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GROUP ACTIONS IN A UNIT-REGULAR RING WITH COMMUTING IDEMPOTENTS

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Abstract. Let R be a ring with unity, X the set of all nonzero, nonunits of R and G the group of all units of R. We will consider some group actions on X by G, the left (resp. right) regular action and the conjugate action. In this paper, by investigating these group actions we can have some results as follows: First, if $E(R)$, the set of all nonzero nonunit idempotents of a unit-regular ring R, is commuting, then $o_\ell(x) = o_r(x)$, $o_c(x) = \{x\}$ for all $x \in X$ where $o_\ell(x)$ (resp. $o_r(x)$, $o_c(x)$) is the orbit of x under the *left regular* (resp. *right regular*, *conjugate*) action on X by G and R is abelian regular. Secondly, if R is a unit-regular ring with unity 1 such that G is a cyclic group and $2 = 1 + 1 \in G$, then G is a finite group. Finally, if R is an abelian regular ring such that G is an abelian group, then R is a commutative ring.

1. Introduction and basic definitions

Let R be a ring with unity, X the set of all nonzero, nonunits of R and G the group of all units of R. In this paper, we will consider some group actions of G on X. We call the action, $((g, x) \longrightarrow gx)$ (resp. $((g, x) \longrightarrow xg^{-1}), ((g, x) \longrightarrow gxg^{-1})$) from $G \times X$ to X, left regular (resp. right regular, conjugate) action. If $\phi : G \times X \longrightarrow X$ is one of the above group actions, then for each $x \in X$ we define the *orbit* of x by $o(x) = \{\phi(g, x) : g \in G\}$ and *stabilizer* of x by $stab(x) = \{g \in G : \phi(g, x) = x\}$. Recall that G is *transitive* on X (or G acts transitively on X) if there is an $x \in X$ with $o(x) = X$ and the group action on X by G is *trivial* if $o(x) = \{x\}$ for all $x \in X$.

A ring R is *von Neumann regular* (or simply *regular*) (resp. *unit-regular*) provided that for any $a \in R$ there exists an element $r \in R$ (resp. $u \in G$) such that $a = ara$ (resp. $a = aua$). A ring R is *strongly regular* provided that for any $a \in R$ there exists an element $r \in R$ such that $a = ra^2$. Also a ring R is *abelian* provided all idempotents in R are central. It is known [1] that R is

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an abelian regular ring if and only if R is strongly regular and that an abelian regular ring is unit-regular.

Throughout this paper, unless stated otherwise, R is a ring with unity 1, G is the group of all units of R and X is the set of all nonzero, nonunits in R. Also for each $x \in X$, $o_\ell(x)$ (resp. $o_r(x)$, $o_c(x)$) is considered as the orbit of x under the left regular (resp. right regular, conjugate) action of G on X. Let $E(R)$ be the set of all nonzero, nonunit idempotents of R. Recall that $E(R)$ is said to be *commuting* if $ef = fe$ for all $e, f \in E(R)$. We use $|\ |$ to denote the cardinality of a set.

It was shown in [3, Lemma 2.3, Theorem 3.3] that R is unit-regular if and only if every orbit under the left regular action is $o_\ell(e)$ for some idempotent $e \in X$ and that if R is a unit-regular ring such that G is a cyclic group and 2 = 1 + 1 $\in G$, then the orbit $o_\ell(e)$ is finite. In Section 2, we show that (1) if R is a unit-regular ring such that $E(R)$ is commuting, then (1) $o_\ell(x) = o_r(x)$ for all $x \in X$ and R is abelian regular ring; (2) if for a ring R such that G is a cyclic group and $2 = 1 + 1 \in G$ there exists an idempotent $e \in X$ such that $2e = (1+1)e \neq 0$, then $o_\ell(1-e)$ (resp. $o_r(1-e)$) is finite; (3) if $X \neq \emptyset$ for a unit-regular ring R such that G is a cyclic group and $2 = 1 + 1 \in G$, then G is finite. We also show that if R is an abelian regular ring such that $E(R)$ is finite, then R is isomorphic to the direct sum of a finite number of division rings.

It was shown in [3, Theorem 3.2] that if R is a unit-regular ring with unity 1 such that G is abelian and $2 = 1 + 1 \in G$, then R is a commutative ring. In Section 3, we show that if R is a ring with unity such that $E(R)$ is commuting, then $o_c(e) = \{e\}$ for all $e \in E(R)$, i.e., $ge = eg$ for all $g \in G$. By using this result we also show that if R is an abelian regular ring such that G is abelian, then R is a commutative ring.

2. Regular action in unit-regular rings

Recall that a nonzero element a in a ring R is said to be a *right zero−divisor* if there exists a nonzero $b \in R$ such that $ba = 0$.

The following theorem has been proved in [2]:

Theorem 2.1. *Let R be a ring such that X is a finite union of orbits under the left regular action on X by G. Then X is the set of all right zero-divisors of R. Moreover, if X is a nonempty finite set, then R is a finite ring.*

Proof. Refer [2, Theorem 2.2]. □

Lemma 2.2. *Let R be a unit-regular ring. Then for all $x \in X$, x is a zero-divisor.*

Proof. Since R is a unit-regular ring, for $x \in X$ there exists an element $g \in G$ with $x = xgx$, and so $x(gx-1) = 0 = (xg-1)x$. If $gx - 1 \in G$, then $x = 0$, which is a contradiction. If $gx - 1 = 0$, then $gx = 1$, and so $x = g^{-1} \in G$, which is also a contradiction. Thus $gx-1 \in X$. Similarly, we have $xg-1 \in X$. Hence x is a zero-divisor. □

Lemma 2.3. *The ring R is unit-regular if and only if every orbit under the left regular action is $o_\ell(e)$ for some idempotent $e \in X$.*

Proof. Refer [3, Lemma 2.3]. □

Corollary 2.4. *The ring R is unit-regular if and only if every orbit under the right regular action is $o_r(e)$ for some idempotent $e \in X$.*

Proof. It follows by an argument similar to that in the proof of [3, Lemma 2.3]. □

Remark 1. Note that if R is a noncommutative ring, then $o_\ell(x) \neq o_r(x)$ for some $x \in X$. For example, let $R = \begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ \mathbb{Z}_2 & \mathbb{Z}_2 \end{pmatrix}$ be the ring of 2×2 matrices over $\mathbb{Z}_2$, a galois field of order 2, and take $x = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in X$. Then $o_\ell(x) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \right\} \neq \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \right\} = o_r(x)$.

Lemma 2.5. *If $E(R)$ is commuting, then $o_\ell(e) \cap E(R) = \{e\}$ (resp. $o_r(e) \cap E(R) = \{e\}$) for all $e \in E(R)$.*

Proof. Let $e_1 \in o_\ell(e) \cap E(R)$. Then $e_1 = ge$ for some $g \in G$. Thus $e_1e = (ge)e = ge = e_1$. Since $e = g^{-1}e_1$, $e = ee_1$. Since $E(R)$ is commuting, $e = ee_1 = e_1e = e_1$. Hence $o_\ell(e) \cap E(R) = \{e\}$. Similarly, we have $o_r(e) \cap E(R) = \{e\}$. □

Corollary 2.6. *Let R be a unit-regular ring. If $E(R)$ is commuting, then for all $x \in X$, $o_\ell(x) \cap E(R) = \{e\}$ (resp. $o_r(x) \cap E(R) = \{f\}$) for some $e \in E(R)$ (resp. $f \in E(R)$).*

Proof. It follows from Lemma 2.3 and Lemma 2.5. □

Note that if R is a unit-regular ring such that $E(R)$ is commuting, then the number of orbits under the left (resp. right) regular action on X by G is equal to the cardinality of $E(R)$ by Lemma 2.3 and Corollary 2.6.

Theorem 2.7. *If $E(R)$ is commuting, then $o_\ell(e) = o_r(e)$ for all $e \in E(R)$.*

Proof. Let $e \in E(R)$ be arbitrary. Then $o_\ell(e) \subseteq o_r(e_1)$ for some $e_1 \in E(R)$. Indeed, if $y \in o_\ell(e)$ is arbitrary, then $y = ge$ for some $g \in G$. Thus $e = g^{-1}y = (g^{-1}y)(g^{-1}y) = e^2$, and then $y = yg^{-1}y$. Let $e_1 = yg^{-1}$. Thus $e_1 \in E(R)$ and $y = e_1g \in o_r(e_1)$. Hence $o_\ell(e) \subseteq o_r(e_1)$. Similarly, we can have that $o_r(e_1) \subseteq o_\ell(e_2)$ for some $e_2 \in E(R)$. Thus $e \in o_\ell(e) \subseteq o_r(e_1) \subseteq o_\ell(e_2)$. Since $E(R)$ is commuting, $o_\ell(e_2) \cap E(R) = \{e_2\}$ and so $e = e_2$. Therefore, $o_\ell(e) \subseteq o_r(e_1) \subseteq o_\ell(e)$, which implies that $o_\ell(e) = o_r(e_1)$, and thus $e_1 = e$ by Lemma 2.5. Consequently, $o_\ell(e) = o_r(e)$ for all $e \in E(R)$. □

Corollary 2.8. *Let R be a unit-regular ring. If $E(R)$ is commuting, then $o_\ell(x) = o_r(x)$ for all $x \in X$.*

Proof. Let $x \in X$ be arbitrary. Then $o_\ell(x) = o_\ell(e) = o_r(e)$ for some $e \in E(R)$ by from Lemma 2.3 and Theorem 2.7. Since $x \in o_r(e)$, $o_r(x) = o_r(e)$. Hence we have $o_\ell(x) = o_r(x)$ for all $x \in X$. □

Lemma 2.9. *Let R be a unit-regular ring. If $o_\ell(x) = o_r(x)$ for all $x \in X$, then R is abelian regular.*

Proof. By [1, Theorem 3.2], it is enough to show that R has no nonzero nilpotent elements. Assume that there exists a nonzero nilpotent element $x \in R$ such that $x^n = 0 \neq x^{n-1}$ for some positive integer n. By Lemma 2.3, $x = ge$ for some idempotent $e \in X$ and some $g \in G$. Since $o_\ell(x) = o_r(x)$, $0 = x^n = he^n$ for some $h \in G$. Thus $e^n = e = 0$, which is a contradiction. □

Corollary 2.10. *Let R be a unit-regular ring. Then $E(R)$ is commuting if and only if R is abelian regular.*

Proof. If $E(R)$ is commuting, then R is abelian regular by Corollary 2.8 and Lemma 2.9. The converse is clear. □

Remark 2. If R is a unit-regular ring in which $X = E(R)$, then R is abelian regular.

Theorem 2.11. *Let R be a unit-regular ring. Then the following are equivalent:*
(1) $X = E(R)$;
(2) the left (resp. right) regular group action on X by G is trivial;
(3) R is a Boolean ring in which $G = \{1\}$.

Proof. (2) $\Rightarrow$ (1). It follows from Lemma 2.3 and Corollary 2.4.
(1) $\Rightarrow$ (3). Suppose that $X = E(R)$. Then $o_\ell(e) = o_r(e) = \{e\}$ for all $e \in E(R)$ by (1) $\Leftrightarrow$ (2). Assume that $G \neq \{1\}$. Then there exist $g, h \in G$ such that $g \neq h$. Since $ge = e = he$ for any $e \in X = E(R)$, $(g-h)e = 0$. If $g - h \in G$, then $e = 0$, a contradiction. Thus $g - h \in X = E(R)$. Since $o_\ell(g-h) = o_r(g-h) = \{g-h\}$, we have $g - h = g(g-h) = (g-h)g$, and so $gh = hg$. Also we have $g - h = g(g-h) = (-h)(g-h)$, and so $g^2 = h^2$. Since $g - h \in X = E(R)$, $g - h = (g-h)^2 = g^2 - 2gh + h^2 = 2g^2 - 2gh = 2(g(g-h)) = 2(g-h)$, and then $g - h = 0$, which is a contradiction. Therefore $G = \{1\}$. Since $X = E(R)$ and $G = \{1\}$, R is a Boolean ring.
(3) $\Rightarrow$ (2). Clear.

□

Example 1. Let $R = \prod_{i=1}^{\infty} \mathbb{Z}_2$ where $\mathbb{Z}_2$ is a galois field of order 2. Then R is a unit-regular ring such that $X = E(R)$, and is equivalently a Boolean ring in which $G = \{1\}$ by Theorem 2.11.

Theorem 2.12. *Let R be an abelian regular ring. If $E(R)$ is finite, then $R \simeq D_1 \times D_2 \times \cdots \times D_n$ where all D_i are division rings for some positive integer n. In fact, $|E(R)| = 2^n$.*

Proof. Since $E(R)$ is finite, there exists a finite number of orbits under the left regular action on X by G by Lemma 2.3. Observe that every left ideal of R is G-invariant and is a union of orbits under the left regular action. Since there exists a finite number of orbits under the left regular action, every left ideal of R is a union of finite number of orbits under the left regular action. Hence R is a left artinian ring. Since $E(R)$ is central, by the Wedderburn-Artin Theorem we have $R \simeq D_1 \times D_2 \times \cdots \times D_n$ where all D_i are division rings for some positive integer n and $|E(R)| = 2^n$. □

Corollary 2.13. *Let R be an abelian regular ring. If $E(R)$ is finite, then. Then the following are equivalent:*
(1) G is finite;
(2) X is finite;
(3) R is finite.

Proof. (1) $\Rightarrow$ (2). Let $|E(R)| = n$. Then X is the union of n orbits $o(x_1), \ldots, o(x_n)$ for some $x_1, \ldots, x_n \in X$ by Corollary 2.6. Since G is finite, X is clearly finite.
(2) $\Rightarrow$ (3). It follows from Theorem 2.1.
(3) $\Rightarrow$ (1). It is clear. □

Theorem 2.14. *Let R be a ring such that G is a cyclic group. If $e \in X$ is an idempotent such that $2e \neq 2(= 1 + 1)$, then the orbit $o_\ell(e)$ (resp. $o_r(e)$) is finite.*

Proof. If $o_\ell(e) = \{e\}$ or $G = \{1\}$ for an idempotent $e \in X$, then $o_\ell(e) = \{e\}$, and so $o_\ell(e)$ is finite. Suppose that $o_\ell(e) \neq \{e\}$ and $G \neq \{1\}$. Then $|o_\ell(e)| > 1$ and $\mathrm{Stab}(e) = \{g \in G | ge = e\}$ is a proper subgroup of G. Let $H = \mathrm{Stab}(e)$ and let a be a generator of G. Since $e \in X$ is an idempotent and $2e \neq 2$, $2e - 1 (\neq 1) \in G$. Thus $(2e - 1)e = e$ implies that $2e - 1 \in H$ and so $H \neq \{1\}$. Since H is a proper subgroup of G, H is generated by a^s for some nonnegative integer s ($s \geq 2$). Since $a^s \in H$, $a^s e = e$. For all $g \in G$, $g = a^m$ for some $m \in \mathbb{Z}$. By the division algorithm for $\mathbb{Z}$, $m = r + qs$ form some $g, r \in \mathbb{Z}$, where $s - 1 \geq r \geq 0$. Thus for all $g \in G$, $ge = a^m e = a^{r+qs} e = a^r e$. Therefore $o_\ell(e) = \{a^r e : 0, 1, \ldots, s - 1\}$ is finite. Similarly, we can show that $o_r(e)$ is finite for an idempotent $e \in X$ such that $2e \neq 2$. □

Corollary 2.15. *Let R be a ring such that G is a cyclic group. If $e \in X$ is an idempotent such that $2e = (1 + 1)e \neq 0$, then $o_\ell(1 - e)$ (resp. $o_r(1 - e)$) is finite.*

Proof. Since $2e \neq 0$, $2(1 - e) \neq 2$. Hence it follows from Theorem 2.14. □

Corollary 2.16. *Let R be a ring such that G is a cyclic group. If there exists an idempotent $e \in X$ such that $2e = (1 + 1)e \neq 0, 2$, then G is a finite group.*

Proof. Since for all $g \in G$, $g = ge + g(1-e) \in o_\ell(e) + o_\ell(1-e)$. Since $2e \neq 0, 2$, both $o_\ell(e)$ and $o_\ell(1-e)$ are finite by Theorem 2.14 and Corollary 2.15. Hence G is a finite group. □

Corollary 2.17. *Let R be a unit-regular ring such that $X \neq \emptyset$. If G is a cyclic group and $2 = 1 + 1 \in G$, then G is a finite group.*

Proof. It follows from Lemma 2.3 and Corollary 2.16. □

Remark 3. Let R be a unit-regular ring such that $X \neq \emptyset$. If G is a cyclic group and $2 = 1 + 1 \in G$, then R is a commutative ring by [3, Theorem 3.2] and G is a finite group by the above Corollary 2.17. Hence we have that every orbit $o_\ell(x) = o_r(x)$ is finite for all $x \in X$. By Lemma 2.3, we have $o_\ell(x) = o_\ell(e)$ for some $e \in E(R)$. Since G is abelian, $stab(x) = stab(e)$. Since $2 \in G$, $2e - 1 \in stab(e)$ and so $stab(e) \neq \{e\}$. Since $o_\ell(x)$ is finite, $|o_\ell(x)| = |o_\ell(e)| = |G|/|stab(e)|$. In particular, if G is a cyclic group of prime order, then $o_\ell(x) = o_r(x) = \{x\}$, i.e., the left (right) regular action on X by G is trivial (which is equivalent to $X = E(R)$ by Theorem 2.11).

3. Conjugate action in unit-regular rings

Theorem 3.1. *Let R be a ring such that G is a cyclic group. If $e \in X$ is an idempotent such that $2e \neq 2(= 1 + 1)$, then the orbit $o_c(e)$ (resp. $o_c(1-e)$) is finite.*

Proof. The proof is similar to that of Theorem 2.14. If $o_c(e) = \{e\}$ or $G = \{1\}$ for an idempotent $e \in X$, then $o_c(e) = \{e\}$, and so $o_c(e)$ is finite. Suppose that $o_c(e) \neq \{e\}$ and $G \neq \{1\}$. Then $|o_c(e)| > 1$ and $stab(e) = \{g \in G | geg^{-1} = e\}$ is a proper subgroup of G. Let $H = stab(e)$ and let a be a generator of G. Since $e \in X$ is an idempotent and $2e \neq 2$, $2e-1(\neq 1) \in G$. Thus $(2e-1)e(2e-1)^{-1} = (2e-1)e(2e-1) = e$ implies that $2e - 1 \in H$ and so $H \neq \{1\}$. Since H is a proper subgroup of G, H is generated by a^s for some nonnegative integer s ($s \geq 2$). Since $a^s \in H$, $a^t e = e$. For all $g \in G$, $g = a^m$ for some $m \in \mathbb{Z}$. By the division algorithm for $\mathbb{Z}$, $m = r + qs$ form some $g, r \in \mathbb{Z}$, where $s - 1 \geq r \geq 0$. Thus for all $g \in G$, $geg^{-1} = a^m e a^{-m} = a^{r+qs} e a^{-(r+qs)} = a^r e a^{-r}$. Therefore $o_c(e) = \{a^r e a^{-r} : 0, 1, \ldots, s-1\}$ is finite. □

Corollary 3.2. *Let R be a ring such that G is a cyclic group. If $e \in X$ is an idempotent such that $2e \neq 0$, then the orbit $o_c(e)$ (resp. $o_c(1-e)$) is finite.*

Proof. Since $2e \neq 0$, $2(1-e) \neq 2$. Hence it follows from Theorem 3.1. □

Lemma 3.3. *If $E(R)$ is commuting, then $o_c(e) \subseteq o_\ell(e)(= o_r(e))$ for all $e \in E(R)$.*

Proof. Since $E(R)$ is commuting, $o_\ell(e) = o_r(e)$ for all $e \in E(R)$ by Theorem 2.7. Let $geg^{-1} \in o_c(e)$ ($\forall g \in G$) be arbitrary. Since $o_\ell(e) = o_r(e)$, $eg^{-1} = he$

for some $h \in G$, and so $geg^{-1} = (gh)e \in o_\ell(e)$. Thus $o_c(e) \subseteq o_\ell(e)(= o_r(e))$ for all $e \in E(R)$. □

Lemma 3.4. *If $E(R)$ is commuting, then $o_c(e) = \{e\}$ for all $e \in E(R)$, i.e., $ge = eg$ for all $g \in G$.*

Proof. Since $E(R)$ is commuting, $o_\ell(e) \cap E(R) = \{e\}$ by Lemma 2.5 and also $o_c(e) \subseteq o_\ell(e)$ by Lemma 3.3 for all $e \in E(R)$. Since $o_c(e) \subseteq E(R)$, $o_c(e) \subseteq o_\ell(e) \cap E(R) = \{e\}$, and so $o_c(e) = \{e\}$. □

Theorem 3.5. *Let R be a unit-regular ring in which $E(R)$ is commuting. If G is an abelian group, then R is a commutative ring.*

Proof. Since $E(R)$ is commuting, $ge = eg$ for all $e \in E(R)$ and all $g \in G$ by Lemma 3.4. Let $x \in X$ and $g \in G$ be arbitrary. Then $x = he_1$ for some $e_1 \in E(R)$ and some $h \in G$ by Lemma 2.3. Since G is abelian, we have $gx = g(he_1) = (gh)e_1 = e_1(gh) = e_1(hg) = (e_1h)g = (he_1)g = xg$. Let $y \in X$ be arbitrary. Then $y = ke_2$ for some $e_2 \in E(R)$ and some $k \in G$ by Lemma 2.3. Since $E(R)$ is commuting, $xy = (he_1)(ke_2) = (hk)(e_1e_2) = (kh)(e_2e_1) = (ke_2)(he_1) = yx$. Consequently, R is commutative. □

Corollary 3.6. *Let R be an abelian regular ring. If G is an abelian group, then R is a commutative ring.*

Proof. It follows from Corollary 2.10 and Theorem 3.5. □

Theorem 3.7. *Let R be an abelian regular ring such that G is a torsion group. Then the following are equivalent:*
(1) The conjugate action on X by G is trivial;
(2) G is abelian;
(3) R is commutative.

Proof. (1)⇒(2). Let $g, h \in G$ be arbitrary. Since the order of g is finite, $1-g \in X$. Since the conjugate action on X by G is trivial, the orbit $o(1-g) = \{1-g\}$, i.e., $h(1-g)h^{-1} = 1-g$ and so $gh = hg$. Hence G is abelian.
(2)⇒(3). It follows from Corollary 3.6.
(3)⇒(1). It is clear. □

Note that (2)⇒(1) in Theorem 3.7 may not be true in a ring which is not an abelian regular ring by the following example:

Example 2. Let $R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in \mathbb{Z}_2 \right\}$. Then R is a noncommutative ring but $G = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\}$ is an abelian group. The orbit of $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in X$ under the conjugate action on X by G is equal to $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \right\} \neq \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\}$, and so the conjugate action on X by G is not trivial.

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