

## OPTIMAL PARAMETERS FOR A DAMPED SINE-GORDON EQUATION

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ABSTRACT. In this paper a parameter identification problem for a damped sine-Gordon equation is studied from the theoretical and numerical perspectives. A spectral method is developed for the solution of the state and the adjoint equations. The Powell's minimization method is used for the numerical parameter identification. The necessary conditions for the optimization problem are shown to yield the bang-bang control law. Numerical results are discussed and the applicability of the necessary conditions is examined.

### 1. Introduction

In this paper, we study the identification problem for parameters  $\alpha, \beta$  and  $\delta$  appearing in the one-dimensional damped sine-Gordon equation

$$\begin{aligned} & y_{tt} + \alpha y_t - \beta y_{xx} + \delta \sin y = f(t, x), \quad (t, x) \in Q = (0, T) \times (0, L), \\ (1) \quad & y(t, 0) = y(t, L) = 0, \quad t \in (0, T), \\ & y(0, x) = y_0(x) \quad \text{and} \quad y_t(0, x) = y_1(x), \quad x \in (0, L). \end{aligned}$$

Equation (1) describes the dynamics of a Josephson junction driven by a current source taking into account the damping effect, see [1]. The sine-Gordon model has attracted an additional interest since it is known to exhibit chaotic behavior for certain values of the governing parameters  $\alpha, \beta$  and  $\delta$ . For details we refer to [8] and [11].

Let  $f \in L^2(Q)$ ,  $y_0 \in H_0^1(0, L)$  and  $y_1 \in L^2(0, L)$ . The identification problem for (1) consists in finding the parameters  $\alpha, \beta$  and  $\delta$  such that the solution of (1) exhibits the desired behavior. More precisely, let  $P = \{q = (\alpha, \beta, \delta) \in \mathbb{R}^3 \mid \beta > 0\}$  be equipped with the Euclidean norm. Let  $P_{ad} \subset P$  be an admissible set of parameters. Define the cost functional  $J(q)$  by

$$(2) \quad J(q) = \int_Q [y(q; t, x) - z_d(t, x)]^2 dx dt, \quad q \in P_{ad},$$

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where  $z_d$  is a given function on  $Q$ . The data  $z_d$  can be thought of as the targeted behavior of (1). The parameter identification problem for (1) with the objective function (2) is to find  $q^* = (\alpha^*, \beta^*, \delta^*) \in P_{ad}$  satisfying

$$(3) \quad J(q^*) = \inf_{q \in P_{ad}} J(q), \quad P_{ad} \subset P.$$

The solution map  $q \rightarrow y(q)$  from  $P$  into  $C([0, T]; L^2(0, L))$  is continuous under proper initial conditions, see [5] or Section 3. Hence it is clear that the minimization problem (3) has at least one solution  $q^*$ , provided  $P_{ad}$  is bounded and closed.

On the other hand, the minimizer  $q^*$  can be characterized by the inequality

$$(4) \quad DJ(q^*)(q - q^*) \geq 0, \quad q \in P_{ad},$$

where  $DJ(q^*)(q - q^*)$  denotes the Gâteaux derivative of  $J(q)$  at  $q = q^*$  in the direction of  $q - q^*$ . The existence of the Gâteaux derivative was proved by us in [5]. We refer to [6, 7] for perturbed-sine Gordon equations or more general nonlinear evolution systems. In Section 3 we show that (4) can be written as

$$(5) \quad (\alpha^* - \alpha)a + (\beta^* - \beta)b + (\delta^* - \delta)c \geq 0, \quad q = (\alpha, \beta, \delta) \in P_{ad},$$

where the constants  $a, b, c$  are expressed through the solutions of the state and adjoint systems. This inequality is a necessary condition for the optimal set of parameters  $q^*$ . Furthermore, if

$$P_{ad} = [\alpha_1, \alpha_2] \times [\beta_1, \beta_2] \times [\delta_1, \delta_2], \quad \beta_1 > 0$$

and all of  $a, b, c$  are not zeros, then the necessary condition (5) becomes

$$\begin{aligned} \alpha^* &= \frac{1}{2}\{\text{sign}(a) + 1\}\alpha_2 - \frac{1}{2}\{\text{sign}(a) - 1\}\alpha_1, \\ \beta^* &= \frac{1}{2}\{\text{sign}(b) + 1\}\beta_1 - \frac{1}{2}\{\text{sign}(b) - 1\}\beta_1, \\ \delta^* &= \frac{1}{2}\{\text{sign}(c) + 1\}\delta_2 - \frac{1}{2}\{\text{sign}(c) - 1\}\delta_1, \end{aligned}$$

which is a bang-bang control law.

The goals of this paper are to develop a spectral method for the solution of the state and the adjoint equations (1) and (29), apply the Powell's minimization method for the identification of the optimal parameters  $q^*$  in (3), and to examine the numerical applicability of the bang-bang control law (5).

The paper is organized as follows. In Section 2 we develop a spectral method for the solution of the state and adjoint equations (1) and (29). In Section 3 the adjoint system is introduced and the necessary condition (5) is established. In Section 4 we examine the parameter identification problem for (3) subject to (1) and introduce a modification of the Powell's minimization method. Finally, in Section 5 we discuss the numerical results.



We need the following well-known error estimate [9]: for any  $s, r \in (-\infty, \infty)$  with  $0 \leq s \leq r$ ,

$$(12) \quad \|P_N u - u\|_s \leq C_2(1 + N^2)^{(s-r)/2} \|u\|_r \quad \text{for } u \in H_0^r(I),$$

where  $P_N : H \rightarrow S_N$  is the projection operator, and  $C_2$  is a constant dependent on  $L$ . Using  $P_N$  the initial value problem (11) can be written in an equivalent form

$$(13) \quad \begin{aligned} y_N'' + \alpha y_N' + \beta A y_N + \delta P_N \sin y_N &= P_N f(t), \quad t \in (0, T), \\ y_N(0) = P_N y_0, \quad y_N'(0) &= P_N y_1. \end{aligned}$$

Now we establish error bounds for finite spectral approximations  $y_N(t)$ .

**Theorem 2.2.** *Let  $r > 0$ . If the solution  $y$  of (7) satisfy  $y \in H_0^r(I)$ , then there is a  $C_3$  such that*

$$|y(t) - y_N(t)| \leq C_3(1 + N^2)^{-r/2}, \quad t \in [0, T].$$

*Proof.* Setting  $Y_N = P_N y$  and applying the projection operator  $P_N$  to (7) we get

$$(14) \quad \begin{aligned} Y_N'' + \alpha Y_N' + \beta A Y_N + \delta P_N \sin y &= P_N f(t), \quad t \in (0, T), \\ Y_N(0) = P_N y_0, \quad Y_N'(0) &= P_N y_1. \end{aligned}$$

Indeed, the first equation of (7) is changed into

$$(15) \quad \frac{d}{dt} \left[ y'(t) + \beta A \int_0^t y(s) ds \right] = f(t) - \delta \sin y(t) - \alpha y'(t).$$

Since the function in  $[\dots]$  and the right side term of (15) belong to  $H$ , it follows that  $\frac{d}{dt}[\dots] \in H$ . Since  $(\frac{\partial u}{\partial t}, e_n) = \frac{d}{dt}(u, e_n)$  for  $u \in H$ , we have

$$(16) \quad \frac{d}{dt} P_N \left[ y'(t) + \beta A \int_0^t y(s) ds \right] = P_N [f(t) - \delta \sin y(t) - \alpha y'(t)].$$

Furthermore by the linearity of  $P_N$  we have

$$(17) \quad \frac{d}{dt} \left[ (P_N y)'(t) + \beta A P_N \int_0^t y(s) ds \right] = P_N f(t) - \delta P_N \sin y(t) - \alpha (P_N y)'(t).$$

Here we used the fact that  $A$  and  $P_N$  are commutable on  $H$ . Differentiating the left side of (17) we get

$$(18) \quad (P_N y)''(t) + \beta A (P_N y)(t) = P_N f(t) - \delta P_N \sin y(t) - \alpha (P_N y)'(t).$$

Subtracting (14) from (13) and setting  $U_N = y_N - Y_N$  we obtain

$$(19) \quad \begin{aligned} U_N'' + \alpha U_N' + \beta A U_N &= \delta P_N (\sin y - \sin y_N), \quad t \in (0, T), \\ U_N(0) = 0, \quad U_N'(0) &= 0. \end{aligned}$$

Multiplying (19) by  $2U'_N$  and using differential properties of the inner product, we get

$$(20) \quad \frac{d}{dt}|U'_N(t)|^2 + 2\alpha|U'_N(t)|^2 + \beta \frac{d}{dt}\|U_N(t)\|^2 = 2\delta(P_N(\sin y(t) - \sin y_N(t)), U'_N(t)).$$

Integrating (19) over  $[0, t]$  by parts, we have

$$(21) \quad \begin{aligned} & |U'_N(t)|^2 + 2\alpha \int_0^t |U'_N(s)|^2 ds + \beta \|U_N(t)\|^2 \\ & = 2\delta \int_0^t (P_N(\sin y(s) - \sin y_N(s)), U'_N(s)) ds. \end{aligned}$$

Now let us estimate the right side term of (21).

$$\begin{aligned} & 2 \int_0^t |(P_N(\sin y(s) - \sin y_N(s)), U'_N(s))| ds \\ & \leq 2 \int_0^t \|P_N\| |\sin y(s) - \sin y_N(s)| |U'_N(s)| ds \\ & \leq 2 \int_0^t (|y(s) - Y(s)| + |U_N(s)|) |U'_N(s)| ds \\ & \leq \int_0^t |y(s) - Y(s)|^2 ds + \int_0^t [k^2 \|U_N(s)\|^2 ds + |U'_N(s)|^2] ds \\ & \leq C_2(1 + N^2)^{-r} \|y\|_{L^2(0,T;H)}^2 + \int_0^t [k^2 \|U_N(s)\|^2 + |U'_N(s)|^2] ds. \end{aligned}$$

Letting  $\tilde{\beta} = \min\{1, \beta\}$ ,  $\tilde{k} = \max\{1, k^2\}$ ,  $k_\beta = \tilde{k}/\tilde{\beta}$  and  $C_\beta = C_2/\tilde{\beta}$  and substituting the above result into (21), we obtain

$$(22) \quad \begin{aligned} & |U'_N(t)|^2 + \|U_N(t)\|^2 \\ & \leq C_\beta(1 + N^2)^{-r} \|y\|_{L^2(0,T;H)}^2 + k_\beta \int_0^t [\|U_N(s)\|^2 + |U'_N(s)|^2] ds. \end{aligned}$$

Applying Bellman-Gronwall Lemma to (22) we get

$$(23) \quad |U'_N(t)|^2 + \|U_N(t)\|^2 \leq C_\beta(1 + N^2)^{-r} \|y\|_{L^2(0,T;H)}^2 e^{k_\beta T}.$$

Now it follows from (12) and (23) that

$$\begin{aligned} |y(t) - y_N(t)| & \leq |y(t) - Y_N(t)| + |U_N(t)| \\ & \leq (1 + N^2)^{-r/2} \left[ C_2 \|y(t)\|_r + C_\beta k^{-1} \|y\|_{L^2(0,T;H)} e^{k_\beta T/2} \right]. \end{aligned}$$

Finally the conclusion is deduced from Theorem 2.1. □

Expanding the functions in (11) into the Fourier sine series, and using  $v = u_k$ ,  $k = 1, 2, \dots, N$  there we get

$$(24) \quad \begin{aligned} & Y''_k + \alpha Y'_k + \beta_k Y_k + \delta S_k(t) = F_k(t), \quad t \in (0, T), \\ & Y_k(0) = Y_{k_0}, \quad Y'_k(0) = Y_{k_1}, \end{aligned}$$

where  $\beta_k = \beta k^2 \pi^2 / L^2$ ,  $S_k(t)$  is the  $k$ -th Fourier sine coefficient of  $P_N \sin y_N(t)$ , and  $Y_k(t), F_k(t), Y_{k_0}$ , and  $Y_{k_1}$  are the Fourier coefficients of the solution  $y_N(t)$  and the corresponding functions in (11). Thus the approximate solutions of (1) are given by

$$y_N(t, x) = \sum_{n=1}^N Y_n(t) e_n(x).$$

Similarly one can define the approximate solutions  $w_N(t) \in S_N$  of the adjoint system (29) by

$$w_N(t, x) = \sum_{n=1}^N W_n(t) e_n(x),$$

where

$$(25) \quad \begin{aligned} W_k'' - \alpha W_k' + \beta_k W_k + \delta C_k(t) W_k &= Y_k(t) - Z_k(t), \quad t \in (0, T), \\ W_k(T) = 0, \quad W_k'(T) &= 0, \end{aligned}$$

and  $C_k(t)$  is the  $k$ -th Fourier sine coefficient of  $P_N \cos y_N(t)$  and  $Z_k(t)$  is the  $k$ -th Fourier sine coefficient of  $P_N z_d(t)$ .

### 3. Optimal parameters

In this section we show the continuity and the Gâteaux differentiability of the objective function  $J(q)$ , as well as establish the necessary condition (5) for an optimal set of coefficients  $q^*$ .

According to Theorem 2.1 given  $q \in P_{ad}$  the weak solution  $y(q)$  of (7) is an element of the space  $W(0, T)$  introduced in (8). By Theorem 3.1 in [5] the solution map  $q \rightarrow y(q) : P_{ad} \rightarrow W(0, T)$  is weakly continuous. In particular, if  $q_n \rightarrow q$  as  $n \rightarrow \infty$  in  $P_{ad} \subset \mathbb{R}^3$ , then  $y(q_n) \rightarrow y(q)$  weakly in  $L^2(0, T; V)$ . Since  $V = H_0^1(I)$  is compactly imbedded in  $H = L^2(I)$  it follows that  $y(q_n) \rightarrow y(q)$  strongly in  $L^2(0, T; H)$ . Thus the objective function  $J(q) = \|y(q) - z_d\|_{L^2(0, T; H)}^2$  is continuous on  $P_{ad}$ .

In fact, we can show that  $J(q)$  is Gâteaux differentiable. By Theorem 3.3 in [5] the map  $q \rightarrow y(q)$  is weakly Gâteaux differentiable at  $q^*$  in the direction of  $q - q^*$ . The proof is based on the transposition method of J. Lions, see [5] for details. According to this theorem the weak Gâteaux derivative  $Dy(q^*)(q - q^*)$  of  $y(q)$  at  $q^*$  in the direction of  $q - q^*$  can be described as follows.

Let  $q = (\alpha, \beta, \delta)$  and  $q^* = (\alpha^*, \beta^*, \delta^*)$ . For  $u \in W(0, T)$  with  $u(T, x) = 0, u_t(T, x) = 0, x \in (0, L)$  and  $u(t, 0) = u(t, L) = 0, t \in (0, T)$  define the linear operator  $\mathcal{L}$  by

$$\mathcal{L}(u) = u'' - \alpha^* u' - \beta^* u_{xx} + \delta^* \cos y(q^*) u.$$

Let

$$X_0 = \{u \in W(0, T) : \mathcal{L}(u) = g, u(T) = u_t(T) = 0, g \in L^2(0, T; H)\}.$$

That is  $X_0$  is the collection of the weak solutions  $u$  of

$$\begin{aligned}
 & u_{tt} - \alpha^* u_t - \beta^* u_{xx} + \delta^* \cos y(q^*) u = g(t, x), \quad (t, x) \in Q, \\
 (26) \quad & u(t, 0) = u(t, L) = 0, \quad t \in (0, T), \\
 & u(T, x) = 0, \quad u_t(T, x) = 0, \quad x \in (0, L).
 \end{aligned}$$

Then the weak Gâteaux derivative  $Dy(q^*)(q - q^*)$  is the function  $z = z(q, q^*) \in L^2(0, T; H)$  that is the unique solution of the integral equation

$$\begin{aligned}
 (27) \quad & \int_Q z(t, x) \mathcal{L}(v)(t, x) dx dt = (\alpha^* - \alpha) \int_Q y_t(q^*; t, x) v(t, x) dx dt \\
 & + (\beta^* - \beta) \int_Q y_x(q^*; t, x) v_x(t, x) dx dt \\
 & + (\delta^* - \delta) \int_Q \sin y(q^*; t, x) v(t, x) dx dt
 \end{aligned}$$

for any  $v \in X_0$ .

From the definition of the functional  $J(q) = \|y(q) - z_d\|_{L^2(0, T; H)}^2$

$$\begin{aligned}
 DJ(q^*)(q - q^*) &= 2(y(q^*) - z_d, Dy(q^*)(q - q^*)) \\
 &= 2 \int_Q [y(q^*; t, x) - z_d(t, x)] z(t, x) dx dt.
 \end{aligned}$$

Thus the necessary condition (4) is

$$(28) \quad \int_Q [y(q^*; t, x) - z_d(t, x)] z(t, x) dx dt \geq 0, \quad q \in P_{ad}.$$

Now let  $w \in X_0$  be the adjoint state. That is  $\mathcal{L}(w) = y(q^*) - z_d$  or

$$\begin{aligned}
 (29) \quad & w_{tt} - \alpha^* w_t - \beta^* w_{xx} + \delta^* \cos y(q^*) w = y(q^*) - z_d \quad \text{in } Q, \\
 & w(t, 0) = w(t, L) = 0, \quad t \in (0, T), \\
 & w(T, x) = 0, \quad w_t(T, x) = 0, \quad x \in (0, L).
 \end{aligned}$$

Using  $w$  in place of  $v$  in (27) allows us to rewrite condition (28) as

$$(30) \quad (\alpha^* - \alpha)a + (\beta^* - \beta)b + (\delta^* - \delta)c \geq 0, \quad q = (\alpha, \beta, \delta) \in P_{ad},$$

where

$$\begin{aligned}
 a &= \int_Q y_t(t, x) w(t, x) dx dt, \\
 b &= \int_Q y_x(x, t) w_x(t, x) dx dt, \\
 c &= \int_Q \sin y(t, x) w(t, x) dx dt.
 \end{aligned}$$

Thus the inequality (30) is a necessary condition for the optimal set  $q^*$ . In case the admissible set  $P_{ad}$  is a box in  $\mathbb{R}^3$  one uses the fact that (30) must

be satisfied for any  $q = (\alpha, \beta, \delta) \in P_{ad}$  to derive the bang-bang control law mentioned in the Introduction.

#### 4. Identification problem

Since our goal is to give a numerical solution to the minimization problem (3) the objective function  $J(q)$  introduced in (2) is replaced with its time-discrete approximation (still named  $J$ ):

$$(31) \quad J(q) = \sum_{j=1}^M |y(q; t_j) - z_d(t_j)|^2, \quad z_d(t_j) \in H, \quad q \in P.$$

The approximate cost functional corresponding to (31) will be given by

$$(32) \quad J_N(q) = \sum_{j=1}^M |y_N(q; t_j) - z_d(t_j)|^2, \quad q \in P,$$

where  $y_N(q)$  is the weak solution of (11). The parameter identification problem for (32) is to find  $q_N^* \in P_{ad}$  such that

$$(33) \quad J_N(q_N^*) = \min_{q \in P_{ad}} J_N(q).$$

As in [5], one can easily prove that the cost functional (33) is continuous on  $P_{ad}$ . Therefore the minimization problem admits a minimum in  $P_{ad}$ .

**Lemma 4.1.** *There exists  $C_4 > 0$  independent on  $N$  such that*

$$|J_N(q) - J(q)| \leq C_4(1 + N^2)^{-r}.$$

*Proof.* Using the orthogonality between  $y(q) - y_N(q)$  and  $S_N$ , one has at  $t = t_j$ :

$$\begin{aligned} |y_N(q) - z_d|^2 - |y(q) - z_d|^2 &= (y_N(q) - y, y_N(q) + y(q) - 2z_d) \\ &= (y_N(q) - y, y(q) - y_N(q) - 2z_d). \end{aligned}$$

Hence by Theorem 2.2 we have

$$\begin{aligned} |J_N(q) - J(q)| &\leq \sum_{j=1}^M |y_N(q; t_j) - y(q; t_j)|^2 + 2 \sum_{j=1}^M |y_N(q; t_j) - y(q; t_j)| |z_d(t_j)| \\ &\leq \left[ C_3^2 M + 2C_3 \sum_{j=1}^M |z_d(t_j)| \right] (1 + N^2)^{-r/2}, \end{aligned}$$

where  $C_4 = C_3^2 M + 2C_3 \sum_{j=1}^M |z_d(t_j)|$ . □

**Theorem 4.2.** *Let  $\{q_N^*\}$  be a sequence satisfying (33) and  $q^*$  be its limit point. Then  $J(q^*) = \min_{q \in P_{ad}} J(q)$ .*



*Proof.* Let  $\{q_{N_k}^*\}$  be a subsequence of  $\{q_N^*\}$  such that  $q_{N_k}^* \rightarrow q^*$ . Since

$$|J_{N_k}(q_{N_k}^*) - J(q^*)| \leq |J_{N_k}(q_{N_k}^*) - J(q_{N_k}^*)| + |J(q_{N_k}^*) - J(q^*)|,$$

it follows from Lemma 4.1 and the continuity of  $y(q)$  that  $\lim_{k \rightarrow \infty} J_{N_k}(q_{N_k}^*) = J(q^*)$ . Let  $\bar{q}$  be such that  $J(\bar{q}) = \min_{q \in P_{ad}} J(q)$ . Since  $J(\bar{q}) = \lim_{N \rightarrow \infty} J_N(\bar{q})$  and  $J_N(q_N^*) \leq J_N(\bar{q})$ , we have  $J(q^*) \leq J(\bar{q})$ . Therefore we conclude that  $J(\bar{q}) = J(q^*)$ .  $\square$

Let  $z_d(t) = P_N z_d(t) = \sum_{n=1}^N Z_n(t) e_n$ . Then the cost functional  $J_N(q)$  is

$$(34) \quad J_N(q) = \frac{L}{2} \sum_{n=1}^N \sum_{j=1}^M [Y_n(q; t_j) - Z_n(t_j)]^2, \quad q \in P_{ad},$$

where  $Y_k(q; t)$  is the solution  $Y_k(t)$  of (24) for the given values of the parameters  $q = (\alpha, \beta, \delta) \in P_{ad}$ . Lemma 4.1 and Theorem 4.2 hold for the cost functional (34).

The minimization problem for  $J_N(q)$  is solved using a modification of Powell's minimization method, see [3]. the advantage of the Powell's method is that it does not require the knowledge of the derivatives (gradients) to minimize functions of several variables, but it still achieves a quadratic convergence near the points of minima.

The method iteratively minimizes a function of  $K$  variables (here  $K = 3$ ). Steps 1-7 describe one iteration of the method.

- (1) Initialize the set of directions  $u_i \in \mathbb{R}^3$  to the standard basis vectors  $e_i$  in  $\mathbb{R}^3$

$$u_i = e_i, \quad i = 1, 2, 3.$$

- (2) Save your starting position as  $q_0 \in \mathbb{R}^3$ .
- (3) For  $i = 1, 2, 3$  move from  $q_0$  along the direction  $u_i$  and find the point of minimum  $p_i$ .
- (4) Re-index the directions  $u_i$ , so that (for the new indices)  $J_N(p_1) \leq J_N(p_2) \leq J_N(p_3) \leq J_N(q_0)$ .
- (5) Move from  $q_0$  along the new direction  $u_1$  and find the point of minimum  $r_1$ . Move from  $r_1$  along the direction  $u_2$  and find the point of minimum  $r_2$ . Move from  $r_2$  along the direction  $u_3$  and find the point of minimum  $r_3$ .
- (6) Set  $v = r_3 - q_0$ .
- (7) Move from  $q_0$  along the direction  $v$  and find the minimum. Call it  $q_0$ . It replaces  $q_0$  from step 2.
- (8) Repeat the above steps until a stopping criterion is satisfied.

If the minimization is to be restricted to a subset  $P_{ad} \subset \mathbb{R}^K$ , then the moves in the one-dimensional minimization steps above are restricted so that the trial points would not leave  $P_{ad}$ .

### 5. Numerical results

In all the numerical experiments we used the values specified in Table 1, and observation times  $t_j = (T/M)j, j = 1, 2, \dots, M$ .

TABLE 1. Parameter values for numerical simulations

Time and spatial intervals	$[0, T] \times [0, L] = [0, 4] \times [0, \pi]$
Admissible set	$P_{ad} = [0.001, 1] \times [0.1, 1] \times [0.1, 1]$
Initial conditions	$y_0(x) = 0$ $y_1(x) = \exp[-100(x - \pi/2)^2]$
Forcing function	$f(t, x) = 0.01$
$N$	16

To simulate the data  $z_d(t, x)$  let  $\hat{q} \in P_{ad}$  be the test set of parameters. The initial value problems (24) and (25) were solved using a Leap-Frog Method with the time step  $h = 0.01$  as follows. Define  $Y_k^j, k = 1, 2, \dots, N$  by

$$Y_k^{-1} = Y_{k_0} - hY_{k_1},$$

$$Y_k^{j+1} = \frac{2Y_k^j - [\beta_k Y_k^j - F_k(t_j) + \delta S_k(t_j)h^2] + (1 - \alpha h/2)Y_k^{j-1}}{1 + \alpha h/2}$$

for  $j = 0, 1, 2, \dots$ . Then  $Y_k^j$  is an approximation of  $Y_k(t)$  at  $t = t_j = hj$ .

Since real data always contain some noise, we set

$$(35) \quad z_d(t, x) = y(\hat{q}; t, x) + \epsilon \eta(x),$$

where  $\eta(x)$  is a random variable uniformly distributed on interval  $[-1, 1]$ , and  $\epsilon$  is a small constant.

Let  $q_0 \in P_{ad}$  be an arbitrarily chosen set of parameters. The Powell's minimization method iteratively generates the sequence  $q_1, q_2, \dots$  as described in Section 3. The stopping criterion for this iterative process was chosen to be

$$(36) \quad \frac{|J_N(q_m) - J_N(q_{m-1})|}{|J_N(q_0)|} < 10^{-6}.$$

Let us consider  $\hat{q} = (0.02, 1, 0.5)$  which is a *typical point* in  $P_{ad}$ . Let  $q_N^* = q_m$  be the set of parameters attained when the Powell's minimization method was terminated according to the stopping criterion (36). The minimizers  $q_N^*$  together with the number of iterations  $m$  are shown in Tables 2, 3 and 4 for various noise levels  $\epsilon$ , and the number of observations  $M$ .

Table 2 shows the identification algorithm is successful without noise. It can be seen that the number of iterations is quite small and the precision of the identification is high even for a few time observation data. Tables 3, 4 show that significant time observation data is required when noise is present.

TABLE 2. Identification results for  $\epsilon = 0$

$M$	Iterations( $m$ )	$q_N^*$	$J_N(q_N^*)$
1	4	(0.01983, 1.00000, 0.49997)	0.000000
4	4	(0.01943, 0.99999, 0.50004)	0.000000
10	4	(0.01959, 0.99999, 0.50005)	0.000000
400	4	(0.02000, 1.00000, 0.50000)	0.000000

TABLE 3. Identification results for  $\epsilon = 0.01$

$M$	Iterations	$q_N^*$	$J_N(q_N^*)$
1	20	(1.00000, 0.60011, 0.10000)	0.000012
4	3	(0.00100, 1.00000, 0.10000)	0.000064
10	6	(0.10379, 1.00000, 0.81481)	0.000161
400	4	(0.01981, 0.99998, 0.50071)	0.006171

TABLE 4. Identification results for  $\epsilon = 0.1$

$M$	Iterations	$q_N^*$	$J_N(q_N^*)$
400	4	(0.01810, 0.99981, 0.50712)	0.617093

Now let us evaluate the values  $a, b, c$ . For  $t_j = jh, h > 0$  put

$$\begin{aligned}
 a_N &= \int_0^T \int_0^L \frac{\partial y_N}{\partial t}(t, x) w_N(t, x) dx dt \\
 &= \frac{L}{2} \sum_{n=1}^N \int_0^T Y'_n(t) W_n(t) dt \\
 &\approx \frac{Lh}{2} \sum_{n=1}^N \sum_{j=1}^M Y'_n(t_j) W_n(t_j) = a_{NM},
 \end{aligned}$$

$$\begin{aligned}
 b_N &= \int_0^T \int_0^L \frac{\partial y_N}{\partial x}(t, x) \frac{\partial w_N}{\partial x}(t, x) dx dt \\
 &= \frac{\pi^2}{2L} \sum_{n=1}^N n^2 \int_0^T Y_n(t) W_n(t) dt \\
 &\approx \frac{\pi^2 h}{2L} \sum_{n=1}^N n^2 \sum_{j=1}^M Y_n(t_j) W_n(t_j) = b_{NM},
 \end{aligned}$$

and

$$\begin{aligned} c_N &= \int_0^T \int_0^L P_N[\sin y_N(t, x)] w_N(t, x) dx dt \\ &= \frac{L}{2} \sum_{n=1}^N \int_0^T S_n(t) W_n(t) dt \\ &\approx \frac{Lh}{2} \sum_{n=1}^N \sum_{j=1}^M S_n(t_j) W_n(t_j) = c_{NM}. \end{aligned}$$

It is clear that

$$\lim_{N, M \rightarrow \infty} a_{N, M} = a, \quad \lim_{N, M \rightarrow \infty} b_{N, M} = b, \quad \lim_{N, M \rightarrow \infty} c_{N, M} = c.$$

For the interior point  $q^* = (0.02, 0.7, 0.5)$  of  $P_{ad}$ , the estimated values are

$$a_{16,400} = 0.360151 \times 10^{-8}, \quad b_{16,400} = -0.103648 \times 10^{-5}, \quad c_{16,400} = 0.473936 \times 10^{-13}.$$

For the boundary point  $q^* = (0.001, 1, 0.1)$  of  $P_{ad}$ , the estimated values are

$$a_{16,400} = -0.04369 \times 10^{-8}, \quad b_{16,400} = 0.188047 \times 10^{-6}, \quad c_{16,400} = 0.208795 \times 10^{-8}.$$

All estimated values can be regarded as zero. If they are not, i.e.,  $b_{16,400} = -0.103648 \times 10^{-5} \not\approx 0$ , then  $\beta^*$  must be 1 by the bang-bang control law. So, we can conclude that the bang-bang control law was not useful in our numerical experiments.

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