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CR-WARPED PRODUCT SUBMANIFOLDS OF NEARLY KAEHLER MANIFOLDS

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Abstract. As warped product manifolds provide an excellent setting to model space time near black holes or bodies with large gravitational field, the study of these manifolds assumes significance in general. B. Y. Chen [4] initiated the study of CR-warped product submanifolds in a Kaehler manifold. He obtained a characterization for a CR-submanifold to be locally a CR-warped product and an estimate for the squared norm of the second fundamental form of CR-warped products in a complex space form (cf [6]). In the present paper, we have obtained a necessary and sufficient conditions in terms of the canonical structures P and F on a CR-submanifold of a nearly Kaehler manifold under which the submanifold reduces to a locally CR-warped product submanifold. Moreover, an estimate for the second fundamental form of the submanifold in a generalized complex space is obtained and thus extend the results of Chen to a more general setting.

1. Introduction

R. L. Bishop and B. O'Neill [1] introduced the notion of warped product manifolds by homothetically warping the product metric of a product manifold $B \times F$ onto the fibers $p \times F$ for each $p \in B$. This generalized product metric appears in differential geometric studies in a natural way. For instance a surface of revolution is a warped product manifold. Moreover, many important submanifolds in real and complex space forms are expressed as warped product submanifolds. In view of its physical applications many research articles have recently appeared exploring existence (or non existence) of warped product submanifolds in known spaces (cf [8], [13], etc). B. Y. Chen [4] initiated the investigations by showing that there does not exist a warped product CR-submanifold $N_\perp \times_f N_T$ in a Kaehler manifold. B. Sahin [11], extending the result of Chen proved that there exists no semi-slant warped product submanifolds in a Kaehler manifold other than CR-warped product submanifolds $N_T \times_f N_\perp$, where N_T and $N_\perp$ are holomorphic and totally real submanifolds

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of the underlying Kaehler manifold. Interesting geometric properties of CR-warped product submanifolds are obtained as well as many examples of these warped product submanifolds are provided in [4]. In view of the interesting geometric features of a nearly Kaehler manifolds and the non existence of CR-products in S^6 (cf [14]), it is worthwhile to study CR-warped products in a nearly Kaehler manifold. In the present paper, we have worked out conditions under which a CR-submanifold reduces to locally a CR-warped product submanifold. Moreover, an inequality for the squared norm of the second fundamental form of CR-warped product submanifolds in a generalized complex space form is obtained.

2. Preliminaries

Let $\overline{M}$ be an almost Hermitian manifold with an almost complex structure J and a Hermitian metric g, i.e.,

$$J^2 = -I, \quad \text{and} \quad g(JU, JV) = g(U, V) \tag{2.1}$$

for all vector fields U, V on $\overline{M}$. If J is parallel with respect to the Levi-Civita connection $\overline{\nabla}$ on $\overline{M}$, i.e.,$\overline{\nabla} J = 0$, then $(\overline{M}, J, g, \overline{\nabla})$ is called a *Kaehler manifold.* A more general structure on $\overline{M}$, namely *nearly Kaehler structure* is defined by a weaker condition namely

$$(\overline{\nabla}_U J)V + (\overline{\nabla}_V J)U = 0. \tag{2.2}$$

A necessary and sufficient condition for a nearly Kaehler manifold to be a Kaehler manifold is the vanishing of the Nijenhuis tensor of J. Any four dimensional nearly Kaehler manifold is a Kaehler manifold. A typical example of a nearly Kaehler non Kaehler manifold is the six dimensional sphere S^6. It has an almost complex structure J defined by vector cross product in the space of purely imaginary Cayley numbers which satisfy the condition (2.2).

There is a more general class of almost Hermitian manifolds than nearly Kaehler manifolds, known as *RK-manifolds.* These are defined as follows.

A RK-manifold $(\overline{M}, J, g, \overline{\nabla})$ is an almost Hermitian manifold for which the curvature tensor $\overline{R}$ is invariant under J, i.e.,

$$\overline{R}(JU, JV, JW, JZ) = \overline{R}(U, V, W, Z)$$

for any $U, V, W, Z \in T\overline{M}$.

An almost Hermitian manifold $\overline{M}$ is *of pointwise constant type* if for any $x \in \overline{M}$ and $U \in T_x\overline{M}$

$$\lambda(U, V) = \lambda(U, W),$$

where $\lambda(U, V) = \overline{R}(U, V, JU, JV) - \overline{R}(U, V, U, V)$ with V and W being tangent vectors at x, orthogonal to U and JU. The manifold $\overline{M}$ is said to be *of constant type* if for any unit vectors $U, V \in T\overline{M}$ with $g(U, V) = g(JU, V) = 0$, $\lambda(U, V)$ is a constant function. Then we have:

Theorem 2.1 ([15]). *Let $\overline{M}$ be an RK-manifold. Then $\overline{M}$ is of pointwise constant type if and only if there exists a function α on $\overline{M}$ such that*

$$\lambda(U,V)=\alpha[g(U,U)g(V,V)-(g(U,V))^2-(g(U,JV))^2]$$

for any $U,V\in T\overline{M}$. Moreover, $\overline{M}$ is of constant type if and only if the above equality holds for a constant α. In this case, α is the constant type of $\overline{M}$.

A *generalized complex space form* is an RK-manifold of constant holomorphic sectional curvature and of constant type. A generalized complex space form of constant holomorphic sectional curvature c and of constant type α is denoted by $\overline{M}(c,\alpha)$. Each complex space form is a generalized complex space form. The converse is not true. The sphere S^6 endowed with the standard nearly Kaehler structure is an example of a generalized complex space form which is not a complex space form.

Let $\overline{M}(c,\alpha)$ be a generalized complex space form of constant holomorphic sectional curvature c and of constant type α. Then the curvature tensor $\overline{R}$ of $\overline{M}(c,\alpha)$ has the following expression

$$\begin{aligned}\overline{R}(U,V)W=&\frac{c+3\alpha}{4}[g(V,W)U-g(U,W)V]\\&+\frac{c-\alpha}{4}[g(U,JW)JV-g(V,JW)JU\\&+2g(U,JV)JW].\end{aligned}\tag{2.3}$$

Let M be a submanifold of an almost Hermitian manifold $\overline{M}$. Then we denote the induced metric on M by the same symbol g whereas the induced Riemannian connection on M by ∇. With these notations, Gauss and Weingarten formulae are written as

$$\overline{\nabla}_U V=\nabla_U V+h(U,V),\tag{2.4}$$

$$\overline{\nabla}_U N=-A_N U+\nabla^{\perp}_U N\tag{2.5}$$

for each $U,V\in TM$ and $N\in T^{\perp}M$, where $\nabla^{\perp}$ denotes the induced connection on the normal bundle $T^{\perp}M$. h and A_N are the second fundamental form and the shape operator of the immersion of M into $\overline{M}$. They are related by

$$g(h(U,V),N)=g(A_N U,V).\tag{2.6}$$

For any $U\in T(M)$ and $N\in T^{\perp}M$, we write

$$JU=PU+FU,\tag{2.7}$$

$$JN=tN+fN,\tag{2.8}$$

where PU and tN are the tangential components of JU and JN, respectively whereas FU and fN are the normal components of JU and JN, respectively.

The covariant differentiation of the tensors $P,F,t,$ and f are defined as

$$(\overline{\nabla}_U P)V=\nabla_U PV-P\nabla_U V,\tag{2.9}$$

$$(\overline{\nabla}_U F)V = \nabla^{\perp}_U FV - F\nabla_U V, \tag{2.10}$$

$$(\overline{\nabla}_U t)N = \nabla_U tN - t\nabla^{\perp}_U N, \tag{2.11}$$

$$(\overline{\nabla}_U f)N = \nabla^{\perp}_U fN - f\nabla^{\perp}_U N. \tag{2.12}$$

On the other hand the covariant derivative of the second fundamental form h is defined as

$$(\overline{\nabla}_U h)(V, W) = \nabla^{\perp}_U h(V, W) - h(\nabla_U V, W) - h(V, \nabla_U W) \tag{2.13}$$

for any $U, V, W \in TM$. Let $\overline{R}$ and R be the curvature tensors of the connections $\overline{\nabla}$ and ∇ on $\overline{M}$ and M, respectively. Then the equations of Gauss and Codazzi are given by

$$\begin{aligned} \overline{R}(U, V, W, Z) = R(U, V, W, Z) &- g(h(U, W), h(V, Z)) \\ &+ g(h(U, Z), h(V, W)), \end{aligned} \tag{2.14}$$

$$[\overline{R}(U, V)W]^{\perp} = (\overline{\nabla}_U h)(V, W) - (\overline{\nabla}_V h)(U, W). \tag{2.15}$$

A submanifold M of $\overline{M}$ is said to be a *CR-submanifold* if there exists on M, a differentiable holomorphic distribution D such that its orthogonal complementary distribution $D^{\perp}$ is totally real, i.e., $JD_x \subset T_x(M)$ and $JD^{\perp}_x \subset T^{\perp}_x(M)$ for each $x \in M$.

For a CR-submanifold of an almost Hermitian manifold $\overline{M}$, we have

$$TM = D \oplus D^{\perp}, \tag{2.16}$$

$$T^{\perp}M = JD^{\perp} \oplus \mu, \tag{2.17}$$

where μ denotes the orthogonal complementary distribution of $JD^{\perp}$ and is an invariant normal subbundle of $T^{\perp}M$ under J.

The orthogonal projections on TM are denoted by B and C, i.e., for any $U \in TM$

$$U = BU + CU, \tag{2.18}$$

where $BU \in D$ and $CU \in D^{\perp}$. It is straight forward to observe that

$$\begin{array}{llll} \text{(a)} & PC = 0, & \text{(b)} & FB = 0, \\ \text{(c)} & t(T^{\perp}M) = D^{\perp}, & \text{(d)} & f(T^{\perp}M) \subset \mu. \end{array} \tag{2.19}$$

Furthermore, for $U, V \in TM$ if we denote by $\mathcal{P}_U V$ and $\mathcal{Q}_U V$, the tangential and normal parts of $(\overline{\nabla}_U J)V$, then by making use of (2.4)-(2.10), we obtain

$$\mathcal{P}_U V = (\overline{\nabla}_U P)V - A_{FV}U - th(U, V), \tag{2.20}$$

$$\mathcal{Q}_U V = (\overline{\nabla}_U F)V + h(U, PV) - fh(U, V). \tag{2.21}$$

Similarly, for $N \in T^{\perp}M$, the tangential and normal parts of $(\overline{\nabla}_U J)N$ are respectively given by

$$\mathcal{P}_U N = (\overline{\nabla}_U t)N + PA_N U - A_{fN}U, \tag{2.22}$$

$$\mathcal{Q}_U N = (\overline{\nabla}_U f)N + h(tN, U) + FA_N U. \tag{2.23}$$

Let (N_1, g_1) and (N_2, g_2) be two Riemannian manifolds and f, a positive differentiable function on N_1. The warped product of N_1 and N_2 is the Riemannian manifold

$$M = N_1 \times_f N_2 = (N_1 \times N_2, g),$$

where $g = g_1 + f^2 g_2$ [1]. N_1 is called the base of M, and N_2 the fibre.

For a warped product manifold $N_1 \times_f N_2$, we denote by D_1 and D_2 the distributions defined by the vectors tangent to the leaves and fibers respectively. In other words, D_1 is obtained by the tangent vectors of N_1 via the horizontal lift and D_2 is obtained by the tangent vectors of N_2 via the vertical lift. In case of CR-warped product submanifolds D_1 and D_2 are replaced by D and $D^\perp$ respectively.

A warped product $N_1 \times_f N_2$ is said to be a trivial warped product if its warping function f is constant. A trivial warped product $N_1 \times_f N_2$ is nothing but a Riemannian product $N_1 \times N_2^f$, where N_2^f is the Riemannian manifold with Riemannian metric $f^2 g_{N_2}$ which is homothetic to the original metric g_{N_2} of N_2.

3. Some basic results

A submanifold M of an almost Hermitian manifold $\overline{M}$ is said to be a CR-*product submanifold* if M is locally a Riemannian product of a holomorphic submanifold N_T and a totally real submanifold $N_\perp$. Thus a CR-submanifold of an almost Hermitian manifold is a CR-product if and only if both the distributions D and $D^\perp$ on M are integrable and their leaves are totally geodesic in M. It is proved that a CR-product in a complex space form is a product of a holomorphic submanifold and a totally real submanifold of complex linear subspaces and there do not exist CR-products in complex hyperbolic spaces. Moreover, CR-submanifolds in complex projective spaces CP^{h+p+hp} are obtained from Segre imbedding in a natural way (cf. [6]).

For a CR-submanifold of a Kaehler manifold, Chen [2] proved:

Theorem 3.1 ([2]). *A* CR-*submanifold of a Kaehler manifold is a* CR-*product if and only if* $\overline{\nabla} P = 0$ *or equivalently* $A_{JD^\perp} D = 0$.

K. A. Khan et.al [10] worked out conditions for the two distributions on a CR-submanifold of a nearly Kaehler manifold to be integrable and parallel, which led to a characterization for a CR-submanifold to be a CR-product in a nearly Kaehler manifold. We recall:

Proposition 3.1 ([10]). *The holomorphic distribution D on a* CR-*submanifold M of a nearly Kaehler manifold is integrable if and only if*

$$\mathcal{Q}_X Y = 0 \quad \textit{and} \quad h(X, JY) = h(JX, Y)$$

for each X, Y in D.

Proposition 3.2 ([10])**.** *The totally real distribution $D^\perp$ on a* CR*-submanifold M of a nearly Kaehler manifold is integrable if and only if*

$$g(\mathcal{P}_Z W, X) = 0, \quad \textit{or equivalently,} \quad g(A_{JZ}W, X) = g(A_{JW}Z, X).$$

CR-submanifolds which are warped products have the forms $N_\perp \times_f N_T$ and $N_T \times_f N_\perp$. These warped product submanifolds are known as warped product CR-submanifold and CR-warped product submanifolds respectively. Chen [4] proved that warped product CR-submanifolds of a Kaehler manifold are trivial, i.e., they are simply CR-products. Recently, the result is extended to the setting of nearly Kaehler manifolds, i.e., warped product CR-submanifolds of nearly Kaehler manifolds are CR-products (cf. [12]). However, many examples of CR-warped product submanifolds of a Kaehler manifold are provided in [4].

Earlier, K. Sekigawa [14] while studying submanifolds of S^6 proved:

Theorem 3.2 ([14])**.** *There does not exist a* CR*-product in S^6.*

This paved way to study CR-warped product submanifolds in S^6 and in a nearly Kaehler manifold in general. Sekigawa [14] obtained an example of a CR-warped product submanifold in S^6. For a more general case, N. Ejiri [7] provided a categorical answer to the existence of warped product submanifolds in S^6. He proved:

Theorem 3.3 ([7])**.** *There exist countably many immersions of $S^1 \times S^{n-1}$ into S^{n+1} such that the induced metric on it is a warped product metric of constant scaler curvature $n(n-1)$.*

Moreover, every Riemannian manifold of constant scalar curvature c can be locally expressed as a warped product whose warping function satisfies $\triangle f = cf$. For example, $S^n(1)$ is locally isometric to $(-\pi/2, \pi/2) \times_{\cos t} S^{n-1}(1)$, R^n is isometric to $(0, \infty) \times_x S^{n-1}(1)$ and $H^{n-1}(-1)$ is locally isometric to $R \times_{e^x} R^{n-1}$.

As far as the intrinsic geometry of a warped product manifold is concerned Bishop and O'Neill obtained:

Theorem 3.4 ([1])**.** *Let $M = N_1 \times_f N_2$ be a warped product manifold. Then*

(i) *N_1 is a totally geodesic submanifold of M and*

(ii) *N_2 is a totally umbilical submanifold of M.*

Moreover,

$$\nabla_U V = \nabla_V U = (U \ln f)V \tag{3.1}$$

and

$$\mathrm{nor}(\nabla_V W) = \frac{-g(V, W)}{f} \nabla f \tag{3.2}$$

for any $U \in D_1$ and $V, W \in D_2$, where $\mathrm{nor}(\nabla_V W)$ *denotes the component of $\nabla_V W$ in D_1 and ∇f denotes the gradient of f and is defined as*

$$g(\nabla f, U) = Uf. \tag{3.3}$$

4. CR-warped products and the canonical structures

In [2], B. Y. Chen obtained various conditions under which a CR-submanifold reduces to a CR-product. In particular, he showed that a CR-submanifold of a Kaehler manifold is a CR-product if and only if $\overline{\nabla}P = 0$. Later he extended the characterization while studying the impact of parallelism of the $(1,1)$ tensor field P on an arbitrary submanifold of a Kaehler manifold. In this case, he proved that $\nabla P = 0$ if and only if the submanifold is locally a Riemannian product of submanifolds which are either holomorphic or totally real or Kaehlerian slant (cf. [3]). As warped product manifolds are generalized version of product manifolds, it is natural to seek analogous conditions under which a CR-submanifold is a CR-warped product submanifold. In this section, we have obtained necessary and sufficient conditions involving P and F, forcing a CR-submanifold to be locally a CR-warped product submanifold. To prove the main theorems, we first obtain some useful relations.

Lemma 4.1. *Let M be a* CR*-warped product submanifold $N_T \times_f N_\perp$ of an almost Hermitian manifold $\overline{M}$. Then*

$$(\overline{\nabla}_Z P)X = (PX \ln f)Z, \tag{4.1}$$

$$(\overline{\nabla}_U P)Z = g(CU, Z)P(\nabla \ln f) \tag{4.2}$$

for each $U \in TM$, $X \in D$ and $Z \in D^\perp$, where $\nabla \ln f$ denotes the gradient of $\ln f$.

Proof. By formula (3.1)

$$\nabla_X Z = \nabla_Z X = (X \ln f)Z,$$

and therefore by making use of formulae (2.9) and (2.19), we obtain (4.1). On the other hand as $PU \in D$ for any $U \in TM$, by formulae (2.9) and (2.19), we find that $(\overline{\nabla}_U P)Z \in D$. Now for any $X \in D$,

$$\begin{aligned} g((\overline{\nabla}_U P)Z, X) &= g(\nabla_U Z, PX) \\ &= -g(Z, \nabla_U PX) \\ &= -g(Z, \nabla_{CU} PX) \\ &= -(PX \ln f)g(CU, Z), \end{aligned}$$

which on taking account of (3.3) gives

$$(\overline{\nabla}_U P)Z = g(CU, Z)P(\nabla \ln f).$$

This proves the lemma. □

Theorem 4.1. *Let $\overline{M}$ be a nearly Kaehler manifold and M be a* CR*-submanifold of $\overline{M}$ with integrable distributions D and $D^\perp$. Then M is a* CR*-warped product submanifold of $\overline{M}$ if and only if*

$$(\overline{\nabla}_U P)U = (PU\mu)CU + ||CU||^2 J\nabla\mu \tag{4.3}$$

for each $U \in TM$ *and* μ *a* C^∞*-function on* M *satisfying* $Z\mu = 0$ *for each* $Z \in D^\perp$*.*

Proof. The relation (4.3) is equivalent to

$$(\overline{\nabla}_U P)V + (\overline{\nabla}_V P)U = (PU\mu)CV + (PV\mu)CU + 2g(CU, CV)J\nabla\mu, \tag{4.4}$$

with $U, V \in TM$. Let M be a CR-warped product submanifold of $\overline{M}$. For any $U \in TM$, we may write

$$(\overline{\nabla}_U P)U = (\overline{\nabla}_{BU} P)BU + (\overline{\nabla}_{CU} P)BU + (\overline{\nabla}_U P)CU. \tag{4.5}$$

The first term in the right hand side of (4.5) is zero as N_T is totally geodesic in M and by Lemma 4.1,

$$(\overline{\nabla}_{CU} P)BU = (PU \ln f)CU, \tag{4.6}$$

$$(\overline{\nabla}_U P)CU = ||CU||^2 P\nabla \ln f. \tag{4.7}$$

From (4.5), (4.6) and (4.7),

$$(\overline{\nabla}_U P)U = (PU \ln f)CU + ||CU||^2 P\nabla \ln f.$$

Conversely, suppose that M is a CR-submanifold of $\overline{M}$ with D and $D^\perp$ involutive on M and such that (4.3) holds for a C^∞-function μ on M with $Z\mu = 0$ for each $Z \in D^\perp$. It follows from (4.4) that

$$(\overline{\nabla}_X P)Y + (\overline{\nabla}_Y P)X = 0 \tag{4.8}$$

for each $X, Y \in D$. Further, as $\overline{M}$ is nearly Kaehler, it follows from (2.2) that

$$\mathcal{P}_U V + \mathcal{P}_V U = 0, \quad \text{and} \quad \mathcal{Q}_U V + \mathcal{Q}_V U = 0 \tag{4.9}$$

for each $U, V \in TM$. From (4.8) and the first part of (4.9), it follows on using (2.20) that $th(X, Y) = 0$. That means

$$h(X, Y) \in \mu \tag{4.10}$$

for each $X, Y \in D$. Now, for $Z \in D^\perp$

$$g(\nabla_X Y, Z) = g(J\overline{\nabla}_X Y, JZ).$$

As D is involutive, on using Proposition 3.1, formula (2.2) and the observation (4.10), the above equation yields

$$g(\nabla_X Y, Z) = 0.$$

This proves that the leaves of D are totally geodesic in M. On the other hand by formulae (2.21) and (4.9)

$$(\overline{\nabla}_X P)Z + (\overline{\nabla}_Z P)X = A_{FZ}X + 2th(X, Z), \tag{4.11}$$

whereas by (4.4),

$$(\overline{\nabla}_X P)Z + (\overline{\nabla}_Z P)X = (PX\mu)Z \tag{4.12}$$

for each $X \in D$ and $Z \in D^\perp$. From (4.11) and (4.12),

$$g(A_{FZ}X, W) - 2g(h(X, Z), JW) = (PX\mu)g(Z, W) \tag{4.13}$$

for each $W \in D^\perp$. As $D^\perp$ is involutive, in view of Proposition 3.2, (4.13) gives

$$g(h(X,Z),JW) = -(PX\mu)g(Z,W)$$

or

$$g(\nabla_Z JX - \mathcal{P}_Z X, W) = (PX\mu)g(Z,W).$$

On taking account of Proposition 3.2, the above relation yields

$$g(\nabla_Z W, JX) = -(PX\mu)g(Z,W). \tag{4.14}$$

Let N_T and $N_\perp$ denote the leaves of D and $D^\perp$ respectively. If h' denotes the second fundamental form of the immersion of $N_\perp$ into M, then (4.14) can be written as

$$g(h'(Z,W),JX) = (JX\mu)g(Z,W) = g(\nabla\mu, JX)g(Z,W).$$

Thus we obtain

$$h'(Z,W) = g(Z,W)\nabla\mu.$$

Thus each leaf $N_\perp$ of $D^\perp$ is totally umbilical in M and as $Z\mu = 0$ for each $Z \in D^\perp$, the mean curvature vector $\nabla\mu$ is parallel on $N_\perp$, i.e., $N_\perp$ is an extrinsic sphere in M. Hence by virtue of a theorem in [9], which states that "If the tangent bundle of a Riemannian manifold M splits into an orthogonal sum $TM = E_0 \oplus E_1$ of non trivial vector subbundles such that E_1 is spherical and its orthogonal complement E_0 is autoparallel, then the manifold M is locally isometric to a warped product $M_0 \times_f M_1$", we get that M is locally a CR-warped product submanifold $N_T \times_f N_\perp$, where $f = e^\mu$. □

In terms of the structure F, we have:

Theorem 4.2. *Let M be a* CR*-submanifold of a nearly Kaehler manifold $\overline{M}$ with totally real distribution being involutive. Then M is locally a* CR*-warped product submanifold if and only if*

$$g((\overline{\nabla}_U F)V, JW) = g(\mathcal{Q}_{CU}CV, JW) - (BV\mu)g(CU,W) \tag{4.15}$$

for each $U, V \in TM$ and $W \in D^\perp$, where μ is a C^∞-function on M such that $Z\mu = 0$ for each $Z \in D^\perp$.

Proof. Let M be a CR-warped product submanifold $N_T \times_f N_\perp$. Then for any $X, Y \in D$ and $W \in D^\perp$, by formula (2.10)

$$g((\overline{\nabla}_X F)Y, JW) = -g(F\nabla_X Y, JW) = -g(\nabla_X Y, W).$$

Therefore, as N_T is totally geodesic in M

$$g((\overline{\nabla}_X F)Y, JW) = 0. \tag{4.16}$$

On the other hand, for any $X \in D$ and $Z, W \in D^\perp$, by formula (2.21)

$$\begin{aligned} g(\overline{\nabla}_X F)Z, JW) &= g(\mathcal{Q}_X Z + fh(X,Z), JW) \\ &= g(\mathcal{Q}_X Z, JW) \\ &= -g(\mathcal{P}_Z JX, W) \\ &= g(JX, \mathcal{P}_Z W). \end{aligned}$$

Thus, by Proposition 3.2, we have

$$g(\overline{\nabla}_X F)Z, JW) = 0. \tag{4.17}$$

Further,

$$g((\overline{\nabla}_Z F)X, JW) = -g(F\nabla_Z X, JW),$$

which on using (3.1) gives that

$$g((\overline{\nabla}_Z F)X, JW = -(X \ln f)g(Z, W), \tag{4.18}$$

and for any $W' \in D^{\perp}$, by formula (2.21) we have

$$\begin{aligned} g((\overline{\nabla}_Z F)W', JW) &= g(\mathcal{Q}_Z W' + fh(Z, W'), JW) \\ &= g(\mathcal{Q}_Z W', JW). \end{aligned} \tag{4.19}$$

Now, for $U, V \in TM$ we may write

$$\begin{aligned} g((\overline{\nabla}_U F)V, JW) = g((\overline{\nabla}_{BU} F)BV, JW) + g((\overline{\nabla}_{BU} F)CV, JW) \\ + g((\overline{\nabla}_{CU} F)BV, JW) + g((\overline{\nabla}_{CU} F)CV, JW). \end{aligned} \tag{4.20}$$

The first two terms in the right hand side of the above equation are zero in view of (4.16) and (4.17) and the remaining terms in view of (4.18) and (4.19) yield (4.15).

Conversely, suppose that M is a CR-submanifold of a nearly Kaehler manifold $\overline{M}$ such that (4.15) holds. Then obviously

$$g((\overline{\nabla}_X F)Y, JW) = 0$$

for each $X, Y \in D$ and $W \in D^{\perp}$. Therefore $g(\nabla_X Y, W) = 0$, that means D is integrable and its leaves are totally geodesic in M. Now, for any $Z, W \in D^{\perp}$, by (4.15) we have

$$g((\overline{\nabla}_Z F)X, JW) = -(X\mu)g(Z, W)$$

or

$$g(\nabla_Z W, X) = -(X\mu)g(Z, W). \tag{4.21}$$

Let $N_{\perp}$ be a leaf of $D^{\perp}$. If ∇' denotes the induced Riemannian connection on $N_{\perp}$ and h', the second fundamental form of the immersion of $N_{\perp}$ into M, then the last equation in view of Gauss formula is written as

$$g(X, \nabla'_Z W + h'(Z, W)) = -(X\mu)g(Z, W)$$

or

$$\begin{aligned} g(X, h'(Z, W)) &= -(X\mu)g(Z, W) \\ &= -g(X, \nabla\mu)g(Z, W), \end{aligned}$$

or

$$h'(Z, W) = -g(Z, W)\nabla\mu.$$

This shows that $N_{\perp}$ is totally umbilical in M and in view of the condition that $Z\mu = 0$ for each $Z \in D^{\perp}$, $\nabla\mu$ is defined on N_T only, i.e., the mean curvature $\nabla\mu$ is parallel on $N_{\perp}$. In other words, $N_{\perp}$ is an extrinsic sphere.

Hence, by a similar argument as given in Theorem 4.1, M is locally isometric to a warped product submanifold with a warping function $f = e^{\mu}$. □

Let $M = N_T \times_f N_\perp$ be a CR-warped product submanifold of a nearly Kaehler manifold $\overline{M}$. In view of the decomposition (2.17), we may write

$$h(U,V) = h_{JD^\perp}(U,V) + h_\mu(U,V) \tag{4.22}$$

for each $U, V \in TM$, where $h_{JD^\perp}(U,V) \in JD^\perp$ and $h_\mu(U,V) \in \mu$.

If $\{e_1, e_2, \ldots, e_n\}$ be a local orthonormal frame of vector fields on M, then we define

$$||h||^2 = \sum_{i,j=1}^{n} g(h(e_i,e_j), h(e_i,e_j)),$$

and for a differentiable function f on M, the *Laplacian* $\triangle f$ of f is defined by

$$\triangle f = \sum_{j=1}^{n} [(e_j(e_j f) - \nabla_{e_j} e_j) f]. \tag{4.23}$$

Proposition 4.1. *Let M be a* CR-*warped product submanifold of a nearly Kaehler manifold $\overline{M}$. Then*

(i) $h_{JD^\perp}(JX,Z) = (X \ln f)JZ$,

(ii) $g(\mathcal{Q}_Z X, JW) = 0$,

(iii) $g(h(JX,Z), Jh(X,Z)) = ||h_\mu(X,Z)||^2 - g(\mathcal{Q}_X Z, Jh_\mu(X,Z))$

for each $X \in D$ and $Z, W \in D^\perp$.

Proof. By Gauss formula,

$$\begin{aligned} h(JX,Z) &= \overline{\nabla}_Z JX - \nabla_Z JX \\ &= (\overline{\nabla}_Z J)X + J\nabla_Z X + Jh(X,Z) - \nabla_Z JX. \end{aligned}$$

Thus, on using (3.1), we obtain

$$h(JX,Z) = \mathcal{P}_Z X + \mathcal{Q}_Z X + (X \ln f)JZ + Jh(X,Z) - (JX \ln f)Z. \tag{4.24}$$

Comparing tangential parts in (4.24), we get

$$\mathcal{P}_Z X = (JX \ln f)Z - Jh_{JD^\perp}(X,Z).$$

Taking product with W in both sides, yields

$$g(\mathcal{P}_Z X, W) = (JX \ln f)g(Z,W) + g(h_{JD^\perp}(X,Z), JW).$$

The left hand side of the last equation is zero in view of Proposition 3.2 and thus the equation reduces to

$$g(h(X,Z), JW) = -(JX \ln f)g(Z,W),$$

or equivalently,

$$h_{JD^\perp}(JX,Z) = (X \ln f)JZ. \tag{4.25}$$

This proves (i). Now, on comparing the normal parts in (4.24), we get

$$h(JX,Z) = \mathcal{Q}_Z X + (X \ln f)JZ + Jh_\mu(X,Z)$$

or

$$h(JX,Z) - Jh_\mu(X,Z) = \mathcal{Q}_Z X + (X \ln f)JZ. \tag{4.26}$$

Taking product with JW in (4.26) and using (4.25), we obtain statement (ii) of the proposition, i.e.,

$$g(\mathcal{Q}_Z X, JW) = 0.$$

Now, by (4.26)

$$g(h(JX,Z), Jh(X,Z)) = g(Jh_\mu(X,Z) + \mathcal{Q}_Z X, Jh_\mu(X,Z)).$$

Or,

$$g(h(JX,Z), Jh(X,Z)) = ||h_\mu(X,Z)||^2 - g(\mathcal{Q}_X Z, Jh_\mu(X,Z)). \tag{4.27}$$

This proves the proposition completely. □

For CR-warped products in nearly Kaehler manifolds, we have the following:

Theorem 4.3. *Let $M = N_T \times_f N_\perp$ be a* CR-*warped product submanifold of a nearly Kaehler manifold $\overline{M}$. Then we have*

(i) *The squared norm of the second fundamental form satisfies*

$$||h||^2 \geq 2q||\nabla \ln f||^2, \tag{4.28}$$

where $\nabla \ln f$ is the gradient of $\ln f$ and q is the dimension of $N_\perp$.

(ii) *If the equality sign in* (4.28) *holds identically, then N_T is totally geodesic submanifold of $\overline{M}$, $N_\perp$ a totally umbilical submanifold of $\overline{M}$ and M is a minimal submanifold of $\overline{M}$.*

Proof. Let $\{X_1, X_2, \ldots, X_p, X_{p+1} = JX_1, \ldots, X_{2p} = JX_p\}$ be a local orthonormal frame of vector fields on N_T and $\{Z_1, Z_2, \ldots, Z_q\}$ a local orthonormal frame on $N_\perp$. Then by definition

$$\begin{aligned} ||h||^2 = &\sum_{i,j=1}^{2p} g(h(X_i,X_j), h(X_i,X_j)) + \sum_{i=1}^{2p}\sum_{r=1}^{q} g(h(X_i,Z_r), h(X_i,Z_r)) \\ &+ \sum_{r,s=1}^{q} g(h(Z_r,Z_s), h(Z_r,Z_s)). \end{aligned} \tag{4.29}$$

Thus,

$$||h||^2 \geq \sum_{i=1}^{2p}\sum_{r=1}^{q} g(h(X_i,Z_r), h(X_i,Z_r))$$

or

$$\begin{aligned} ||h||^2 &\geq \sum_{i=1}^{2p}\sum_{r=1}^{q} (X_i \ln f)^2 g(Z_r, Z_r) \\ &\geq 2q||\nabla \ln f||^2. \end{aligned}$$

This verifies the assertion (i). If the equality sign in (4.28) holds, then by (4.29) and (4.25), we obtain

$$h(D,D)=0,\quad h(D^{\perp},D^{\perp})=0 \quad \text{and} \quad h(D,D^{\perp})\subset JD^{\perp}, \tag{4.30}$$

since N_T is a totally geodesic submanifold of M, the first condition in (4.30) implies that N_T is totally geodesic in $\overline{M}$. Moreover as $N_\perp$ is totally umbilical in M, the second condition in (4.30) implies that $N_\perp$ is totally umbilical in $\overline{M}$. It also follows from (4.30) that M is minimal in $\overline{M}$. □

5. CR-warped products in a generalized complex space form

Throughout the section, we denote by $\overline{M}(c,\alpha)$ a generalized complex space form of constant curvature c and of constant type α. Let $M = N_T \times_f N_\perp$ be a CR-warped product submanifold of $\overline{M}(c,\alpha)$.

Various inequalities involving the squared norm of the second fundamental form as well as that of the mean curvature vector of a CR-warped product submanifolds in real and complex space forms are obtained (cf. [5], [6] etc). For example, every CR-warped product $N_T \times_f N_\perp$ in a complex space form $\overline{M}(4c)$ satisfies the general inequality

$$||h||^2 \geq 2q\{||\nabla \ln f||^2 + \triangle \ln f\} + 4pqc, \tag{5.1}$$

where $p = \dim_C N_T$, $q = \dim_R N_\perp$ [6].

Now, as the class of generalized complex space forms includes nearly Kaehler manifolds of constant holomorphic sectional curvature, it is interesting to study similar estimates in the setting of generalized complex space forms. To this end we prove:

Theorem 5.1. *Let $M = N_T \times_f N_\perp$ be a* CR*-warped product submanifold of a generalized complex space form $\overline{M}(c,\alpha)$. Then we have*

$$||h||^2 \geq 2q\left\{||\nabla \ln f||^2 + \frac{1}{2}(\triangle \ln f) + \frac{p(c-\alpha)}{4}\right\}, \tag{5.2}$$

where h denotes the second fundamental form of the immersion of M into $\overline{M}(c,\alpha)$, $\nabla \ln f$ is the gradient of $\ln f$, $\triangle \ln f$ is the Laplacian of $\ln f$, $2p$ and q are the real dimensions of N_T and $N_\perp$ respectively.

Proof. For $X \in D$ and $Z \in D^\perp$, by formula (2.3), we have

$$\overline{R}(X,JX,Z,JZ) = \left(\frac{\alpha - c}{2}\right) g(X,X)g(Z,Z). \tag{5.3}$$

On the other hand by Codazzi equation

$$\begin{aligned}\overline{R}(X,JX,Z,JZ) =& g(\nabla^{\perp}_X h(JX,Z),JZ) - g(h(\nabla_X JX,Z),JZ)\\ &- g(h(JX,\nabla_X Z),JZ) - g(\nabla^{\perp}_{JX} h(X,Z),JZ)\\ &+ g(h(\nabla_{JX}X,Z),JZ) + g(h(X,\nabla_{JX}Z),JZ).\end{aligned} \tag{5.4}$$

Now,

$$g(\nabla^{\perp}_X h(JX,Z),JZ) = Xg(h(JX,Z),JZ) - g(h(JX,Z),\overline{\nabla}_X JZ). \tag{5.5}$$

The first term in the right hand side of (5.5), on using formulae (4.25) and (3.1) takes the form

$$Xg(h(JX,Z),JZ) = (X(X\ln f) + 2(X\ln f)^2)g(Z,Z), \tag{5.6}$$

whereas the second term is written as

$$\begin{aligned} g(h(JX,Z),\overline{\nabla}_X JZ) &= g(h(JX,Z),\mathcal{Q}_X Z + J\nabla_X Z + Jh(X,Z)) \\ &= g(h(JX,Z),\mathcal{Q}_X Z) + (X\ln f)^2 g(Z,Z) \\ &\quad + g(h(JX,Z),Jh(X,Z)). \end{aligned}$$

Making use of (4.27), the above equality takes the form

$$\begin{aligned} g(h(JX,Z),\overline{\nabla}_X JZ) = g(h(JX,Z) - Jh(X,Z),\mathcal{Q}_X Z) \\ + (X\ln f)^2 g(Z,Z) + ||h_\mu(X,Z)||^2. \end{aligned}$$

On using (4.26), the above relation can be written as

$$g(h(JX,Z),\overline{\nabla}_X JZ) = (X\ln f)^2 g(Z,Z) + ||h_\mu(X,Z)||^2 - ||\mathcal{Q}_X Z||^2. \tag{5.7}$$

On substituting from (5.6) and (5.7), equation (5.5) yields

$$\begin{aligned} g(\nabla^{\perp}_X h(JX,Z),JZ) =&(X(X\ln f) + (X\ln f)^2)g(Z,Z) \\ &+ ||\mathcal{Q}_X Z||^2 - ||h_\mu(X,Z)||^2. \end{aligned} \tag{5.8}$$

Similarly, we obtain

$$\begin{aligned} g(\nabla^{\perp}_{JX} h(X,Z),JZ) =& - (JX(JX\ln f) + (JX\ln f)^2)g(Z,Z) \\ &- ||\mathcal{Q}_{JX} Z||^2 + ||h_\mu(JX,Z)||^2. \end{aligned} \tag{5.9}$$

By formulae (3.1) and (4.25), we have

$$g(h(JX,\nabla_X Z),JZ) = (X\ln f)^2 g(Z,Z), \tag{5.10}$$

and

$$g(h(X,\nabla_{JX} Z),JZ) = -(JX\ln f)^2 g(Z,Z). \tag{5.11}$$

Further by making use of formula (3.1) and the fact that N_T is totally geodesic in M, we get

$$g(h(\nabla_{JX} X,Z),JZ) = -(J\nabla_{JX} X)(\ln f)g(Z,Z), \tag{5.12}$$

and,

$$g(h(\nabla_X JX,Z),JZ) = -((J\nabla_X JX)\ln f)g(Z,Z).$$

The right hand side of the above equation, on making use of the fact that N_T is totally geodesic in M and the formula (3.1) reduces to $-g(\nabla_Z J\nabla_X JX, Z)$. Thus, by using Gauss formula, we get

$$\begin{aligned}(5.13)\quad g(h(\nabla_X JX, Z), JZ) = &-g(\nabla_X X, \nabla_Z Z) - g(Jh(X, JX), \nabla_Z Z)\\ = &((\nabla_X X)\ln f)g(Z,Z) + ((\nabla_{JX} JX)\ln f)g(Z,Z)\\ &-((J\nabla_{JX} X \ln f)g(Z,Z).\end{aligned}$$

Let $\{X_1, X_2, \ldots, X_p, X_{p+1} = JX_1, \ldots, X_{2p} = JX_p\}$ and $\{Z_1, Z_2, \ldots, Z_q\}$ be a local frame of orthonormal vector fields on N_T and $N_\perp$ respectively. Choosing X, Z as basic vector fields and substituting from (5.8)-(5.13) into (5.4), we obtain

$$\begin{aligned}&\overline{R}(X_i, JX_i, Z_r, JZ_r)\\ &= (X_i(X_i \ln f) + JX_i(JX_i \ln f))g(Z_r, Z_r)\\ &\quad - ((\nabla_{X_i} X_i)\ln f + (\nabla_{JX_i} JX_i)\ln f)g(Z_r, Z_r)\\ &\quad + ||\mathcal{Q}_{X_i} Z_r||^2 + ||\mathcal{Q}_{JX_i} Z_r||^2 - ||h_\mu(X_i, Z_r)||^2 - ||h_\mu(JX_i, Z_r)||^2.\end{aligned}$$

Summing both sides over $i = 1, 2, \ldots, p$ and $r = 1, 2, \ldots, q$ and making use of (5.3) and (4.23), we obtain

$$(5.14)\qquad \frac{pq(\alpha - c)}{2} = ||\mathcal{Q}_D D^\perp||^2 - ||h_\mu(D, D^\perp)||^2 + q\triangle \ln f,$$

where we have used the following notations

$$||\mathcal{Q}_D D^\perp||^2 = \sum_{i=1}^{2p}\sum_{r=1}^{q} ||\mathcal{Q}_{X_i} Z_r||^2,$$

and,

$$||h_\mu(D, D^\perp)||^2 = \sum_{i=1}^{2p}\sum_{r=1}^{q} ||h_\mu(X_i, Z_r)||^2.$$

Further, if we denote $\sum_{i=1}^{2p}\sum_{r=1}^{q} ||h_{JD^\perp}(X_i, Z_r)||^2$ by $||h_{JD^\perp}(D, D^\perp)||^2$, then from (4.29), we have

$$(5.15)\qquad ||h_{JD^\perp}(D, D^\perp)||^2 = 2q||\nabla \ln f||^2,$$

whereas from (5.14), we have

$$(5.16)\qquad ||h_\mu(D, D^\perp)||^2 = ||\mathcal{Q}_D D^\perp||^2 + \frac{pq(c - \alpha)}{2} + q\triangle \ln f.$$

On adding (5.15) and (5.16), we get

$$||h(D, D^\perp)||^2 = 2q||\nabla \ln f||^2 + q\triangle \ln f + ||\mathcal{Q}_D D^\perp||^2 + \frac{pq(c - \alpha)}{2}.$$

Hence, $||h||^2$ for the CR-warped product submanifold of a generalized complex space form $\overline{M}(c, \alpha)$ satisfy

$$||h||^2 \geq 2q \left\{ ||\nabla \ln f||^2 + \frac{1}{2} \triangle \ln f + \frac{p(c-\alpha)}{4} \right\}.$$

The above inequality generalizes the inequality (5.1).

In particular for the CR-warped product of S^6, the above inequality reduces to

$$||h||^2 \geq 2q \left\{ ||\nabla \ln f||^2 + \frac{\triangle \ln f}{2} \right\},$$

which improves the inequality (4.28). □

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