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ON CONDITIONS PROVIDED BY NILRADICALS

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Abstract. A ring R is called *IFP*, due to Bell, if $ab = 0$ implies $aRb = 0$ for $a, b \in R$. Huh et al. showed that the IFP condition is not preserved by polynomial ring extensions. In this note we concentrate on a generalized condition of the IFPness that can be lifted up to polynomial rings, introducing the concept of *quasi-IFP* rings. The structure of quasi-IFP rings will be studied, characterizing quasi-IFP rings via minimal strongly prime ideals. The connections between quasi-IFP rings and related concepts are also observed in various situations, constructing necessary examples in the process. The structure of minimal noncommutative (quasi-)IFP rings is also observed.

1. Quasi-IFP rings and related concepts

Throughout every ring is associative with identity unless otherwise stated.

Given a ring R, $J(R)$, $N_*(R)$, $N^*(R)$, and $N(R)$ denote the Jacobson radical, the prime radical, the upper nilradical (i.e., sum of nil ideals), and the set of all nilpotent elements in R, respectively. Note $N_*(R) \subseteq N^*(R) \subseteq N(R)$. Based on Artin and Wedderburn, the Wedderburn radical of a ring R means the sum of all nilpotent ideals in R (in spite of this sum being not a radical, it was given the name), written by $N_0(R)$. X denotes a nonempty set of commuting indeterminates over rings. Let R be a ring. The polynomial ring over R with X is denoted by $R[X]$, and if X is a singleton, say $X = \{x\}$, then we write $R[x]$ in place of $R[\{x\}]$. The n by n matrix ring over a ring R is denoted by $\mathrm{Mat}_n(R)$, and e_{ij} denotes the n by n matrix with (i, j)-entry 1 and zero elsewhere.

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$r_R(-)$ (resp. $\ell_R(-)$) is used for the right (resp. left) annihilator over a ring R, i.e., $r_R(S) = \{a \in R \mid sa = 0 \text{ for all } s \in S\}$ (resp. $\ell_R(S) = \{b \in R \mid bs = 0 \text{ for all } s \in S\}$), where $S \subseteq R$ or S is a subset of a right (resp. left) R-module. Write $r_R(a)$ (resp. $\ell_R(a)$) in place of $r_R(\{a\})$ (resp. $\ell_R(\{a\})$). $a \in R$ is said to be right (resp. left) regular if $r_R(a) = 0$ (resp. $\ell_R(a) = 0$). A *regular* element is defined to be both left and right regular. $a \in R$ is called a left (resp. right) zero-divisor if $r_R(a) \neq 0$ (resp. $\ell_R(a) \neq 0$). A zero-divisor means an element that is neither right nor left regular. A domain means a ring whose nonzero elements are regular.

A prime ideal P of a ring R is called *completely prime* if R/P is a domain. According to Kim et al. [17], a ring is called *nil-semisimple* if it has no nonzero nil ideals. Nil-semisimple rings are clearly semiprime, but semiprime rings need not be nil-semisimple as can be seen by [13, Example 1.2 and Proposition 1.3]. Due to Rowen [22, Definition 2.6.5], an ideal P of a ring R is called *strongly prime* if P is prime and R/P is nil-semisimple. Maximal ideals and completely prime ideals are clearly strongly prime. Nil-semisimple rings need not be prime as can be seen by direct products of reduced rings; and prime rings also need not be nil-semisimple as can be seen by [13, Example 1.2 and Proposition 1.3]. Note that any strongly prime ideal contains a minimal strongly prime ideal. $N^*(R)$ of a ring R is the unique maximal nil ideal of R by [22, Proposition 2.6.2], and with the help of [22, Proposition 2.6.7] we have

$$\begin{aligned} N^*(R) &= \{a \in R \mid RaR \text{ is a nil ideal of } R\} \\ &= \bigcap\{P \mid P \text{ is a (minimal) strongly prime ideal of } R\}. \end{aligned}$$

A ring R is called *reduced* if $N(R) = 0$. Due to Marks [20], a ring R is called *NI* if $N^*(R) = N(R)$. Reduced rings are clearly NI. Note that R is NI if and only if $N(R)$ forms an ideal if and only if $R/N^*(R)$ is reduced. Hong et al. [9, Corollary 13] showed that a ring R is NI if and only if every minimal strongly prime ideal of R is completely prime. According to Birkenmeier et al. [3], a ring R is called *2-primal* when $N_*(R) = N(R)$. It is obvious that a ring R is 2-primal if and only if $R/N_*(R)$ is reduced. Birkenmeier et al. [3, Proposition 2.6] proved that a ring R is 2-primal if and only if so is $R[X]$. Shin showed that a ring R is 2-primal if and only if every minimal prime ideal of R is completely prime [23, Proposition 1.11]. 2-primal rings are clearly NI, but the converse need not hold by Hwang et al. [13, Example 1.2] or Marks [20, Example 2.2].

There are various conditions between the commutativity and the NIness. Among them we concentrate on the *insertion-of-factors-property* (simply *IFP*) and the 2-primalness. Due to Bell [2], a right (or left) ideal I of a ring R is said to have the *IFP* if $ab \in I$ implies $aRb \subseteq I$ for $a, b \in R$. So a ring R is called *IFP* if the zero ideal of R has the IFP. Shin [23] used the term *SI* for the IFP; while IFP rings are also known as *semicommutative* in Narbonne's paper

[21]. Reduced rings are IFP by a simple computation. IFP rings are 2-primal by [23, Theorem 1.5].

A ring is called *Abelian* if each idempotent is central. IFP rings are Abelian by a simple computation, but quasi-IFP rings need not be Abelian as can be seen by 2 by 2 upper triangular matrix rings over reduced rings.

In this note a ring R will be called *quasi-IFP* provided that $\sum_{i=0}^{n} Ra_iR$ is nilpotent whenever $\sum_{i=0}^{n} a_ix^i \in R[x]$ is nilpotent. As we see below, IFP rings are quasi-IFP and this implication is irreversible.

Lemma 1.1. (1) *A ring R is IFP if and only if $r_R(S)$ is an ideal of R for any $S \subseteq R$ if and only if $\ell_R(S)$ is an ideal of R for any $S \subseteq R$.*

(2) *IFP rings are quasi-IFP.*

(3) *Quasi-IFP rings are 2-primal.*

(4) *Subrings of quasi-IFP rings are quasi-IFP.*

Proof. (1) and (2) are proved by [23, Lemma 1.2] and [8, Lemma 1.1(4)] respectively.

(3) Let R be a quasi-IFP ring and $0 \neq a \in N(R)$. Then RaR is nilpotent and so $RaR \subseteq N_*(R)$, getting $N_*(R) = N(R)$.

(4) Let R be a quasi-IFP ring and S be a subring of R. Let $\sum_{i=0}^{n} a_ix^i \in N(S[x])$, then $\sum_{i=0}^{n} a_ix^i \in N(R[x])$. Since R is quasi-IFP, $\sum_{i=0}^{n} Ra_iR$ is nilpotent. Immediately $\sum_{i=0}^{n} Sa_iS$ is nilpotent. □

The implications in Lemma 1.1(2),(3) are irreversible by the following. Given a ring R, the n by n upper triangular matrix ring over R is denoted by $U_n(R)$.

Example 1.2. (1) There exists a quasi-IFP ring but not IFP. Let $R = U_n(S)$ over a reduced ring S with $n \geq 2$. Note $R[X] \cong U_n(S[X])$. Note

$$N_*(R[X]) = \{A \in R \mid \text{ the diagonal of } A \text{ is zero } \}[X] = N_0(R[X]).$$

So we get a reduced factor ring $\frac{R[X]}{N_*(R[X])} \cong \oplus_{i=1}^{n} T_i$ with $T_i = S[X]$ for all i, entailing $N_*(R[X]) = N(R[X])$. Hence $\sum_{i=0}^{m} Ra_iR$ is nilpotent (actually $(\sum_{i=0}^{m} Ra_iR)^n = 0$) whenever $\sum_{i=0}^{m} a_ix^i \in N(R[x])$. Thus R is quasi-IFP but R is not IFP since R is non-Abelian.

(2) There exists a 2-primal ring but not quasi-IFP. Let K be a field and $K\{x, y\}$ be the free algebra generated by noncommuting indeterminates x, y over K. The following construction is essentially due to [17, Example 2.4(2)]. Consider an infinite word

$$w = yxyxxyxxxyxxxxyxxxxx \cdots = \prod_{i=1}^{\infty} yx^i.$$

Let I be the ideal of $K\{x, y\}$ generated by the set of all words each of which is not a subword of w, and define $R = K\{x, y\}/I$. $y + I$ is nilpotent, however $R(y + I)R$ is non-nilpotent by the computation in [17, Example 2.4(2)]. Thus R is not quasi-IFP, but 2-primal by the argument in [17, Example 2.4(2)].

In the following lemma we find various condition equivalent to the quasi-IFPness. Lambek [11] called a ring R *symmetric* when $rst = 0$ implies $rts = 0$ for all $r, s, t \in R$, proving that a ring R is symmetric if and only if $r_1 r_2 \cdots r_n = 0$, with n any positive integer, implies $r_{\sigma(1)} r_{\sigma(2)} \cdots r_{\sigma(n)} = 0$ for any permutation σ of the set $\{1, 2, \ldots, n\}$ and $r_i \in R$ [18, Proposition 1]. Symmetric rings are IFP obviously, and reduced rings are symmetric by [18, Section 1(G)].

Lemma 1.3. *For a ring R the following conditions are equivalent*:

(1) *R is quasi-IFP*;
(2) *$\sum_{i=0}^{n} Ra_iR$ is nilpotent whenever $a_0 + \sum_{i=1}^{n} a_iP_i \in R[X]$ is nilpotent, where each P_i is a finite product of indeterminates in X*;
(3) *$N(R) = N_0(R)$ (i.e., RaR is nilpotent for any $a \in N(R)$)*;
(4) *R is 2-primal with $N_0(R) = N_*(R)$*;
(5) *R is NI with $N_0(R) = N^*(R)$*;
(6) *Every minimal prime ideal of R is completely prime with*
$$N_0(R) = N_*(R);$$
(7) *Every minimal strongly prime ideal of R is completely prime with*
$$N_0(R) = N^*(R);$$
(8) *$R/N_0(R)$ is a subdirect product of domains*;
(9) *$R/N_0(R)$ is a reduced ring*;
(10) *$R/N_0(R)$ is a symmetric ring with $N_0(R) = N_*(R)$.*

Proof. (1)⇒(3), (3)⇒(4), (4)⇒(5), (5)⇒(3), (7)⇒(8), (8)⇒(9), (9)⇒(3), and (2)⇒(1) are obvious. (4)⇔(6) and (5)⇔(7) are obtained from [23, Proposition 1.11] and [9, Corollary 13] respectively.

(1)⇒(2): Let R be quasi-IFP and $a_0 + \sum_{i=1}^{n} a_iP_i \in N(R[X])$. Since R is 2-primal by Lemma 1.1(3), $\frac{R[X]}{N_*(R[X])} \cong \frac{R}{N_*(R)}[X]$ is reduced and so $N(R[X]) \subseteq N_*(R[X]) = N_*(R)[X]$, concluding that each a_i is in $N_*(R)$ for $i = 0, \ldots, n$. Since R is quasi-IFP, each Ra_iR is nilpotent and thus $\sum_{i=0}^{n} Ra_iR$ is nilpotent.

(3)⇒(1): Let $N(R) = N_0(R)$ and $\sum_{i=0}^{n} a_ix^i \in N(R[x])$. Then $N_*(R) = N_0(R)$ and $\frac{R[x]}{N_*(R[x])} \cong \frac{R}{N_*(R)}[x]$ is reduced, concluding that $N(R[x]) \subseteq N_*(R[x]) = N_*(R)[x] = N_0(R)[x]$. Thus each a_i is in $N_0(R)$ for $i = 0, \ldots, n$; hence each Ra_iR is nilpotent, entailing $\sum_{i=0}^{n} Ra_iR$ is nilpotent.

Reduced rings are symmetric by [18], obtaining (9)⇒(10). Symmetric rings are 2-primal, and so $R/N_0(R)$ is reduced when $R/N_0(R)$ is symmetric with $N_0(R) = N_*(R)$. Thus (10)⇒(9) holds. □

By Lemma 1.3(3), the quasi-IFPness is equal to the W_1-reducedness in [17]. The subcondition "$N_0(R) = N_*(R)$" in the condition (10) in Lemma 1.3 is not superfluous by the following. Let R be an algebra over a commutative ring S. The *Dorroh extension* of R by S, write $R \oplus_D S$, is the ring $R \times S$ with operations $(r_1, s_1) + (r_2, s_2) = (r_1 + r_2, s_1 + s_2)$ and $(r_1, s_1)(r_2, s_2) = (r_1r_2 + s_1r_2 + s_2r_1, s_1s_2)$, where $r_i \in R$ and $s_i \in S$.

Example 1.4. According to [17, Example 2.4(3)], let S be the factor ring of the polynomial ring $\mathbb{Z}_2[t_1, t_2, \ldots]$ with t_i's a set of commuting indeterminates over $\mathbb{Z}_2$ modulo the ideal generated by $\{t_i^2 \mid i = 1, 2, \ldots\}$ and $T = \begin{pmatrix} N_0(S) & S \\ N_0(S) & N_0(S) \end{pmatrix}$, where $\mathbb{Z}_2$ is the field of integers modulo 2. Set $R = T \oplus_D \mathbb{Z}_2$. By the computation in [16, Example 2.4(3)], $N_0(R) = \begin{pmatrix} N_0(S) & N_0(S) \\ N_0(S) & N_0(S) \end{pmatrix} \oplus_D 0$ but $N_0(R) \subsetneqq N_*(R) = T \oplus_D 0$. Then $R/N_0(R) \cong \begin{pmatrix} 0 & \mathbb{Z}_2 \\ 0 & 0 \end{pmatrix} \oplus_D \mathbb{Z}_2$. Note that $(a, b) \in R/N_0(R)$ is invertible if $b = 1$. Suppose $(a_1, b_1)(a_2, b_2)(a_3, b_3) = 0$ for $0 \neq (a_i, b_i) \in R/N_0(R)$ with $i = 1, 2, 3$. Then at least two of b_i's must be zero and so we get $(a_1, b_1)(a_3, b_3)(a_2, b_2) = (c, 0)(d, 0)$ for some $c, d \in \begin{pmatrix} 0 & \mathbb{Z}_2 \\ 0 & 0 \end{pmatrix}$, entailing $(a_1, b_1)(a_3, b_3)(a_2, b_2) = 0$. Thus $R/N_0(R)$ is symmetric, but R is not quasi-IFP by the computation in [17, Example 2.4(3)].

The subcondition "$N_0(R) = N_*(R)$" ($N_0(R) = N^*(R)$) in the condition (4) ((5)) in Lemma 1.3 is also not superfluous by Example 1.4.

The n by n upper (lower) triangular matrix rings over a quasi-IFP ring R are also quasi-IFP by [17, Corollary 3.8(1)]. However given any ring A, $\mathrm{Mat}_n(A)$ is not quasi-IFP by Lemma 1.3 when $n \geq 2$. To see that we take the following two examples:

(1) e_{12} and e_{21} are nilpotent but $e_{12} + e_{21}$ is non-nilpotent.

(2) Let $a_0 = e_{12}$, $a_1 = e_{11} - e_{22}$, and $a_2 = -e_{21}$ and $f(x) = a_0 + a_1 x + a_2 x^2 \in \mathrm{Mat}_n(A)[x]$. Then $f(x)^2 = 0$ but $\sum_{i=0}^{2} \mathrm{Mat}_n(A) a_i \mathrm{Mat}_n(A)$ is non-nilpotent.

In the remainder of this section we study conditions under which various concepts near at the quasi-IFPness coincide.

Semiprime 2-primal rings are reduced and so we obtain the following equivalences by Lemma 1.3.

Proposition 1.5. *Let R be a semiprime ring. Then the following conditions are equivalent*:

(1) *R is quasi-IFP*;
(2) *R is IFP*;
(3) *R is reduced*;
(4) *R is 2-primal.*

When R is a semiprime ring we may conjecture that a ring R is NI if and only if R is reduced, based on Proposition 1.5. However there is a semiprime NI ring but not reduced as we see in [13, Example 1.2].

The *index of nilpotency* of a nilpotent element x in a ring R is the least positive integer n such that $x^n = 0$. The *index of nilpotency* of a subset I of R is the supremum of the indices of nilpotency of all nilpotent elements in I. If such a supremum is finite, then I is said to be *of bounded index of nilpotency.* If R is of bounded index of nilpotency, then R is NI if and only if it is 2-primal by [13, Proposition 1.4]. So we get the following from Proposition 1.5.

Proposition 1.6. *Let R be a semiprime ring of bounded index of nilpotency. Then the following conditions are equivalent*:

(1) *R is quasi-IFP*;
(2) *R is IFP*;
(3) *R is reduced*;
(4) *R is 2-primal*;
(5) *R is NI.*

The condition "of bounded index of nilpotency" in Proposition 1.6 is not superfluous by [13, Example 1.2] (this ring is semiprime but not of bounded index of nilpotency); while, the condition "semiprime" in Proposition 1.6 is also not superfluous by Example 1.4 (this ring is of bounded index of nilpotency but not semiprime). However "R being quasi-IFP" is not equivalent to them by [17, Example 2.4(3)].

A ring R is called *von Neumann regular* if for each $a \in R$ there exists $x \in R$ such that $a = axa$. A ring is called *right* (resp. *left*) *duo* if every right (resp. left) ideal is two-sided. Duo means the two-sided duo. Right or left duo rings are IFP by Lemma 1.1(1), but not conversely as can be seen by [16, Proposition 1.2] and the ring of all matrices of the form $\begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix}$ over a reduced ring. Von Neumann regular rings need not be quasi-IFP in spite of being semiprime. $\text{Mat}_n(R)$ is von Neumann regular by [7, Lemma 1.6] over a von Neumann regular ring R, but it is not quasi-IFP by Proposition 1.7 below when $n \geq 2$. In the following we see some conditions under which von Neumann regular rings can be near-IFP.

Proposition 1.7. *Let R be a von Neumann regular ring. Then the following conditions are equivalent*:
(1) *R is right* (*left*) *duo*;
(2) *R is reduced*;
(3) *R is Abelian*;
(4) *R is IFP*;
(5) *R is quasi-IFP*;
(6) *R is 2-primal*;
(7) *R is NI.*

Proof. Von Neumann regular rings are semiprimitive (hence semiprime) by [7, Corollary 1.2], and so we have the result by [7, Theorem 3.2] and Proposition 1.5. □

A ring R is called *strongly regular* if for each $a \in R$ there exists $x \in R$ such that $a = a^2x$. A ring is strongly regular if and only if it is Abelian and von Neumann regular [7, Theorem 3.5]. From Proposition 1.7 we obtain a similar result to [7, Theorem 3.5].

Corollary 1.8. *A ring is strongly regular if and only if it is quasi-IFP and von Neumann regular.*

A ring R is called *π-regular* if for each $a \in R$ there exist a positive integer $n = n(a)$, depending on a, and $x \in R$ such that $a^n = a^nxa^n$. Von Neumann regular rings are clearly π-regular but the converse need not hold as can be seen by the 2 by 2 upper triangular matrix rings over division rings. From Proposition 1.7, one may conjecture that π-regular 2-primal (or NI) rings are quasi-IFP. But the following erases the possibility.

Example 1.9. Set $R = T \oplus_D \mathbb{Z}_2$ be the ring in Example 1.4. Then R is not quasi-IFP. Note that $N_*(R) = T \oplus_D 0$ and $R/N_*(R) \cong \mathbb{Z}_2$, getting that R is 2-primal. Now we will show that R is π-regular. Take $(t, 0) \in N_*(R)$, then $(t, 0) \in J(R)$ and so $(t, 1) = (0, 1) - (t, 0)$ is invertible. Consequently each element in R is either nilpotent or invertible, and thus R is π-regular. Note that R is not von Neumann regular since $J(R)$ is nonzero.

In Example 1.9, let $(n, 1)^2 = (n, 1)$. Then $n^2 = n$ and so n must be zero because n is nilpotent; hence all idempotents in R are 0 and 1, obtaining that R is Abelian. Consequently the ring R in Example 1.8 is Abelian π-regular that is 2-primal but not quasi-IFP. While, Badawi proved that Abelian π-regular rings are NI [1, Theorem 3]. So one may ask whether Abelian π-regular rings may be 2-primal. We also answer that negatively in the following.

Example 1.10. Let S be a division ring and denote by U_n the 2^n by 2^n upper triangular matrix ring over a ring S, where n is a positive integer. Consider a subring of U_n

$$D_n = \{M \in U_n \mid \text{ the diagonal entries of } M \text{ are equal}\}.$$

Define a map $\sigma : D_n \to D_{n+1}$ by $A \mapsto \left(\begin{smallmatrix} A & 0 \\ 0 & A \end{smallmatrix}\right)$. Then D_n can be considered as a subring of D_{n+1} via σ (i.e., $A = \sigma(A)$ for $a \in D_n$). Set R be the direct limit of the direct system (D_n, σ_{ij}), where $\sigma_{ij} = \sigma^{j-i}$. Then R is a semiprime ring by [15, Theorem 2.2]. Note

$$N^*(R) = N(R) = \{M \in R \mid \text{ the diagonal entries of } M \text{ are zero}\}.$$

Let $A \in R$. Then $A \in D_n$ for some n and so A is either invertible (when the diagonal of A is nonzero) or nilpotent (when the diagonal of A is zero); hence R is π-regular. Moreover R is Abelian by [10, Lemma 2]. However R is not 2-primal since $N_*(R) = 0$ and $N(R) \neq 0$.

Proposition 1.11. *Let R be an one-sided Goldie ring or R satisfies ACC on left and right annihilators. Then the following conditions are equivalent*:

(1) *R is quasi-IFP*;
(2) *R is 2-primal*;
(3) *R is NI.*

Proof. If R is an one-sided Goldie ring or satisfies ACC on left and right annihilators, then nil ideals are nilpotent by [4, Theorem 1.3.4] and [19]. So NI rings are quasi-IFP by Lemma 1.3 since $N(R) = N_0(R)$. □

In the following we see another equivalent condition of IFP rings.

Proposition 1.12. *A ring R is IFP if and only if every finitely generated subring of R is IFP.*

Proof. It suffices to show the sufficiency. Let the necessity hold and assume on the contrary that R is not IFP. Then there are $a, b, c \in R$ such that $ac = 0$ but $abc \neq 0$. Consider the subring S of R generated by a, b, c. Then S is IFP by the necessity and so $abc = 0$, a contradiction. Thus R is IFP. □

However this result need not hold for quasi-IFP rings as follows. Due to Huh et al. [11], a ring is called *locally finite* if every finite subset in it generates a finite semigroup multiplicatively. A ring R is locally finite if and only if each finite subset of R generates a finite subring (not necessarily with identity) if and only if R/I and I are both locally finite for a proper ideal of R [11, Proposition 2.1 and Theorem 2.2].

Proposition 1.13. *For a locally finite ring R the following conditions are equivalent*:

(1) *R is NI*;
(2) *Every finitely generated subring of R is quasi-IFP*;
(3) *Every finitely generated subring of R is 2-primal*;
(4) *Every finitely generated subring of R is NI*;
(5) *For every finitely generated subring S of R, $S/J(S)$ is a finite direct product of finite fields*

Proof. Since R is locally finite, every finitely generated subring of R is finite. So we have the equivalences from (2) to (5) with the help of [17, Proposition 2.3], noting $N^*(S) = N_*(S) = J(S) = N_0(S)$ for a finite ring S. (1)⇒(4) and (4)⇒(1) are obtained by [13, Proposition 2.4(2)] and [13, Lemma 2.1] respectively. □

The quasi-IFPness and 2-primalness cannot be equivalent to the conditions in Proposition 1.13 by the following.

Example 1.14. We use the construction and computation in [13, Example 1.2]. Let S be a finite field, n be a positive integer and R_n be the 2^n by 2^n upper triangular matrix ring over S. Then each R_n is a finite ring, and quasi-IFP by Lemma 1.3. According to [13, Example 1.2], define a map $\sigma : R_n \to R_{n+1}$ by $A \mapsto \left(\begin{smallmatrix} A & 0 \\ 0 & A \end{smallmatrix}\right)$. Then R_n can be considered as a subring of R_{n+1} via σ (i.e., $A = \sigma(A)$ for $A \in R_n$). Let $D = \{R_n, \sigma_{nm}\}$ (with $\sigma_{nm} = \sigma^{m-n}$ whenever $n \leq m$) be the direct system over $I = \{1, 2, \ldots\}$, and $R = \varinjlim R_n$ be the direct limit of D. Then R is an NI ring by [13, Proposition 1.1], but not 2-primal by the computation in [13, Example 1.2]. Considering any finitely generated subring S of R, it is clearly a subring of R_n for some n; hence S is quasi-IFP by Lemma 1.1(4).

2. Structure of quasi-IFP rings

In this section we study various properties of quasi-IFP rings. In [17, Theorem 2.5] we see a characterization of quasi-IFP rings relating to minimal prime ideals. We furthermore find another characterization of quasi-IFP rings via strongly prime ideals as follows. To do that we introduce the following concepts which are essentially due to Shin [23]. Let R be a ring and P be a strongly prime ideal of R.

$$M(P) = \{a \in R \mid aRb \subseteq N_0(R) \text{ for some } b \in R \backslash P\};$$
$$M_P = \{a \in R \mid ab \in N_0(R) \text{ for some } b \in R \backslash P\};$$
$$\overline{M}_P = \{a \in R \mid a^m \in M_P \text{ for some positive integer } m\}.$$

Note that $M(P) \subseteq P$, $N(R) \subseteq \overline{M}_P$, and $M(P) \subseteq M_P \subseteq \overline{M}_P$. Given a multiplicative monoid X in $R\backslash 0$, if Q is an ideal of R maximal with respect to the property $Q \cap X = \emptyset$, then Q is a strongly prime ideal of R [13, Lemma 2.2].

Theorem 2.1. *Given a ring R the following conditions are equivalent*:

(1) *R is quasi-IFP*;
(2) *For any minimal strongly prime ideal P of R we have that $M(P) = M_P = \overline{M}_P = P$, and if $a \in R$ and $aRb \subseteq N_0(R)$ for some $b \in R\backslash P$, then $a \in N_0(R)$.*

Proof. We use Lemma 1.3 freely and apply the proof of [23, Theorem 1.8] to strongly prime ideals and $N_0(R)$.

(1)⇒(2): Let P be a strongly prime ideal of R. We already have $M(P) \subseteq M_P \subseteq \overline{M}_P$. Let $a \in \overline{M}_P$. Then $a^k b \in N_0(R)$ for some $b \in R\backslash P$ and positive integer k. Since R is quasi-IFP, $R/N_0(R)$ is reduced and so we have $aRb \subseteq N_0(R)$ by [18, Section 1(G)] and this implies $a \in M(P)$; hence we get $M(P) = M_P = \overline{M}_P$. Next we will show

$$M(P) = \bigcap \{Q \mid Q \text{ is a strongly prime ideal of } R \text{ with } Q \subseteq P\}.$$

If Q is a strongly prime ideal of R with $Q \subseteq P$, then $M(P) \subseteq M(Q) \subseteq Q$; hence we have $M(P) \subseteq \bigcap\{Q \mid Q \text{ is a strongly prime ideal of } R \text{ with } Q \subseteq P\}$.

Conversely, let $a \notin M(P)$. Then the multiplicative subset $S = \{a, a^2, a^3, \ldots\}$ of R does not contain any element of $N_0(R)$ (if $a^n \in N_0(R)$, then $a \in N_0(R)$ by the reducedness of $R/N_0(R)$) because $M(P) = M_P = \overline{M}_P$. Define

$$T = \{a^{t_0} b_1 a^{t_1} b_2 \cdots a^{t_{m-1}} b_m a^{t_m} \notin N_0(R) \mid m \in \mathbb{Z}^+, b_i \in R\backslash P,$$
$$t_j \in \mathbb{Z}^+ \text{for } j = 1, 2, \ldots, m-1 \text{ and } t_0, t_m \in \{0\} \cup \mathbb{Z}^+\},$$

where $\mathbb{Z}^+$ is the set of all positive integers. Then T contains both S and $R\backslash P$ (so $1 \in T$). We claim that T is a multiplicative monoid in $R\backslash 0$. Consider $x, y \in T$. If $x, y \in S$, then $xy \in S \subseteq T$ clearly. If $x = a^s \in S, y = a^{t_0} b_1 a^{t_1} b_2 \cdots b_m a^{t_m} \in T$, then $xy \notin N_0(R)$. For, if $xy \in N_0(R)$, then $xy = a^{s+t_0} b_1 a^{t_1} b_2 \cdots b_m a^{t_m} \in N_0(R)$. Since $R/N_0(R)$ is reduced we have $[(a^{s+t_0+\cdots+t_m})(b_1 \cdots b_m)]^{m+1} \in$

$N_0(R)$ with the help of [18, Section 1(G)], entailing $(a^{s+t_0+\cdots+t_m})(b_1\cdots b_m)\in N_0(R)$. But P is prime and so there exist $r_1,\ldots,r_{m-1}\in R$ such that $b_1r_1\cdots b_{m-1}r_{m-1}b_m\in R\backslash P$. Letting $k=s+t_0+\cdots+t_m$ and $b=b_1r_1\cdots b_{m-1}r_{m-1}b_m$, then we have $a^kb\in N_0(R)$ because $R/N_0(R)$ is reduced; hence $a\in\overline{M}_P$ and then $a\in M(P)$ since $M(P)=M_P=\overline{M}_P$. This is a contradiction, concluding $xy\notin N_0(R)$ and $xy\in T$. For the cases of $(x,y\in T)$ and $(x\in T, y\in S)$ we can get $xy\notin N_0(R)$ (so $xy\in T$) by similar computations. Thus T is a multiplicative monoid in $R\backslash 0$. By [13, Lemma 2.2] there exists a strongly prime ideal of R, say J, that is disjoint from T. Then $a\notin J$ and $J\subseteq P$, obtaining $M(P)\supseteq\bigcap\{Q\mid Q$ is a prime ideal of R with $Q\subseteq P\}$. Thus $M(P)=\bigcap\{Q\mid Q$ is a strongly prime ideal of R with $Q\subseteq P\}$. Now suppose that P is any minimal strongly prime ideal of R. Then we get $M(P)=P$ from $M(P)=\bigcap\{Q\mid Q$ is a strongly prime ideal of R with $Q\subseteq P\}$. Next assume that $a\in R$ and $aRb\subseteq N_0(R)$ for some $b\in R\backslash P$. Note $aRb\subseteq N_0(R)\subseteq P$; hence $a\in P$ since P is prime. Since P is arbitrarily taken, we get $a\in N^*(R)$. But R is quasi-IFP, and so we have $N_0(R)=N^*(R)$. Thus $a\in N_0(R)$.

(2)⇒(1): Suppose that the condition (2) holds. Let $a\in N(R)$ with $a^m=0$. Then $a\in\overline{M}_P$, and so $a\in M_P=M(P)$ for any minimal strongly prime ideal P of R. Thus $aRb\subseteq N_0(R)$ for some $b\in R\backslash P$ and hence $a\in N_0(R)$. Thus $N(R)=N_0(R)$ and so R is quasi-IFP by Lemma 1.3. □

We apply Theorem 2.1 to the non-quasi-IFP ring in Example 1.2(2). Let R be the ring in Example 1.2(2). Note that $R(y+I)R$, say P, is the unique minimal strongly prime ideal of R with $P=N_*(R)=N^*(R)=N(R)=\overline{M}_P$ since $R/P\cong K[x]$. However $y+I\notin M_P$ and thus R is not quasi-IFP by Theorem 2.1.

Huh et al. showed that $R[x]$ need not be IFP when R is an IFP ring [12, Example 2]. But $R[X]$ can be quasi-IFP over a quasi-IFP ring R as in the following.

Proposition 2.2. *If R is a quasi-IFP ring, then so is $R[X]$.*

Proof. Let R be a quasi-IFP ring and $f=a_0+\sum_{i=1}^n a_iP_i\in N(R[X])$, where each P_i is a finite product of indeterminates in X. Then by Lemma 1.3(2), $\sum_{i=0}^n Ra_iR$ is nilpotent and thus

$$\sum_{i=0}^n R[X]\left(Ra_0R+\sum_{i=1}^n Ra_iRP_i\right)R[X]$$

is also nilpotent. But $R[X]fR[X]\subseteq\sum_{i=0}^n R[X](Ra_0R+\sum_{i=1}^n Ra_iRP_i)R[X]$ and so $R[X]fR[X]$ is nilpotent; hence $R[X]$ is quasi-IFP by Lemma 1.3. □

Proposition 2.2 is also obtained from [17, Theorem 3.2], but the preceding proof is a simpler one. From Proposition 2.2, it is natural to ask whether the quasi-IFPness is preserved by power series ring extensions. However the answer is negative by [17, Example 3.3]. But if a ring R is 2-primal with nilpotent $N_0(R)$, then the power series ring over R is quasi-IFP by [17, Corollary 3.5(1)].

We next obtain a useful method by which given rings are examined to be quasi-IFP. A proper subrings S (possibly without identity) of given a ring can be defined to be quasi-IFP when S satisfies the conditions in Lemma 1.3. The following is equal to [17, Proposition 3.10], but here we take another proof using the properties of $N_0(R)$.

Proposition 2.3. *Let R be a ring and I be a proper ideal of R. If R/I and I are both quasi-IFP rings with $N_0(I)$ nilpotent, then R is quasi-IFP.*

Proof. We use Lemma 1.3 freely. Since I is quasi-IFP, $N(I) = N_0(I)$. We first claim $N_0(I)$ is an ideal of R contained in $N_0(R)$. Notice $\langle N_0(I)\rangle^3 \subseteq N_0(I)$ by the Andrunakievich Lemma [5, p. 107], where $\langle N_0(I)\rangle$ is the ideal of R generated by $N_0(I)$. But since $I/N_0(I)$ is nil-semisimple from $N(I) = N_0(I)$, we get $\langle N_0(I)\rangle^3 = N_0(I)$, concluding that $N_0(I)$ is an ideal of R. Next for $a \in N_0(I)$, we have $(aI)^m = 0$ for some positive integer m. Then we have $(aR)^{2m} = 0$ from $(aRaR)^m \subseteq (aI)^m = 0$, obtaining $a \in N_0(R)$.

Now take $0 \neq d \in N(R)$ with $d^k = 0$. Since R/I is quasi-IFP, $N(R/I) = N_0(R/I)$ and so $(dR)^\ell \subseteq I$ for some positive integer ℓ. Note $(dR)^\ell d^k = 0 \subseteq N_0(I)$. Since $k \geq 2$, $((dR)^\ell d^{k-1})^3 \subseteq N_0(I)$ by the reducedness of $I/N_0(I)$, entailing $(dR)^\ell d^{k-1} \subseteq N_0(I)$. Inductively we get $(dR)^\ell d \subseteq N_0(I)$. It then follows, from $N_0(I)$ being nilpotent, that $(dR)^{(\ell+1)h} = ((dR)^\ell dR)^h = 0$ for some positive integer h, concluding $d \in N_0(R)$. Thus R is quasi-IFP. $\square$

With the help of Proposition 2.3, the upper (lower) triangular matrix rings over quasi-IFP rings are also quasi-IFP. If the ideal I is nilpotent, then we can obtain the following from Proposition 2.3.

Corollary 2.4. *Let R be a ring and I be a proper ideal of R. If R/I is quasi-IFP and I is nilpotent, then R is quasi-IFP.*

In Corollary 2.4, consider a weaker condition "I is nil" instead of the condition "I is nilpotent". However there can be counterexamples as we see in the following.

Example 2.5. Let T be a reduced ring, n be a positive integer and R_n be the 2^n by 2^n upper triangular matrix ring over T. Define a map $\sigma : R_n \to R_{n+1}$ by $A \mapsto \left(\begin{smallmatrix} A & 0 \\ 0 & A \end{smallmatrix}\right)$. Then R_n can be considered as a subring of R_{n+1} via σ (i.e., $A = \sigma(A)$ for $A \in R_n$). Let R be the direct limit of the direct system (R_n, σ_{ij}), where $\sigma_{ij} = \sigma^{j-i}$. Put

$$I = \{M \in R \mid \text{each diagonal entry of } M \text{ is zero}\}.$$

Then I is a nil ideal of R such that R/I is reduced (hence quasi-IFP). But R is not 2-primal (hence not quasi-IFP) by the computation in [13, Example 1.2].

$GF(p^n)$ means the Galois field of order p^n and $\mathbb{Z}_n$ means the ring of integers modulo n. Due to Kim et al. [14], a ring R is called *strongly right AB* if every right annihilator of R is bounded, i.e., it contains a nonzero ideal of R. IFP rings are strongly right AB by Lemma 1.1(1) and not conversely by [14].

The following construction is due to Xue [25, Example 2]. Let $A\{x, y\}$ be the free algebra generated by x, y over a ring A.

Let $B_1 = GF(2)\{x, y\}/(x^3, y^3, yx, x^2 - xy, y^2 - xy)$, where $(x^3, y^3, yx, x^2 - xy, y^2 - xy)$ is the ideal of $GF(2)\{x, y\}$ generated by $x^3, y^3, yx, x^2 - xy, y^2 - xy$.

Let $\mathbb{Z}_4 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}\}$ be the ring of integers modulo 4 and

$$B_2 = \mathbb{Z}_4\{x, y\}/(x^3, y^3, yx, x^2 - xy, x^2 - \bar{2}, y^2 - \bar{2}, \bar{2}x, \bar{2}y),$$

where $(x^3, y^3, yx, x^2 - xy, x^2 - \bar{2}, y^2 - \bar{2}, \bar{2}x, \bar{2}y)$ is the ideal of $\mathbb{Z}_4\{x, y\}$ generated by $x^3, y^3, yx, x^2 - xy, x^2 - \bar{2}, y^2 - \bar{2}, \bar{2}x, \bar{2}y$.

Let B_3 be the ring of all matrices of the form $\begin{pmatrix} a & b \\ 0 & a^2 \end{pmatrix}$ over $GF(2^2)$.

Xue showed that each B_i is a noncommutative duo ring of order 16 [24, Proposition 3]. Note that $Ch(B_1) = 2 = Ch(B_3)$, $Ch(B_2) = 4$ and $|J(B_1)| = 8$, $|J(B_3)| = 4$. Thus $B_1 \ncong B_2$, $B_1 \ncong B_3$ and $B_2 \ncong B_3$. Xue also showed that any minimal noncommutative duo ring is isomorphic to B_i for some $i = 1, 2, 3$ by [25, Theorem 3]. The term "minimal" means "having smallest cardinality".

Theorem 2.6. (1) *Every minimal noncommutative quasi-IFP ring is isomorphic to $U_2(GF(2))$.*

(2) *Every minimal noncommutative IFP ring R is isomorphic to B_k for some $k \in \{1, 2, 3, 4\}$, where B_i's are the rings above for $i = 1, 2, 3$ and*

$$B_4 = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \mid a, b, c, d \in GF(2) \right\}.$$

Proof. Eldridge proved that if R is a finite noncommutative ring of order p^3 for a positive prime p, then R is isomorphic to $U_2(GF(p))$ [6, Proposition], and that if R is a ring of finite order whose factorization is cube free, then R is commutative [6, Theorem]. Thus every minimal noncommutative ring is isomorphic to $U_2(GF(2))$.

(1) Note that $U_2(GF(2))$ is quasi-IFP by Lemma 1.3, and so we conclude that $U_2(GF(2))$ is the minimal noncommutative quasi-IFP ring up to isomorphism.

(2) Recall that one-sided duo rings are IFP and IFP rings are strongly right AB. The result (1) is non-available to IFP rings since IFP rings are Abelian. But Kim et al. proved that B_4 is a minimal noncommutative strongly right AB ring [14, Theorem 2.6(1)]; hence the order of R must be 16 because B_4 is also IFP by [16, Proposition 1.2].

Xue showed that each B_i $(i = 1, 2, 3)$ is a noncommutative duo (hence IFP) ring of order 16 [24, Proposition 3]. Based on this result and [17, Theorem 2.6(1)], every minimal noncommutative IFP ring R is isomorphic to B_k for some $k \in \{1, 2, 3, 4\}$ up to isomorphism. □

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