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ON REGULAR NEAR-RINGS WITH (m, n)-POTENT CONDITIONS

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ABSTRACT. Jat and Choudhari defined a near-ring R with left bipotent or right bipotent condition in 1979. Also, we can define a near-ring R as subcommutative if $aR = Ra$ for all a in R. From these above two concepts it is natural to investigate the near-ring R with the properties $aR = Ra^2$ (resp. $a^2R = Ra$) for each a in R. We will say that such is a near-ring with $(1, 2)$-potent condition (resp. a near-ring with $(2, 1)$-potent condition). Thus, we can extend a general concept of a near-ring R with (m, n)-potent condition, that is, $a^m R = Ra^n$ for each a in R, where m, n are positive integers.

We will derive properties of near-ring with $(1, n)$ and $(n, 1)$-potent conditions where n is a positive integer, any homomorphic image of (m, n)-potent near-ring is also (m, n)-potent, and we will obtain some characterization of regular near-rings with (m, n)-potent conditions.

1. Introduction

The concept of Von Neumann regularity of near-rings have been studied by many authors Beidleman, Choudhari, Goyal, Heatherly, Ligh, Mason, Murty, and Szeto. Their main results are suggested in the book of Pilz [13].

In 1980, Mason introduced the notions of left regularity, right regularity and strong regularity of near-rings.

He proved that for zero-symmetric near-ring with identity, the concepts of left regularity, strong left regularity and strong right regularity of near-rings are all equivalent. Moreover, in 1984, the concept of strong regularity was studied by Murty.

The Von Neumann regularity of rings and its generalization were studied by Fisher, Snider, Hirano, Tominaga, Savaga, Li, Schein and Ohori. In 1985, Ohori investigated the characterization of π-regularity and strong π-regularity of rings.

The concepts of Von Neumann regularity and π-regularity are the same meaning as in ring theory, but the concept of strong regularity in near-rings is different meaning in rings [8], [11].

A near-ring R is an algebraic system $(R, +, \cdot)$ with two binary operations $+$ and $\cdot$ such that $(R, +)$ is a group (not necessarily abelian) with a zero element 0, $(R, \cdot)$ is a semigroup and $(a + b)c = ac + bc$ for all a, b, c in R. If R has a unity 1, then R is called *unitary*.

A near-field is a unitary near-ring with every nonzero element is invertible.

A near-ring R with the extra axiom $a0 = 0$ for all $a \in R$ is said to be *zero symmetric*. An element d in R is called *distributive* if $d(a + b) = da + db$ for all a and b in R.

We will use the following notations: Given a near-ring R, $R_0 = \{a \in R \mid a0 = 0\}$ which is called the *zero symmetric part* of R, $R_c = \{a \in R \mid a0 = a\}$ which is called the *constant part* of R. The set of all distributive elements in R is denoted by R_d.

Obviously, we see that R_0 and R_c are subnear-rings of R, but R_d is a semigroup under multiplication. Clearly, near-ring R is zero symmetric, in case $R = R_0$ also, in case $R = R_c$, R is called a *constant* near-ring.

In 1979, Jat and Choudhari defined a near-ring R to be left bipotent (resp. right bipotent) if $Ra = Ra^2$ (resp. $aR = a^2R$) for each a in R. Also, we can define a near-ring R as subcommutative if $aR = Ra$ for all a in R like as in ring theory. Obviously, every commutative near-ring is subcommutative. From these above two concepts it is natural to investigate the near-ring R with the properties $aR = Ra^2$ (resp. $a^2R = Ra$) for each a in R. We will say that such is a near-ring with $(1, 2)$-potent condition (resp. a near-ring with $(2, 1)$-potent condition). Thus, from this motivation, we can extend a general concept of a near-ring R with (m, n)-potent condition, that is, $a^m R = Ra^n$ for each a in R, where m, n are positive integers.

First, we will derive properties of near-ring with $(1, 2)$ and $(2, 1)$-potent conditions, also $(1, n)$ and $(n, 1)$-potent conditions where n is a positive integer. Any homomorphic image of (m, n)-potent near-ring is also (m, n)-potent, and every $(1, n)$-potent or $(n, 1)$-potent near-ring has the strong IFP.

Next, we will obtain every idempotent of (m, n)-potent near-ring is central, and find some characterization of regular near-rings with (m, n)-potent conditions.

For the remainder of basic concepts and results on near-rings, we will refer to [10] and [13].

2. Results on (m, n)-potent near-rings

Let R and S be two near-rings. Then a mapping f from R to S is called a *near-ring homomorphism* if (i) $f(a + b) = f(a) + f(b)$, (ii) $f(ab) = f(a)f(b)$. We can replace homomorphism by monomorphism, epimorphism, isomorphism, endomorphism and automorphism as in ring theory [1].

A (two sided) *ideal* of a near-ring R is a subset I of R such that (i) $(I, +)$ is a normal subgroup of $(R, +)$, (ii) $a(I + b) - ab \subset I$ for all $a, b \in R$, (iii) $(I + a)b - ab \subset I$ for all $a, b \in R$, equivalently, $IR \subset I$. If I satisfies (i) and (ii) then it is called a *left ideal* of R. If I satisfies (i) and (iii) then it is called a *right ideal* of R.

We say that a near-ring R has the insertion of factors property (briefly, IFP) provided that for all a, b, x in R with $ab = 0$ implies $axb = 0$, and R has the strong IFP if every homomorphic image of R has the IFP, equivalently, for any ideal I of R, for all a, b, x in R with $ab \in I$ implies $axb \in I$, which are introduced in [13].

Also, we say that R is reduced if R has no nonzero nilpotent elements, that is, for each a in R, $a^n = 0$, for some positive integer n implies $a = 0$. McCoy [9] proved that R is reduced iff for each a in R, $a^2 = 0$ implies $a = 0$.

A near-ring R is called *reversible* if for any $a, b \in R$, $ab = 0$ implies $ba = 0$, and R is said to be *strongly reversible* if for any $a, b \in R$ and for each ideal I of R, $ab \in I$ implies $ba \in I$. On the other hand, we say that R has the *reversible IFP* in case R has the IFP and is reversible.

A (two-sided) *R-subgroup* of R is a subset H of R such that (i) $(H, +)$ is a subgroup of $(R, +)$, (ii) $RH \subset H$ and (iii) $HR \subset H$. If H satisfies (i) and (ii) then it is called a *left R-subgroup* of R. If H satisfies (i) and (iii) then it is called a *right R-subgroup* of R. In case, $(H, +)$ is normal in above, we say that *normal R-subgroup*, *normal left R-subgroup* and *normal right R-subgroup* instead of R-subgroup, left R-subgroup and right R-subgroup, respectively. Note that normal right R-subgroups of R are the same concepts of right ideals of R.

Also, a subset H of R together with (i) $RH \subset H$ and (ii) $HR \subset H$ is called an *R-subset* of R. If this H satisfies (i) then it is called a *left R-subset* of R, and H satisfies (ii) then it is called a *right R-subset* of R.

A near-ring R is called *left regular (resp. right regular)* if for each a in R, there exists an element x in R such that

$$a = xa^2 (resp.\ a = a^2 x).$$

A near-ring R is called *strongly left regular* if R is left regular and regular, similarly, we can define strongly right regular. A strongly left regular and strongly right regular near-ring is called *strongly regular near-ring*.

A near-ring R is called *left κ-regular (resp. right κ-regular)* if for each a in R, there exists an element x in R such that

$$a^n = xa^{n+1} (resp.\ a^n = a^{n+1}x)$$

for some positive integer n. A left κ-regular and κ-right regular near-ring is called *κ-regular near-ring*.

An integer group $(\mathbb{Z}_2, +)$ modulo 2 with the multiplication rule: $0 \cdot 0 = 0 \cdot 1 = 0$, $1 \cdot 0 = 1 \cdot 1 = 1$ is a near-field. Obviously, this near-field is isomorphic to

$M_c(\mathbb{Z}_2)$. All other near-fields are zero-symmetric. Consequently, we get the following important statements.

Lemma 2.1. [13] *Let R be a near-field. Then $R \cong M_c(\mathbb{Z}_2)$ or R is zero-symmetric.*

In our subsequent discussion of near-fields, we will exclude the silly near-field $M_c(\mathbb{Z}_2)$ of order 2. Evidently, every near-field is simple.

Lemma 2.2. [13] *Let R be a near-ring. Then the following statements are equivalent: (1) R is a near-field.*
(2) $R_d \neq 0$ and for each nonzero element a in R, $Ra = R$.
(3) R has a left identity and R is R-simple as an R-group.

Now, we shall give the notion of an (m, n)-potent near-ring and illustrate this concept with suitable examples.

Definition 1. We say that a near-ring R has the (m, n)-potent condition if for all a in R, there exist positive integers m, n such that $a^m R = R a^n$. We shall refer to such a near-ring as an (m, n)-potent near-ring.

Obviously, every (m, n)-potent near-ring is zero-symmetric. On the other hand, from the Lemmas 2.1 and 2.2, we obtain the following examples (1), (2), and including other three examples.

Example 1. (1) Every near-field is an (m, n)-potent near-ring for all positive integers m, n.
(2) The direct sum of near-fields is an (m, n)-potent near-ring for all positive integers m, n.
(3) Every subcommutative near-ring is an $(1, 1)$-potent near-ring.
(4) Every Boolean subcommutative near-ring is an (m, n)-potent near-ring for all positive integers m, n.
(5) Let $R = \{0, a, b, c\}$ be a Klein 4-group under addition. This is a near-ring with the following multiplication table (p. 408 [13]):

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	b	c	a
b	0	c	a	b
c	0	a	b	c

This near-ring is $(1, 1)$, $(1, 4)$, $(2, 2)$, $(2, 4)$, $(3, 3)$, $(4, 1)$, $(4, 2)$, $(4, 4)$-potent, but not Boolean.

A near-ring R is called *left S-unital (resp. right S-unital)* if for each a in R, $a \in Ra$ (resp. $a \in aR$).

Lemma 2.3. *Let R be a zero-symmetric and reduced near-ring. Then R has the reversible IFP.*

Proof. Suppose that a, b in R such that $ab = 0$. Then, since R is zero-symmetric, we have

$$(ba)^2 = baba = b0a = b0 = 0$$

Reducedness implies that $ba = 0$.

Next, assume that for all a, b, x in R with $ab = 0$. Then

$$(axb)^2 = axbaxb = ax0xb = ax0 = 0$$

This implies $axb = 0$, by reducedness. Hence R has the reversible IFP. □

Lemma 2.4. [13] *$R = R_0$ if and only if every left ideal of R is a left R-subgroup of R.*

Proposition 2.5. *Let R be an $(n, n+2)$-potent reduced near-ring, for some positive integer n. Then R is a left κ-regular near-ring.*

Proof. Suppose R is an $(n, n+2)$-potent reduced near-ring. Then for any a in R, we have that

$$a^n R = R a^{n+2}$$

This implies that $a^{n+1} \in a^n R = R a^{n+2}$. Hence there exists x in R such that $a^{n+1} = x a^{n+2}$, that is, $(a^n - x a^{n+1})a = 0$. From Lemma 2.3, we see that $a(a^n - x a^{n+1}) = 0$. Also, we can compute that $a^n(a^n - x a^{n+1}) = 0$ and $x a^{n+1}(a^n - x a^{n+1}) = 0$. Thus from the equation

$$(a^n - x a^{n+1})^2 = a^n(a^n - x a^{n+1}) - x a^{n+1}(a^n - x a^{n+1}) = 0 - 0 = 0$$

and reducedness, we see that $a^n = x a^{n+1}$. Consequently, R is a left κ-regular near-ring. □

Corollary 2.6. *Let R be an $(1, 3)$-potent reduced near-ring. Then R is a left regular near-ring.*

Proposition 2.7. *Let R be an $(1, 2)$-potent near-ring. (1) If R is reduced, then R is a left S-unital near-ring.*

(2) If R is right S-unital, then R is a left regular and reduced near-ring.

Proof. Since R is an $(1, 2)$-potent near-ring, consider the equality, $aR = Ra^2$ for each a in R.

(1) From $a^2 \in aR = Ra^2$, there exists x in R such that $a^2 = xa^2$. This implies that $(a - xa)a = 0$. Since R is zero-symmetric and reduced, Lemma 2.3 guarantees that $a(a - xa) = 0$ and $xa(a - xa) = 0$. Hence we have the equation

$$(a - xa)^2 = a(a - xa) - xa(a - xa) = 0 - 0 = 0$$

Reducedness implies that $(a - xa) = 0$, that is, $a = xa$, for some $x \in R$. Therefore R is left S-unital.

(2) Since R is right S-unital and has $(1,2)$-potent condition, for each $a \in R$, $a \in aR = Ra^2$. Thus $a = xa^2$, for some $x \in R$. Also, in this equation, $a^2 = 0$ implies that $a = 0$. Hence R is a left regular and reduced near-ring. $\square$

Proposition 2.8. *Let R be an $(2,1)$-potent near-ring. (1) If $R = R_d$ is reduced, then R is a right S-unital near-ring.*
(2) If R is left S-unital, then R is a right regular and reduced near-ring.

Proof. This proof is an analogue of the proof in Proposition 2.6. $\square$

Theorem 2.9. *Every homomorphic image of an (m,n)-potent near-ring is also an (m,n)-potent near-ring.*

Proof. Let R be an (m,n)-potent near-ring and let $f : R \longrightarrow R'$ be a near-ring epimorphism. Consider an equality $a^m R = Ra^n$, for all $a \in R$, where m, n are positive integers.

We must show that for all $a' \in R'$, $a'^m R' = R'a'^n$, for some positive integers m, n. Let a', $x' \in R'$. Then there exist a, $x \in R$ such that $a' = f(a)$ and $x' = f(x)$. So we get the following equations:

$$a'^m x' = f(a)^m f(x) = f(a^m) f(x) = f(a^m x) = f(ya^n) = f(y)f(a)^n = f(y)a'^n,$$

where $a^m x \in a^m R = Ra^n$, so that there exist $y \in R$ such that $a^m x = ya^n$. This implies that $a'^m R' \subset R'a'^n$.

In a similar fashion, we obtain that $R'a'^n \subset a'^m R'$. Therefore our desired result is completed. $\square$

Now, we shall discuss the behavior of R-subgroups and ideals of $(1,n)$-potent near-ring. To start with, we have the following:

Theorem 2.10. *Every left R-subgroup of an $(1,n)$-potent near-ring R is an R-subgroup.*

Proof. Let A be a left R-subgroup of R. Then we see that $RA \subset A$. To show that $AR \subset A$, let $ar \in AR$, where $a \in A$, $r \in R$. Since R has $(1,n)$-potent condition, we have $ar \in aR = Ra^n$. This implies that

$$ar = sa^n = (sa^{n-1})a \in Ra \subset RA \subset A,$$

for some s in R. Hence A is an R-subgroup of R. $\square$

From the Lemma 2.4 and Theorem 2.10, we obtain the following statements.

Corollary 2.11. *(1) Every left ideal of an $(1,n)$-potent near-ring R is an ideal of R.*
(2) Every left ideal of an $(1,n)$-potent near-ring R is an R-subgroup of R.

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