0} & \text{if } x \notin A. \end{cases}$$

Let $a,\, b \in S$. Suppose $\overline{\chi}_A(ab) < \min^{i}\{\overline{\chi}_A(a), \overline{\chi}_A(b)\}$, then $\overline{\chi}_A(a) = \overline{\chi}_A(b) = \overline{1}$ and $\overline{\chi}_A(ab) = \overline{0}$. This implies that $a,\ b \in A$. Since A is a subsemigroup of S, $ab \in S$ and hence $\overline{\chi}_A(ab) = \overline{1}$, a contradiction. Thus $\overline{\chi}_A(ab) \geq \min^i\{\overline{\chi}_A(a),\ \overline{\chi}_A(b)\}$ for all $a, b \in S$.

Conversely, assume that $\overline{\chi}_A$ is an i-v fuzzy subsemigroup of S. Let $a, b \in A$. Then $\overline{\chi}_A(a) = \overline{1} = \overline{\chi}_A(b)$. As $\overline{\chi}_A$ is an i-v fuzzy subsemigroup, $\min^i\{\overline{\chi}_A(a),\ \overline{\chi}_A(b)\} = \overline{1} \leq \overline{\chi}_A(ab)$. This implies that $\overline{\chi}_A(ab) = \overline{1}$ and hence $ab \in A$. Thus A is a subsemigroup of S. □

Proposition 2.15. *Let $\overline{\lambda}$ be an i-v fuzzy subset of S. $\overline{\lambda}$ is an i-v fuzzy left ideal (resp. subsemigroup, right ideal) of S, if and only if $\overline{S}\ \overline{\lambda} \leq \overline{\lambda}$ (resp. $\overline{\lambda}\,\overline{\lambda} \leq \overline{\lambda}$, $\overline{\lambda}\,\overline{S} \leq \overline{\lambda}$).*

Proof. Let $x \in S$. Assume that $\overline{\lambda}$ is an i-v fuzzy left ideal of S. If $(\overline{S}\ \overline{\lambda})(x) = \overline{0}$, then it is clear that $(\overline{S}\ \overline{\lambda})(x) \leq \overline{\lambda}(x)$. Otherwise, there exist $a,\ b \in S$ such that $x = ab$. Then, since $\overline{\lambda}$ is an i-v fuzzy left ideal of S, we have

$$\begin{aligned}(\overline{S}\ \overline{\lambda})(x) &= \sup_{x=ab}{}^i\{\min{}^i\{\overline{S}(a), \overline{\lambda}(b)\}\}\\ &= \sup_{x=ab}{}^i\{\min{}^i\{\overline{1}, \overline{\lambda}(b)\}\}\\ &= \sup_{x=ab}{}^i\{\overline{\lambda}(b)\}\\ &\leq \sup_{x=ab}{}^i\{\overline{\lambda}(ab)\}\\ &= \overline{\lambda}(x)\end{aligned}$$

and so $\overline{S}\,\overline{\lambda} \leq \overline{\lambda}$.

Conversely, assume that $\overline{S}\ \overline{\lambda} \leq \overline{\lambda}$, for any i-v fuzzy subset $\overline{\lambda}$ of S. Let $x, y, z \in S$ such that $z = xy$. Then we have

$$\begin{aligned}\overline{\lambda}(xy) = \overline{\lambda}(z) &\geq (\overline{S}\ \overline{\lambda})(z)\\ &= \sup_{z=pq}{}^i\{\min{}^i\{\overline{S}(p), \overline{\lambda}(q)\}\}\\ &\geq \min{}^i\{\overline{S}(x), \overline{\lambda}(y)\}\\ &= \min{}^i\{\overline{1}, \overline{\lambda}(y)\}\\ &= \overline{\lambda}(y).\end{aligned}$$

Hence $\overline{\lambda}$ is an i-v fuzzy left ideal of S. □

Lemma 2.16. *Let $\overline{\lambda}$ and $\overline{\mu}$ be any i-v fuzzy subsemigroups (resp. right ideals, left ideals, two-sided ideals) of S. Then $\overline{\lambda} \cap \overline{\mu}$ is also an i-v fuzzy subsemigroup (resp. right ideal, left ideal, two-sided ideal) of S.*

Proof. Let $\overline{\lambda}$ and $\overline{\mu}$ be any i-v fuzzy subsemigroups of S. Let $a,\ b \in S$. Then

$$\begin{aligned}(\overline{\lambda} \cap \overline{\mu})(ab) &= \min{}^i\{\overline{\lambda}(ab),\ \overline{\mu}(ab)\}\\ &\geq \min{}^i\{\min{}^i\{\overline{\lambda}(a),\ \overline{\lambda}(b)\}, \min{}^i\{\overline{\mu}(a),\ \overline{\mu}(b)\}\}\\ &= \min{}^i\{\min{}^i\{\overline{\lambda}(a),\ \overline{\mu}(a)\},\ \min{}^i\{\overline{\lambda}(b),\ \overline{\mu}(b)\}\}\\ &= \min{}^i\{(\overline{\lambda} \cap \overline{\mu})(a),\ (\overline{\lambda} \cap \overline{\mu})(b)\}.\end{aligned}$$

Thus $\overline{\lambda} \cap \overline{\mu}$ is an i-v fuzzy subsemigroup of S. □

The following lemma can easily be proved.

Lemma 2.17. *Let A and B be nonempty subsets of S. Then the following properties hold.*
(i) $\overline{\chi}_A \cap \overline{\chi}_B = \overline{\chi}_{A\cap B}$.
(ii) $\overline{\chi}_A \overline{\chi}_B = \overline{\chi}_{AB}$.

Lemma 2.18. *If $\overline{\lambda}$ is an i-v fuzzy right (left) ideal of S, then $\overline{\lambda} \cup (\overline{S}\,\overline{\lambda})$ is an i-v fuzzy ideal of S.*

Proof. Suppose $\overline{\lambda}$ is an i-v fuzzy right ideal of S. Then

$$\begin{aligned}\overline{S}\,(\overline{\lambda} \cup (\overline{S}\,\overline{\lambda})) &= (\overline{S}\,\overline{\lambda}) \cup (\overline{S}\,(\overline{S}\overline{\lambda})) \text{ by Lemma 2.12 (i)}\\ &= (\overline{S}\,\overline{\lambda}) \cup ((\overline{S}\,\overline{S})\,\overline{\lambda})\\ &\leq (\overline{S}\,\overline{\lambda}) \cup (\overline{S}\,\overline{\lambda}) = \overline{S}\,\overline{\lambda}\\ &\leq \overline{\lambda} \cup (\overline{S}\,\overline{\lambda}).\end{aligned}$$

Thus $\overline{\lambda} \cup (\overline{S}\,\overline{\lambda})$ is an i-v fuzzy left ideal of S by Proposition 2.15. Also

$$\begin{aligned}(\overline{\lambda} \cup (\overline{S}\,\overline{\lambda}))\,\overline{S} &= (\overline{\lambda}\,\overline{S}) \cup (\overline{S}\,\overline{\lambda})\overline{S} \text{ by Lemma 2.12 (i)}\\ &= (\overline{\lambda}\,\overline{S}) \cup (\overline{S}\,(\overline{\lambda}\,\overline{S}))\\ &\leq (\overline{\lambda}\,\overline{S}) \cup (\overline{S}(\overline{\lambda}\,\overline{S}))\\ &\leq \overline{\lambda} \cup (\overline{S}\,\overline{\lambda}), \text{ since } \overline{\lambda} \text{ is an i-v fuzzy right ideal of } S.\end{aligned}$$

Hence $\overline{\lambda} \cup (\overline{S}\,\overline{\lambda})$ is an i-v fuzzy right ideal of S. Therefore $\overline{\lambda} \cup (\overline{S}\,\overline{\lambda})$ is an i-v fuzzy ideal of S. □

Theorem 2.19. *Let $\overline{\lambda}$ be an i-v fuzzy subset of S. $\overline{\lambda} = [f^-, f^+]$ is an i-v fuzzy subsemigroup (resp. right ideal, left ideal, two-sided ideal) of S, if and only if f^- and f^+ are fuzzy subsemigroups (resp. right ideal, left ideal, two-sided ideal) of S.*

Proof. Assume that $\overline{\lambda}$ is an i-v fuzzy subsemigroup of S. For any $x, y \in S$, we have

$$\begin{aligned}[f^-(xy),\, f^+(xy)] &= \overline{\lambda}(xy)\\ &\geq \min{}^i\{\overline{\lambda}(x),\, \overline{\lambda}(y)\}\\ &= \min{}^i\{[f^-(x),\, f^+(x)],\, [f^-(y),\, f^+(y)]\}\\ &= [\min\{f^-(x),\, f^-(y)\},\, \min\{f^+(x),\, f^+(y)\}].\end{aligned}$$

It follows that $f^-(xy) \geq \min\{f^-(x),\, f^-(y)\}$ and $f^+(xy) \geq \min\{f^+(x),\, f^+(y)\}$. Thus f^- and f^+ are fuzzy subsemigroups of S.

Conversely, assume that f^- and f^+ are fuzzy subsemigroups of S and let $x, y \in S$. Then

$$\begin{aligned}\overline{\lambda}(xy) &= [f^-(xy), f^+(xy)]\\ &\geq [\min\{f^-(x),\, f^-(y)\}, \min\{f^+(x), f^+(y)\}]\\ &= \min{}^i\{[f^-(x),\, f^+(x)],\, [f^-(y), f^+(y)]\}\\ &= \min{}^i\{\overline{\lambda}(x),\, \overline{\lambda}(y)\}.\end{aligned}$$

Thus $\overline{\lambda}$ is an i-v fuzzy subsemigroup of S. □

Theorem 2.20. *Let $\overline{\lambda}$ be an i-v fuzzy subset of S. $\overline{\lambda}$ is an i-v fuzzy subsemigroup (resp. right ideal, left ideal, two-sided ideal) of S if and only if $\overline{U}(\overline{\lambda} : [r_1, r_2])$ is a subsemigroup (resp. right ideal, left ideal, two-sided ideal) of S.*

Proof. Assume that $\overline{\lambda}$ is an i-v fuzzy subset of S and let $[r_1, r_2] \in D[0, 1]$ such that $x, y \in \overline{U}(\overline{\lambda} : [r_1, r_2])$. Then

$$\begin{aligned}\overline{\lambda}(xy) &\geq \min{}^i\{\overline{\lambda}(x), \overline{\lambda}(y)\}\\ &\geq \min{}^i\{[r_1, r_2], [r_1, r_2]\}\\ &= [r_1, r_2]\end{aligned}$$

Thus $xy \in \overline{U}(\overline{\lambda} : [r_1, r_2])$. Hence $\overline{U}(\overline{\lambda} : [r_1, r_2])$ is a subsemigroup of S.

Conversely, assume that $\overline{U}(\overline{\lambda} : [r_1, r_2])$ is a subsemigroup of S for all $[r_1, r_2] \in D[0, 1]$. Let $x, y \in S$. Suppose $\overline{\lambda}(xy) < \min^i\{\overline{\lambda}(x), \overline{\lambda}(y)\}$. Then there exists an interval $\overline{a} = [a_1, a_2] \in D[0, 1]$ such that $\overline{\lambda}(xy) < [a_1, a_2] < \min^i\{\overline{\lambda}(x), \overline{\lambda}(y)\}$. This implies that $\overline{\lambda}(x) > [a_1, a_2]$ and $\overline{\lambda}(y) > [a_1, a_2]$. Then we have $x, y \in \overline{U}(\overline{\lambda} : [a_1, a_2])$ and since $\overline{U}(\overline{\lambda} : [a_1, a_2])$ is a subsemigroup of S, $xy \in \overline{U}(\overline{\lambda} : [a_1, a_2])$. Hence, $\overline{\lambda}(xy) > [a_1, a_2]$, a contradiction. Thus $\overline{\lambda}(xy) \geq \min^i\{\overline{\lambda}(x), \overline{\lambda}(y)\}$ for all $x, y \in S$. □

3. I-v fuzzy bi-ideals and I-v fuzzy quasi-ideals of a semigroup

In this section we introduce i-v fuzzy bi-ideals and i-v fuzzy quasi-ideals in a semigroup S. We also give a characterization for semigroups.

Definition 3.1. An i-v fuzzy subsemigroup $\overline{\lambda}$ of S is called an *i-v fuzzy bi-ideal* of S if $\overline{\lambda}(xyz) \geq \min^i\{\overline{\lambda}(x), \overline{\lambda}(z)\}$ for all $x, y, z \in S$.

Proposition 3.2. *Let A be a nonempty subset of S. A is a bi-ideal of S if and only if $\overline{\chi}_A$ is an i-v fuzzy bi-ideal of S.*

Proof. Assume that A is a bi-ideal of S. By Proposition 2.14 $\overline{\chi}_A$ is an i-v fuzzy subsemigroup of S. Suppose that $\overline{\chi}_A(xyz) < \min^i\{\overline{\chi}_A(x), \overline{\chi}_A(z)\}$ for some $x, y, z \in S$. Then $\overline{\chi}_A(x) = \overline{1}$ and $\overline{\chi}_A(z) = \overline{1}$. This implies that $x, z \in A$ and since A is a bi-ideal of S, $xyz \in ASA \subseteq A$. Thus $\overline{\chi}_A(xyz) = \overline{1}$, a contradiction. Hence $\overline{\chi}_A(xyz) \geq \min^i\{\overline{\chi}_A(x), \overline{\chi}_A(z)\}$ for all $x, y, z \in S$. Therefore $\overline{\chi}_A$ is an i-v fuzzy bi-ideal of S.

Conversely, assume that $\overline{\chi}_A$ is an i-v fuzzy bi-ideal of S. Then by Proposition 2.14 A is a subsemigroup of S. Let $a = xyz \in ASA$ such that $x, z \in A$. Then we have

$$\begin{aligned}\overline{\chi}_A(a) &= \overline{\chi}_A(xyz)\\ &\geq \min{}^i\{\overline{\chi}_A(x), \overline{\chi}_A(z)\}\\ &= \min{}^i\{\overline{1}, \overline{1}\}\\ &= \overline{1}.\end{aligned}$$

Hence $\overline{\chi}_A(a) = \overline{1}$ and so $a = xyz \in A$. Thus $ASA \subseteq A$ which implies that A is a bi-ideal of S. □

Proposition 3.3. *Let $\overline{\lambda}$ be an i-v fuzzy subsemigroup of S. $\overline{\lambda}$ is an i-v fuzzy bi-ideal of S if and only if $\overline{\lambda}\,\overline{S}\,\overline{\lambda} \leq \overline{\lambda}$.*

Proof. Assume that $\overline{\lambda}$ is an i-v fuzzy bi-ideal of S and let $a \in S$. In the case when $(\overline{\lambda}\,\overline{S}\,\overline{\lambda})(a) = \overline{0} \leq \overline{\lambda}(a)$, otherwise there exist $x, y, p, q \in S$ such that $a = xy$ and $x = pq$. Since $\overline{\lambda}$ is an i-v fuzzy bi-ideal of S, we have $\overline{\lambda}(pqy) \geq \min^i\{\overline{\lambda}(p), \overline{\lambda}(y)\}$. Therefore

$$\begin{aligned}
(\overline{\lambda}\,\overline{S}\,\overline{\lambda})(a) &= \sup_{a=xy}{}^i\{\min^i\{(\overline{\lambda}\,\overline{S})(x), \overline{\lambda}(y)\}\\
&= \sup_{a=xy}{}^i\{\min^i\{\sup_{x=pq}{}^i\{\min^i\{\overline{\lambda}(p), \overline{S}(q)\}, \overline{\lambda}(y)\}\}\}\\
&= \sup_{a=xy}{}^i\{\min^i\{\sup_{x=pq}{}^i\{\overline{\lambda}(p)\}, \overline{\lambda}(y)\}\}\\
&= \sup_{x=pqy}{}^i\{\min^i\{\overline{\lambda}(p), \overline{\lambda}(y)\}\}\\
&\leq \sup_{a=pqy}{}^i\{\overline{\lambda}(pqy)\}\\
&= \overline{\lambda}(a)
\end{aligned}$$

and so we have $\overline{\lambda}\,\overline{S}\,\overline{\lambda} \leq \overline{\lambda}$.

Conversely, assume that $\overline{\lambda}\,\overline{S}\,\overline{\lambda} \leq \overline{\lambda}$ holds for any i-v fuzzy subsemigroup $\overline{\lambda}$. Let $a, x, y, z \in S$ such that $a = xyz$. Then we have

$$\begin{aligned}
\overline{\lambda}(xyz) &= \overline{\lambda}(a)\\
&\geq (\overline{\lambda}\,\overline{S}\,\overline{\lambda})(a)\\
&= \sup_{a=bc}{}^i\{\min^i\{(\overline{\lambda}\,\overline{S})(b), \overline{\lambda}(c)\}\}\\
&\geq \min^i\{\overline{\lambda}(x), \overline{S}(y), \overline{\lambda}(z)\}\\
&= \min^i\{\overline{\lambda}(x), \overline{\lambda}(z)\}.
\end{aligned}$$

Thus $\overline{\lambda}(xyz) \geq \min^i\{\overline{\lambda}(x), \overline{\lambda}(z)\}$ for all $x, y, z \in S$. Hence $\overline{\lambda}$ is an i-v fuzzy bi-ideal of S. □

Lemma 3.4. *Let $\overline{\lambda}$ and $\overline{\mu}$ be any i-v fuzzy subset and i-v fuzzy bi-ideal of S respectively. Then the products $\overline{\lambda}\,\overline{\mu}$ and $\overline{\mu}\,\overline{\lambda}$ are i-v fuzzy bi-ideals of S.*

Proof. Since $\overline{\mu}$ is an i-v fuzzy bi-ideal of S, by Proposition 3.3 we have

$$\begin{aligned}
(\overline{\lambda}\,\overline{\mu})\,(\overline{\lambda}\,\overline{\mu}) &= \overline{\lambda}(\overline{\mu}\,\overline{\lambda}\,\overline{\mu})\\
&\leq \overline{\lambda}\,(\overline{\mu}\,\overline{S}\,\overline{\mu})\\
&\leq \overline{\lambda}\,\overline{\mu}.
\end{aligned}$$

Hence, it follows from Proposition 2.15 that $(\overline{\lambda}\,\overline{\mu})$ is an i-v fuzzy subsemigroup of S. We have

$$\begin{aligned}(\overline{\lambda}\,\overline{\mu})\,\overline{S}\,(\overline{\lambda}\,\overline{\mu}) &= \overline{\lambda}\,\overline{\mu}\,(\overline{S}\,\overline{\lambda})\,\overline{\mu}\\ &\leq \overline{\lambda}\,\overline{\mu}\,(\overline{S}\,\overline{S})\,\overline{\mu}\\ &\leq \overline{\lambda}\,(\overline{\mu}\,\overline{S}\,\overline{\mu})\\ &\leq \overline{\lambda}\,\overline{\mu}.\end{aligned}$$

Thus it follows from Proposition 3.3 that $\overline{\lambda}\ \overline{\mu}$ is an i-v fuzzy bi-ideal of S. Similarly, it can be shown that $\overline{\mu}\,\overline{\lambda}$ is an i-v fuzzy bi-ideal of S. □

Lemma 3.5. *Let $\overline{\lambda}$ and $\overline{\mu}$ be two i-v fuzzy bi-ideals of S. Then $\overline{\lambda}\cap\overline{\mu}$ is an i-v fuzzy bi-ideal of S.*

Proof. Let a, b and x be elements of S. Then

$$\begin{aligned}(\overline{\lambda}\cap\overline{\mu})(ab) &= \min{}^i\{\overline{\lambda}(ab), \overline{\mu}(ab)\}\\ &\geq \min{}^i\{\min{}^i\{\overline{\lambda}(a), \overline{\lambda}(b)\}, \min{}^i\{\overline{\mu}(a), \overline{\mu}(b)\}\}\\ &= \min{}^i\{\min{}^i\{\overline{\lambda}(a), \overline{\mu}(a)\}, \min{}^i\{\overline{\lambda}(b), \overline{\mu}(b)\}\}, \text{ by Lemma 2.12}\\ &= \min{}^i\{(\overline{\lambda}\cap\overline{\mu})(a), (\overline{\lambda}\cap\overline{\mu})(b)\}\end{aligned}$$

and

$$\begin{aligned}(\overline{\lambda}\cap\overline{\mu})(axb) &= \min{}^i\{\overline{\lambda}(axb), \overline{\mu}(axb)\}\\ &\geq \min{}^i\{\min{}^i\{\overline{\lambda}(a), \overline{\lambda}(b)\}, \min{}^i\{\overline{\mu}(a), \overline{\mu}(b)\}\}\\ &= \min{}^i\{\min{}^i\{\overline{\lambda}(a), \overline{\mu}(a)\}, \min{}^i\{\overline{\lambda}(b), \overline{\mu}(b)\}\}\\ &= \min{}^i\{(\overline{\lambda}\cap\overline{\mu})(a), (\overline{\lambda}\cap\overline{\mu})(b)\}.\end{aligned}$$

Hence $\overline{\lambda}\cap\overline{\mu}$ is an i-v fuzzy bi-ideal of S.

□

We now introduce the notion of i-v fuzzy interior ideal of S.

Definition 3.6. An i-v fuzzy subset $\overline{\lambda}$ of S is called an *i-v fuzzy interior ideal* of S if $\overline{\lambda}(xay) \geq \overline{\lambda}(a)$ for all $x, a, y \in S$.

Proposition 3.7. *Let A be a non empty subset of S. Then A is an interior ideal of S if and only if $\overline{\chi}_A$ is an i-v fuzzy interior ideal of S.*

Proof. Let A be an interior ideal of S. Suppose $\overline{\chi}_A(xay) < \overline{\chi}_A(a)$ for some $x, a, y \in S$. Then $\overline{\chi}_A(a) = \overline{1}$ and $\overline{\chi}_A(xay) = \overline{0}$. Since $\overline{\chi}_A(a) = \overline{1}$, $a \in A$ and A is an interior ideal of S, $xay \in SAS \subseteq A$. Thus $\overline{\chi}_A(xay) = \overline{1}$, a contradiction. Hence $\overline{\chi}_A(xay) \geq \overline{\chi}_A(a)$ for all $x, a, y \in S$.

Conversely, assume that $\overline{\chi}_A$ is an i-v fuzzy interior ideal of S. Let $z = xay$ such that $x, a, y \in S$ and $a \in A$. Then $\overline{\chi}_A(xay) \geq \overline{\chi}_A(a) = \overline{1}$. This implies that $\overline{\chi}_A(xay) = \overline{1}$ and so $xay \in A$. Hence we have $SAS \subseteq A$ and so A is an interior ideal of S. □

Proposition 3.8. *Let $\overline{\lambda}$ be an i-v fuzzy subset of S. $\overline{\lambda}$ is an i-v fuzzy interior ideal of S if and only if $\overline{S}\,\overline{\lambda}\,\overline{S} \leq \overline{\lambda}$.*

Proof. Assume that $\overline{\lambda}$ is an i-v fuzzy interior ideal of S. Let $z \in S$. If there exist elements x, y, u, $v \in S$ such that $z = xy$ and $x = uv$, then since $\overline{\lambda}(uvy) \geq \overline{\lambda}(v)$, we have

$$\begin{aligned}(\overline{S}\,\overline{\lambda}\,\overline{S})(z) &= \sup_{z=xy}{}^{i}\{\min{}^{i}\{(\overline{S}\,\overline{\lambda})(x),\, \overline{S}(y)\}\} \\ &= \sup_{z=xy}{}^{i}\{\min{}^{i}\{\sup_{x=uv}{}^{i}\{\min{}^{i}\{\overline{S}(u),\, \overline{\lambda}(v)\}\}, \overline{S}(y)\}\} \\ &\leq \sup_{z=xy}{}^{i}\{\min{}^{i}\{\sup_{x=uv}{}^{i}\{\overline{\lambda}(v)\}, \overline{1}\}\} \\ &= \sup_{z=uvy}{}^{i}\{\overline{\lambda}(v)\} \\ &\leq \sup_{z=uvy}{}^{i}\{\overline{\lambda}(uvy)\} \\ &= \overline{\lambda}(z).\end{aligned}$$

In the other case, $(\overline{S}\,\overline{\lambda}\,\overline{S})(z) = \overline{0} \leq \overline{\lambda}(z)$. Therefore $\overline{S}\,\overline{\lambda}\,\overline{S} \leq \overline{\lambda}$.

Conversely, assume that $\overline{S}\,\overline{\lambda}\,\overline{S} \leq \overline{\lambda}$ holds for any i-v fuzzy subset $\overline{\lambda}$ of S. Let $x, a, y \in S$. Then we have

$$\begin{aligned}\overline{\lambda}(xay) &\geq (\overline{S}\,\overline{\lambda}\,\overline{S})(xay) \\ &= \sup_{xay=pq}{}^{i}\{\min{}^{i}\{(\overline{S}\,\overline{\lambda})(p), \overline{S}(q)\}\} \\ &\geq \min{}^{i}\{(\overline{S}\,\overline{\lambda})(xa), \overline{S}(y)\}\} \\ &= (\overline{S}\,\overline{\lambda})(xa) \\ &= \sup_{xa=cd}{}^{i}\{\min{}^{i}\{\overline{S}(c), \overline{\lambda}(d)\}\} \\ &\geq \min{}^{i}\{\overline{S}(x), \overline{\lambda}(a)\} = \overline{\lambda}(a).\end{aligned}$$

Consequently, $\overline{\lambda}$ is an i-v fuzzy interior ideal of S. □

Lemma 3.9. *Every i-v fuzzy ideal of S is an i-v fuzzy interior ideal of S.*

Proof. Let $\overline{\lambda}$ be an i-v fuzzy ideal of S. Consider

$$\begin{aligned}\overline{S}\,\overline{\lambda}\,\overline{S} &= (\overline{S}\,\overline{\lambda})\,\overline{S} \\ &\leq \overline{\lambda}\,\overline{S}, \text{ since } \overline{\lambda} \text{ is an i-v fuzzy left ideal} \\ &\leq \overline{\lambda}, \text{ since } \overline{\lambda} \text{ is an i-v fuzzy right ideal.}\end{aligned}$$

Thus $\overline{\lambda}$ is an i-v fuzzy interior ideal of S. □

The converse of the above lemma is not true in general as shown by the following example.

Example 3.10. Let $S = \{\, a,\, b,\, c,\, d \,\}$ be a semigroup with the multiplication table given below:

•	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

Let $\overline{\lambda}$ be an i-v fuzzy subset of S such that $\overline{\lambda}(a) = [0.6, 0.8]$, $\overline{\lambda}(c) = [0.4, 0.5]$, $\overline{\lambda}(b) = \overline{\lambda}(d) = [0.1, 0.2]$. Then $\overline{\lambda}$ is an i-v fuzzy interior ideal of S, but it is not an i-v fuzzy ideal S. In fact, $\overline{\lambda}(xyz) = \overline{\lambda}(a) = [0.6, 0.8] \geq \overline{\lambda}(y)$ for all $x, y, z \in S$. Thus $\overline{\lambda}$ is an i-v fuzzy interior ideal of S. But since $\overline{\lambda}(dc) = \overline{\lambda}(b) = [0.1, 0.2] < \overline{\lambda}(c) = [0.4, 0.5]$, $\overline{\lambda}$ is not an i-v fuzzy left ideal of S, that is, it is not an i-v fuzzy ideal of S.

Now we introduce the notion of i-v fuzzy quasi-ideal of S and examine the conditions on semigroups through i-v fuzzy subsets.

Definition 3.11. An i-v fuzzy subset $\overline{\lambda}$ of a semigroup S is called an *i-v fuzzy quasi-ideal* of S if $(\overline{\lambda}\,\overline{S}) \cap (\overline{S}\,\overline{\lambda}) \leq \overline{\lambda}$.

Lemma 3.12. *For any nonempty subset A of S, $\overline{\chi}_A$ is an i-v fuzzy quasi-ideal of S if and only if A is a quasi-ideal of S.*

Proof. Suppose $\overline{\chi}_A$ is an i-v fuzzy quasi-ideal of S. Let $x \in SA \cap AS$. Then $x \in SA$ and $x \in AS$. This implies that $x = sa$ and $x = a_1s_1$ for some $a, a_1 \in A$ and $s, s_1 \in S$. Now

$$\begin{aligned}
[(\overline{S}\,\overline{\chi}_A) \cap (\overline{\chi}_A\,\overline{S})](x) &= \min{}^i\{(\overline{S}\,\overline{\chi}_A)(x), (\overline{\chi}_A\,\overline{S})(x)\} \\
&\geq \min{}^i\{\sup_{x=yz}{}^i\{\min{}^i\{\overline{S}(y), \overline{\chi}_A(z)\}\}, \\
&\qquad\qquad \sup_{x=pq}{}^i\{\min{}^i\{\overline{\chi}_A(p), \overline{S}(q)\}\}\} \\
&\geq \min{}^i\{\min{}^i\{\overline{S}(s), \overline{\chi}_A(a)\}, \min{}^i\{\overline{\chi}_A(a_1), \overline{S}(s_1)\}\} \\
&= \min{}^i\{\overline{\chi}_A(a), \overline{\chi}_A(a_1)\} \\
&= \min{}^i\{\overline{1}, \overline{1}\} \\
&= \overline{1}.
\end{aligned}$$

Since $\overline{\chi}_A$ is an i-v fuzzy quasi-ideal, $\overline{\chi}_A(x) \geq ((\overline{S}\,\overline{\chi}_A) \cap (\overline{\chi}_A\,\overline{S}))(x) \geq \overline{1}$. This implies that $\overline{\chi}_A(x) = \overline{1}$ and hence $x \in A$. Thus $SA \cap AS \subseteq A$ and so A is a quasi-ideal of S.

Conversely, assume that A is a quasi-ideal of S. Suppose $\overline{\chi}_A(x) < ((\overline{S}\,\overline{\chi}_A) \cap (\overline{\chi}_A\,\overline{S}))(x)$ for some $x \in S$. This means that $\overline{\chi}_A(x) = \overline{0}$ and $((\overline{S}\,\overline{\chi}_A) \cap (\overline{\chi}_A\,\overline{S}))(x) = \overline{1}$. This implies that $(\overline{\chi}_A\,\overline{S})(x) = \overline{1}$ and $(\overline{S}\,\overline{\chi}_A)(x) = \overline{1}$. By Lemma 2.17 $(\overline{\chi}_{AS})(x) = \overline{1}$ and $(\overline{\chi}_{SA})(x) = \overline{1}$. Thus $x \in AS \cap SA$. Since A is a quasi-ideal, $x \in AS \cap SA \subseteq A$ and hence $x \in A$, a contradiction. Therefore, $\overline{\chi}_A(x) \geq ((\overline{\chi}_A\,\overline{S}) \cap (\overline{S}\,\overline{\chi}_A))(x)$ for all $x \in S$. It follows that $\overline{\chi}_A$ is an i-v fuzzy quasi-ideal of S. □

Lemma 3.13. *Every i-v fuzzy quasi-ideal of S is an i-v fuzzy subsemigroup of S.*

Proof. Let $\overline{\lambda}$ be an i-v fuzzy quasi-ideal of S. Since $\overline{\lambda} \leq \overline{S}\,\overline{\lambda}\,\overline{\lambda} \leq \overline{S}\,\overline{\lambda}$ and $\overline{\lambda}\,\overline{\lambda} \leq \overline{\lambda}\,\overline{S}$. Hence $\overline{\lambda}\,\overline{\lambda} \leq (\overline{S}\,\overline{\lambda}) \cap (\overline{\lambda}\,\overline{S}) \leq \overline{\lambda}$, as $\overline{\lambda}$ is an i-v fuzzy quasi-ideal of S. Thus $\overline{\lambda}$ is an i-v fuzzy subsemigroup of S. □

Lemma 3.14. *Every i-v fuzzy left (right, two-sided) ideal of S is an i-v fuzzy quasi-ideal of S.*

Proof. Let $\overline{\lambda}$ be an i-v fuzzy left ideal of S. Then we have $\overline{S}\ \overline{\lambda} \leq \overline{\lambda}$ and $(\overline{S}\ \overline{\lambda}) \cap (\overline{\lambda}\ \overline{S}) \leq \overline{S}\ \overline{\lambda} \leq \overline{\lambda}$. This means that $\overline{\lambda}$ is an i-v fuzzy quasi-ideal of S. □

However, the converse of the above lemma is not true in general, which is demonstrated by the following example.

Example 3.15. Let S and $\overline{\lambda}$ be as in Example 2.10. Then $\overline{\lambda}$ is an i-v fuzzy quasi-ideal of S and $\overline{\lambda}$ is not an i-v fuzzy left (right, two-sided) ideal of S. In fact, $((\overline{S}\,\overline{\lambda}) \cap (\overline{\lambda}\,\overline{S}))(x) \leq \overline{\lambda}(x)$ for all $x \in S$. Thus $\overline{\lambda}$ is an i-v fuzzy quasi-ideal of S. But since,

$$\begin{aligned}(\overline{\lambda}\,\overline{S})(2) &= \sup_{2=ab}{}^{i}\{\min{}^{i}\{\overline{\lambda}(a), \overline{S}(b)\}\} \\ &= \min{}^{i}\{\overline{\lambda}(1), \overline{S}(2)\} \text{ as } 2 = 1.2 \\ &= \min{}^{i}\{\overline{\lambda}[0.6, 0.7], [1, 1]\} \\ &= [0.6, 0.7] \\ &\not\leq \overline{\lambda}(2) = [0.1, 0.2]\end{aligned}$$

and

$$\begin{aligned}(\overline{S}\,\overline{\lambda})(3) &= \min{}^{i}\{\overline{S}(3), \overline{\lambda}(1)\} \text{ as } 3 = 3.1 \\ &= [0.6, 0.7] \\ &\not\leq \overline{\lambda}(3) \\ &= [0.3, 0.4].\end{aligned}$$

Thus $\overline{\lambda}$ is neither an i-v fuzzy right ideal nor an i-v fuzzy left ideal of S. That is $\overline{\lambda}$ is not an i-v fuzzy ideal of S.

However in the case when S is regular, the converse of Lemma 3.14 is true, which is shown in Theorem 4.6.

Lemma 3.16. *Let $\overline{\lambda}$ and $\overline{\mu}$ be any i-v fuzzy right ideal and any i-v fuzzy left ideal of S, respectively. Then $\overline{\lambda} \cap \overline{\mu}$ is an i-v fuzzy quasi-ideal of S.*

Proof. Since $\overline{\lambda}$ is an i-v fuzzy right ideal and $\overline{\mu}$ is an i-v fuzzy left ideal, we have $\overline{\lambda}\,\overline{S} \leq \overline{\lambda}$ and $\overline{S}\,\overline{\mu} \leq \overline{\mu}$. Now

$$\begin{aligned}((\overline{\lambda} \cap \overline{\mu})\overline{S}) \cap (\overline{S}(\overline{\lambda} \cap \overline{\mu})) &\leq (\overline{\lambda}\,\overline{S}) \cap (\overline{S}\,\overline{\mu}), \text{ since } \overline{\lambda} \cap \overline{\mu} \leq \overline{\lambda}, \overline{\mu} \\ &\leq \overline{\lambda} \cap \overline{\mu}.\end{aligned}$$

Thus $\overline{\lambda} \cap \overline{\mu}$ is an i-v fuzzy quasi-ideal of S. □

Corollary 3.17. *For any nonempty i-v fuzzy subset $\overline{\lambda}$ of S, $(\overline{\lambda} \cup (\overline{S}\,\overline{\lambda})) \cap (\overline{\lambda} \cup (\overline{\lambda}\,\overline{S}))$ is an i-v fuzzy quasi-ideal of S.*

Lemma 3.18. *Every i-v fuzzy quasi-ideal of S is an i-v fuzzy bi-ideal of S.*

Proof. Let $\overline{\lambda}$ be any i-v fuzzy quasi-ideal of S. Then we have

$$\begin{aligned}\overline{\lambda}\,\overline{\lambda} &= (\overline{\lambda}\,\overline{\lambda}) \cap (\overline{\lambda}\,\overline{\lambda}) \\ &\leq (\overline{\lambda}\,\overline{S}) \cap (\overline{S}\,\overline{\lambda}) \\ &\leq \overline{\lambda}.\end{aligned}$$

Also

$$\begin{aligned}\overline{\lambda}\,\overline{S}\,\overline{\lambda} &\leq \overline{\lambda}\,\overline{S}\,\overline{S} \\ &\leq \overline{\lambda}\,\overline{S} \\ \overline{\lambda}\,\overline{S}\,\overline{\lambda} &\leq \overline{S}\,\overline{S}\,\overline{\lambda} \\ &\leq \overline{S}\,\overline{\lambda}.\end{aligned}$$

We have $\overline{\lambda}\,\overline{S}\,\overline{\lambda} \leq (\overline{\lambda}\,\overline{S}) \cap (\overline{S}\,\overline{\lambda}) \leq \overline{\lambda}$. This implies that $\overline{\lambda}$ is an i-v fuzzy bi-ideal of S. □

However, converse of Lemma 3.18 is not true in general as shown in the following example.

Example 3.19. Let $S = \{0, 1, 2, 3\}$ be a semigroup with the following multiplication table:

$\bullet$	0	1	2	3
0	0	0	0	0
1	0	0	0	0
2	0	0	0	1
3	0	0	1	2

Define $\overline{\lambda} : S \to D[0, 1]$ as $\overline{\lambda}(0) = [0.8, 0.9], \overline{\lambda}(1) = [0.4, 0.5]$, $\overline{\lambda}(2) = [0.6, 0.7]$ and $\overline{\lambda}(3) = [0.1, 0.2]$. Then $\overline{\lambda}$ is not an i-v fuzzy quasi-ideal of S. But $\overline{\lambda}$ is an i-v fuzzy bi-ideal of S. For, $(\overline{\lambda}\,\overline{S}\,\overline{\lambda})(0) = [0.8, 0.9]$, $(\overline{\lambda}\,\overline{S}\,\overline{\lambda})(1) = [0.1, 0.2], (\overline{\lambda}\,\overline{S}\,\overline{\lambda})(2) = [0, 0] = \overline{0}$ and $(\overline{\lambda}\,\overline{S}\,\overline{\lambda})(3) = [0, 0] = \overline{0}$. Thus $(\overline{\lambda}\,\overline{S}\,\overline{\lambda})(x) \leq \overline{\lambda}(x)$ for all $x \in S$. Also $(\overline{\lambda}\,\overline{\lambda})(0) = [0.8, 0.9]$, $(\overline{\lambda}\,\overline{\lambda})(1) = [0.1, 0.2], (\overline{\lambda}\,\overline{\lambda})(2) = [0.1, 0.2], (\overline{\lambda}\,\overline{\lambda})(3) = [0, 0] = \overline{0}$. Thus $(\overline{\lambda}\,\overline{\lambda})(x) \leq \overline{\lambda}(x)$ for all $x \in S$. So $\overline{\lambda}$ is an i-v fuzzy bi-ideal of S.

$$\begin{aligned}&(\overline{\lambda}\,\overline{S})(0) = [0.8, 0.9] = (\overline{S}\,\overline{\lambda})(0) \\ &(\overline{\lambda}\,\overline{S})(1) = [0.6, 0.7] = (\overline{S}\,\overline{\lambda})(1) \\ &(\overline{\lambda}\,\overline{S})(2) = [0.1, 0.2] = (\overline{S}\,\overline{\lambda})(2) \\ &(\overline{\lambda}\,\overline{S})(3) = \overline{0} = (\overline{S}\,\overline{\lambda})(3) \\ &((\overline{\lambda}\,\overline{S}) \cap (\overline{S}\,\overline{\lambda}))(0) = [0.8, 0.9] < \overline{\lambda}(0) \\ &((\overline{\lambda}\,\overline{S}) \cap (\overline{S}\,\overline{\lambda}))(1) = [0.6, 0.7] \nleq \overline{\lambda}(1) = [0.4, 0.5] \\ &((\overline{\lambda}\,\overline{S}) \cap (\overline{S}\,\overline{\lambda}))(2) = [0.1, 0.2] < \overline{\lambda}(2) = [0.6, 0.7] \\ &((\overline{\lambda}\,\overline{S}) \cap (\overline{S}\,\overline{\lambda}))(3) = \overline{0} < \overline{\lambda}(3) = [0.1, 0.2].\end{aligned}$$

Thus $\overline{\lambda}$ is not an i-v fuzzy quasi-ideal of S.

Lemma 3.20. *Suppose that $\overline{\lambda}$ or $\overline{\mu}$ is an i-v fuzzy quasi-ideal of S. Then $\overline{\lambda}\,\overline{\mu}$ is an i-v fuzzy bi-ideal of S.*

Proof. Let $\overline{\lambda}$ be an i-v fuzzy quasi-ideal of S and $\overline{\mu}$ be an i-v fuzzy subset S. Thus we have, $\overline{\lambda}\,\overline{S}\,\overline{\lambda} \leq \overline{\lambda}$. Hence

$$(\overline{\lambda}\,\overline{\mu})\,(\overline{\lambda}\,\overline{\mu}) = (\overline{\lambda}\,\overline{\mu}\,\overline{\lambda})\,\overline{\mu} \\ \leq (\overline{\lambda}\,\overline{\mu}).$$

Also

$$(\overline{\lambda}\,\overline{\mu})\,\overline{S}\,(\overline{\lambda}\,\overline{\mu}) = (\overline{\lambda}\,(\overline{\mu}\,\overline{S})\,\overline{\lambda})\,\overline{\mu} \\ \leq (\overline{\lambda}\,(\overline{S}\,\overline{S})\,\overline{\lambda})\,\overline{\mu} \\ \leq (\overline{\lambda}\,\overline{S}\,\overline{\lambda})\,\overline{\mu} \\ \leq \overline{\lambda}\,\overline{\mu}.$$

This implies that $\overline{\lambda}\,\overline{\mu}$ is an i-v fuzzy bi-ideal of S. □

Corollary 3.21. *The product of two i-v fuzzy quasi-ideals of S is an i-v fuzzy bi-ideal of S.*

4. Regular semigroups

In this section, we characterize a regular semigroup in terms of i-v fuzzy ideals, i-v fuzzy interior ideals, i-v fuzzy quasi-ideals and i-v fuzzy bi-ideals. We find the equivalent conditions on regular semigroups.

Theorem 4.1. *For a semigroup S, the following conditions are equivalent.*
(1) *S is regular.*
(2) *$R \cap L = RL$ for every right ideal R of S and every left ideal L of S.*
(3) *$Q = QSQ$ for every quasi-ideal Q of S.*

Proof. The equivalence of (1) and (2) is due to Iseki[[6], Theorem 1] and the equivalence of (1) and (3) follows from Steinfeld[[10], p.10]. □

Theorem 4.2. *The following conditions are equivalent.*
(1) *S is regular.*
(2) *$\overline{\lambda}\,\overline{\mu} = \overline{\lambda} \cap \overline{\mu}$ for every i-v fuzzy right ideal $\overline{\lambda}$ and every i-v fuzzy left ideal $\overline{\mu}$ of S.*

Proof. (1) ⇒ (2). Let $\overline{\lambda}$ and $\overline{\mu}$ be any i-v fuzzy right ideal and i-v fuzzy left ideal of a regular semigroup S respectively. Let $x \in S$. Then

$$(\overline{\lambda}\,\overline{\mu})(x) = \sup_{x=ab}{}^{i}\{\min{}^{i}\{\overline{\lambda}(a),\,\overline{\mu}(b)\}\} \\ \leq \min{}^{i}\{\overline{\lambda}(ab), \overline{\mu}(ab)\} \\ = (\overline{\lambda} \cap \overline{\mu})(x)$$

and so $\overline{\lambda}\,\overline{\mu} \leq \overline{\lambda} \cap \overline{\mu}$. Again let $a \in S$. Then since S is regular, there exists $x \in S$ such that $a = axa$. Now

$$(\overline{\lambda}\,\overline{\mu})(a) = \sup_{a=pq}{}^{i}\{\min{}^{i}\{\overline{\lambda}(p),\,\overline{\mu}(q)\}\} \\ \geq \min{}^{i}\{\overline{\lambda}(a),\,\overline{\mu}(xa)\} \\ \geq \min{}^{i}\{\overline{\lambda}(a),\,\overline{\mu}(a)\} \\ = (\overline{\lambda} \cap \overline{\mu})(a)$$

and hence $\overline{\lambda}\,\overline{\mu} \geq \overline{\lambda} \cap \overline{\mu}$. Thus $\overline{\lambda} \cap \overline{\mu} = \overline{\lambda}\,\overline{\mu}$.

(2) $\Rightarrow$ (1). Assume that (2) holds. Let R and L be the right ideal and left ideal of S respectively. Since $\overline{\chi}_R$ and $\overline{\chi}_L$ are respectively i-v fuzzy right ideal and i-v fuzzy left ideal, $\overline{\chi}_R \cap \overline{\chi}_L = \overline{\chi}_R\,\overline{\chi}_L$. By Lemma 2.17, $\overline{\chi}_{R\cap L} = \overline{\chi}_{RL}$. Thus $RL = R \cap L$ and hence by Theorem 4.1, S is regular. □

Theorem 4.3. *Let $\overline{\lambda}$ be an i-v fuzzy subset of a regular semigroup S. Then the following conditions are equivalent.*

(1) *$\overline{\lambda}$ is an i-v fuzzy ideal of S.*

(2) *$\overline{\lambda}$ is an i-v fuzzy interior ideal of S.*

Proof. (1) $\Rightarrow$ (2) follows by Lemma 3.9.

(2) $\Rightarrow$ (1). Assume that (2) holds. Let $a, b \in S$. Then since S is regular, there exist elements $x, y \in S$ such that $a = axa$ and $b = byb$. Thus we have,

$\overline{\lambda}(ab) = \overline{\lambda}((axa)b) = \overline{\lambda}((ax)ab) \geq \overline{\lambda}(a)$ and

$\overline{\lambda}(ab) = \overline{\lambda}(a(byb)) = \overline{\lambda}(ab(yb)) \geq \overline{\lambda}(b)$.

This implies that $\overline{\lambda}$ is an i-v fuzzy ideal of S. □

Theorem 4.4. *For a semigroup S, the following conditions are equivalent.*

(1) *S is regular.*

(2) *$\overline{\lambda} = \overline{\lambda}\,\overline{S}\,\overline{\lambda}$ for every i-v fuzzy bi-ideal $\overline{\lambda}$ of S.*

(3) *$\overline{\lambda} = \overline{\lambda}\,\overline{S}\,\overline{\lambda}$ for every i-v fuzzy quasi-ideal $\overline{\lambda}$ of S.*

Proof. (1) $\Rightarrow$ (2). Let $\overline{\lambda}$ be any i-v fuzzy bi-ideal of S and $a \in S$. Since S is regular, there exists $x \in S$ such that $a = axa$. Then

$$\begin{aligned}(\overline{\lambda}\,\overline{S}\,\overline{\lambda})(a) &= \sup_{a=yz}{}^{i}\{\min{}^{i}\{\overline{\lambda}(y), (\overline{S}\,\overline{\lambda})(z)\}\}\\ &\geq \min{}^{i}\{\overline{\lambda}(a), \sup_{xa=pq}{}^{i}\{\min{}^{i}\{\overline{S}(p), \overline{\lambda}(q)\}\}\}\\ &= \min{}^{i}\{\overline{\lambda}(a), \overline{\lambda}(a)\}\\ &= \overline{\lambda}(a)\end{aligned}$$

and hence $\overline{\lambda}\,\overline{S}\,\overline{\lambda} \geq \overline{\lambda}$. Since $\overline{\lambda}$ is an i-v fuzzy bi-ideal of S, $\overline{\lambda}\,\overline{S}\,\overline{\lambda} \leq \overline{\lambda}$. Thus $\overline{\lambda}\,\overline{S}\,\overline{\lambda} = \overline{\lambda}$ and (1) $\Rightarrow$ (2).

(2) $\Rightarrow$ (3). Since any i-v fuzzy quasi-ideal of S is a fuzzy bi-ideal of S, by Lemma 3.18, (2) $\Rightarrow$ (3).

(3) $\Rightarrow$ (1). Let Q be any quasi-ideal in S and let $x \in Q$. Then we have

$$\begin{aligned}\overline{\chi}_{QSQ}(x) &= (\overline{\chi}_Q\,\overline{\chi}_S\,\overline{\chi}_Q)(x)\\ &= \overline{\chi}_Q(x)\\ &= \overline{1}.\end{aligned}$$

This implies that $x \in QSQ$. Thus $Q \subseteq QSQ$. On the other hand, since Q is a quasi-ideal of S, $QS \cap SQ \subseteq Q$, $QSQ \subseteq QSS \subseteq QS$ and $QSQ \subseteq SSQ \subseteq SQ$. Hence $QSQ \subseteq QS \cap SQ \subseteq Q$. Therefore $QSQ = Q$ and by Theorem 4.1, S is regular. □

Theorem 4.5. *For a regular semigroup S, the following conditions are equivalent.*

(1) *Every bi-ideal of S is a right (left, two-sided) ideal of S.*

(2) *Every i-v fuzzy bi-ideal of S is an i-v fuzzy right (left, two-sided) ideal of S.*

Proof. (1) $\Rightarrow$ (2). Let $\overline{\lambda}$ be any i-v fuzzy bi-ideal of S and $a, b \in S$. Since the set aSa is a bi-ideal of S, by the assumption it is a right ideal of S. Since S is regular, $ab \in (aSa)S \subseteq aSa$. Thus there exists an element $x \in S$ such that $ab = axa$. Since $\overline{\lambda}$ is an i-v fuzzy bi-ideal of S,

$\overline{\lambda}(ab) = \overline{\lambda}(axa) \geq \min^i\{\overline{\lambda}(a), \overline{\lambda}(a)\} = \overline{\lambda}(a)$.

Thus $\overline{\lambda}$ is an i-v fuzzy right ideal of S and hence (1) $\Rightarrow$ (2).

(2) $\Rightarrow$ (1). Let A be any bi-ideal of S. By Proposition 3.2, $\overline{\chi}_A$ is an i-v fuzzy bi-ideal of S. Hence by assumption $\overline{\chi}_A$ is an i-v fuzzy right ideal of S. Thus by Proposition 2.14, A is a right ideal of S. Hence (2) $\Rightarrow$ (1). □

Theorem 4.6. *For any semigroup S, the following conditions are equivalent.*

(1) *S is regular.*

(2) *$\overline{\lambda} \cap \overline{\mu} = \overline{\mu}\,\overline{\lambda}\,\overline{\mu}$ for every i-v fuzzy ideal $\overline{\lambda}$ and every i-v fuzzy bi-ideal $\overline{\mu}$ of S.*

(3) *$\overline{\lambda} \cap \overline{\mu} = \overline{\mu}\,\overline{\lambda}\,\overline{\mu}$ for every i-v fuzzy ideal $\overline{\lambda}$ and every i-v fuzzy quasi-ideal $\overline{\mu}$ of S.*

Proof. (1) $\Rightarrow$ (2). Let $\overline{\lambda}$ and $\overline{\mu}$ be an i-v fuzzy ideal and an i-v fuzzy bi-ideal of S respectively. Then $\overline{\mu}\,\overline{\lambda}\,\overline{\mu} \leq \overline{\mu}\,\overline{S}\,\overline{\mu} \leq \overline{\mu}$. Again $\overline{\mu}\,\overline{\lambda}\,\overline{\mu} \leq \overline{S}\,\overline{\lambda}\,\overline{S} \leq \overline{\lambda}$. Thus $\overline{\mu}\,\overline{\lambda}\,\overline{\mu} \leq \overline{\lambda} \cap \overline{\mu}$. On the other hand let $a \in S$. Then since S is regular, there exists $x \in S$ such that $a = axa = axaxa$. As $\overline{\lambda}$ is an i-v fuzzy ideal of S, $\overline{\lambda}(xax) \geq \overline{\lambda}(a)$. Then we have

$$\begin{aligned}
(\overline{\mu}\,\overline{\lambda}\,\overline{\mu})(a) &= \sup_{a=yz}{}^{i}\{\min{}^i\{\overline{\mu}(y),\, \overline{\lambda}\,\overline{\mu}(z)\}\} \\
&\geq \min{}^i\{\overline{\mu}(a),\, (\overline{\lambda}\,\overline{\mu})(xaxa)\} \\
&= \min{}^i\{\overline{\mu}(a),\, \sup_{xaxa=pq}{}^{i}\{\min{}^i\{\overline{\lambda}(p),\, \overline{\mu}(q)\}\}\} \\
&\geq \min{}^i\{\overline{\mu}(a),\, \min{}^i\{\overline{\lambda}(xax), \overline{\mu}(a)\}\} \\
&\geq \min{}^i\{\overline{\mu}(a),\, \min{}^i\{\overline{\lambda}(a),\, \overline{\mu}(a)\}\} \\
&= \min{}^i\{\overline{\mu}(a),\, \overline{\lambda}(a)\} \\
&= (\overline{\mu} \cap \overline{\lambda})(a)
\end{aligned}$$

and so $\overline{\mu}\,\overline{\lambda}\,\overline{\mu} \geq \overline{\mu} \cap \overline{\lambda}$ and hence $\overline{\mu}\,\overline{\lambda}\,\overline{\mu} = \overline{\mu} \cap \overline{\lambda}$. Thus (1) $\Rightarrow$ (2).

(2) $\Rightarrow$ (3) is clear.

(3) $\Rightarrow$ (1). Let (3) hold. Let $\overline{\mu}$ be any i-v fuzzy quasi-ideal of S. $\overline{S}$ itself being an i-v fuzzy bi-ideal of S, $\overline{\mu} = \overline{\mu} \cap \overline{S} = \overline{\mu}\,\overline{S}\,\overline{\mu}$. Thus by Theorem 4.4, S is regular and so (3) $\Rightarrow$ (1). □

Theorem 4.7. *Every i-v fuzzy ideal of a regular semigroup is idempotent.*

Proof. Let $\overline{\mu}$ be an i-v fuzzy ideal of a regular semigroup S. Then by Proposition 2.15, $\overline{\mu}\,\overline{S}\,\overline{\mu} \leq \overline{\mu}$. Hence $\overline{\mu}$ is an i-v fuzzy bi-ideal of S. Since S is regular, by Theorem 4.4, we have $\overline{\mu} = \overline{\mu}\,\overline{S}\,\overline{\mu} \leq \overline{\mu}\,\overline{\mu} \leq \overline{\mu}\,\overline{S} \leq \overline{\mu}$ and so $\overline{\mu} = \overline{\mu}\,\overline{\mu}$. Thus $\overline{\mu}$ is idempotent. □

Theorem 4.8. *For a semigroup S, the following conditions are equivalent.*

(1) *S is regular.*

(2) *$\overline{\mu} \cap \overline{\lambda} = \overline{\mu}\,\overline{\lambda}\,\overline{\mu}$ for every i-v fuzzy quasi-ideal $\overline{\mu}$ and every fuzzy ideal $\overline{\lambda}$ of S.*

(3) *$\overline{\mu} \cap \overline{\lambda} = \overline{\mu}\,\overline{\lambda}\,\overline{\mu}$ for every i-v fuzzy quasi-ideal $\overline{\mu}$ and every i-v fuzzy interior ideal $\overline{\lambda}$ of S.*

(4) *$\overline{\mu} \cap \overline{\lambda} = \overline{\mu}\,\overline{\lambda}\,\overline{\mu}$ for every i-v fuzzy bi-ideal $\overline{\mu}$ and every i-v fuzzy ideal $\overline{\lambda}$ of S.*

(5) *$\overline{\mu} \cap \overline{\lambda} = \overline{\mu}\,\overline{\lambda}\,\overline{\mu}$ for every i-v fuzzy bi-ideal $\overline{\mu}$ and every i-v fuzzy interior ideal $\overline{\lambda}$ of S.*

Proof. (1) $\Rightarrow$ (5). Assume that (1) holds. Let $\overline{\mu}$ and $\overline{\lambda}$ be any i-v fuzzy bi-ideal and any i-v fuzzy interior ideal of S respectively. Then $\overline{\mu}\,\overline{\lambda}\,\overline{\mu} \leq \overline{\mu} \cap \overline{\lambda}$. Let $a \in S$. Since S is regular, there exists $x \in S$ such that $a = axa$ $(= axaxa)$. Then

$$
\begin{aligned}
(\overline{\mu}\,\overline{\lambda}\,\overline{\mu})(a) &= \sup_{a=yz}{}^{i}\{\min{}^{i}\{\overline{\mu}(y),\, (\sup_{a=yz}{}^{i}\{\min{}^{i}\{\overline{\mu}(y),\, (\overline{\lambda}\,\overline{\mu})(z)\}\} \\
&\geq \min{}^{i}\{\overline{\mu}(a),\, (\overline{\lambda}\,\overline{\mu})(xaxa)\}\} \\
&= \min{}^{i}\{\overline{\mu}(a),\, \sup_{xaxa=pq}{}^{i}\{\min{}^{i}\{\overline{\lambda}(p),\, \overline{\mu}(q)\}\}\} \\
&\geq \min{}^{i}\{\overline{\mu}(a),\, \min{}^{i}\{\overline{\lambda}(xax), \overline{\mu}(a)\}\} \\
&\geq \min{}^{i}\{\overline{\mu}(a),\, \min{}^{i}\{\overline{\lambda}(a),\, \overline{\mu}(a)\}\}, \\
&\qquad \text{since } \overline{\lambda} \text{ is an i-v fuzzy interior ideal, } \overline{\lambda}(xax) \geq \overline{\lambda}(a). \\
&= \min{}^{i}\{\overline{\mu}(a), \overline{\lambda}(a)\} \\
&= (\overline{\mu} \cap \overline{\lambda})(a)
\end{aligned}
$$

and so $\overline{\mu}\,\overline{\lambda}\,\overline{\mu} \geq \overline{\mu} \cap \overline{\lambda}$. Hence $\overline{\mu}\,\overline{\lambda}\,\overline{\mu} = \overline{\mu} \cap \overline{\lambda}$. Thus (1) $\Rightarrow$ (5).

By Theorem 4.6, (1), (2), (3) and (4) are equivalent. It is clear that (5) $\Rightarrow$ (4) $\Rightarrow$ (3) $\Rightarrow$ (2). □

Theorem 4.9. *For a semigroup S, the following conditions are equivalent.*

(1) *S is regular.*

(2) *$\overline{\lambda} \cap \overline{\mu} \leq \overline{\lambda}\,\overline{\mu}$ for every i-v fuzzy right ideal $\overline{\lambda}$ and every i-v fuzzy bi-ideal $\overline{\mu}$ of S.*

(3) *$\overline{\lambda} \cap \overline{\mu} \leq \overline{\lambda}\,\overline{\mu}$ for every i-v fuzzy right ideal $\overline{\lambda}$ and every i-v fuzzy quasi-ideal $\overline{\mu}$ of S.*

(4) *$\overline{\mu} \cap \overline{\nu} \leq \overline{\mu}\,\overline{\nu}$ for every i-v fuzzy left ideal $\overline{\nu}$ and every i-v fuzzy bi-ideal $\overline{\mu}$ of S.*

(5) *$\overline{\mu} \cap \overline{\nu} \leq \overline{\mu}\,\overline{\nu}$ for every i-v fuzzy left ideal $\overline{\nu}$ and every i-v fuzzy quasi-ideal $\overline{\mu}$ of S.*

(6) $\overline{\lambda}\cap\overline{\mu}\cap\overline{\nu}\leq\overline{\lambda}\,\overline{\mu}\,\overline{\nu}$ *for every i-v fuzzy right ideal* $\overline{\lambda}$, *every i-v fuzzy left ideal* $\overline{\nu}$ *and every i-v fuzzy bi-ideal* $\overline{\mu}$ *of* S.

(7) $\overline{\lambda}\cap\overline{\mu}\cap\overline{\nu}\leq\overline{\lambda}\,\overline{\mu}\,\overline{\nu}$ *for every i-v fuzzy right ideal* $\overline{\lambda}$, *every i-v fuzzy left ideal* $\overline{\nu}$ *and every i-v fuzzy quasi-ideal* $\overline{\mu}$ *of* S.

Proof. (1) ⇒ (2). Let $\overline{\lambda}$ and $\overline{\mu}$ be an i-v fuzzy right ideal and an i-v fuzzy bi-ideal of S respectively. Suppose $a\in S$. Since S is regular, there exists $x\in S$ such that $a=axa=axaxa$. Then

$$\begin{aligned}(\overline{\lambda}\,\overline{\mu})(a) &= \sup_{a=xy}{}^{i}\{\min{}^{i}\{\overline{\lambda}(x),\,\overline{\mu}(y)\}\}\\ &\geq \min{}^{i}\{\overline{\lambda}(axax),\overline{\mu}(a)\}\\ &= \min{}^{i}\{\overline{\lambda}(a(xax)),\overline{\mu}(a)\}\\ &\geq \min{}^{i}\{\overline{\lambda}(a),\overline{\mu}(a)\}=(\overline{\lambda}\cap\overline{\mu})(a).\end{aligned}$$

Thus (1) ⇒ (2). It can be seen in a similar way that (1) ⇒ (4).
Clearly (2) ⇒ (3) and (4) ⇒ (5).

Now assume that (3) holds. Let $\overline{\lambda}$ be an i-v fuzzy right ideal and $\overline{\mu}$ be an i-v fuzzy left ideal of S. Since every i-v fuzzy left ideal is an i-v fuzzy quasi-ideal of S, by (3), we have $\overline{\lambda}\cap\overline{\mu}\leq\overline{\lambda}\,\overline{\mu}$. Again since $\overline{\lambda}$ is an i-v fuzzy right ideal and $\overline{\mu}$ is an i-v fuzzy left ideal of S, we have $\overline{\lambda}\,\overline{\mu}\leq\overline{\lambda}\cap\overline{\mu}$. Thus $\overline{\lambda}\,\overline{\mu}=\overline{\lambda}\cap\overline{\mu}$. By Theorem 4.2, S is regular hence we obtain (3) ⇒ (1). Similarly (4) ⇒ (1). Again assume that (1) holds. Let $\overline{\lambda},\overline{\nu}$ and $\overline{\mu}$ be an i-v fuzzy right ideal, an i-v fuzzy left ideal and an i-v fuzzy bi-ideal of S respectively and let $a\in S$. Since S is regular, there exists $x\in S$ such that $a=axa=axaxa$. Then

$$\begin{aligned}(\overline{\lambda}\,\overline{\mu}\,\overline{\nu})(a) &= \sup_{a=yz}{}^{i}\{\min{}^{i}\{\overline{\lambda}(y),\,(\overline{\mu}\,\overline{\nu})(z)\}\}\\ &\geq \min{}^{i}\{\overline{\lambda}(ax),(\overline{\mu}\,\overline{\nu})(axa)\}\\ &\geq \min{}^{i}\{\overline{\lambda}(a),\min{}^{i}\{\overline{\mu}(a),\overline{\nu}(a)\}\}\\ &= \min{}^{i}\{\overline{\lambda}(a),\overline{\mu}(a),\overline{\nu}(a)\}\\ &= (\overline{\lambda}\cap\overline{\mu}\cap\overline{\nu})(a)\end{aligned}$$

and hence $\overline{\lambda}\cap\overline{\mu}\cap\overline{\nu}\leq\overline{\lambda}\,\overline{\mu}\,\overline{\nu}$ and so (1) ⇒ (6). It is clear that (6) ⇒ (7).

Finally, assume that (7) holds. Let $\overline{\lambda}$ and $\overline{\nu}$ respectively be an i-v fuzzy right ideal and an i-v fuzzy left ideal of S. Since $\overline{S}$ is itself an i-v fuzzy quasi-ideal of S, we have

$(\overline{\lambda}\cap\overline{\nu})=\overline{\lambda}\cap\overline{S}\cap\overline{\nu}\leq\overline{\lambda}\,\overline{S}\,\overline{\nu}\leq\overline{\lambda}\,\overline{\nu}$, by Proposition 2.15.

Clearly $\overline{\lambda}\,\overline{\nu}\leq\overline{\lambda}\cap\overline{\nu}$. hence $\overline{\lambda}\,\overline{\nu}=\overline{\lambda}\cap\overline{\nu}$. It follows from Theorem 4.2, that S is regular and (7) ⇒ (1). □

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