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AN UPPER BOUND ON THE NUMBER OF PARITY CHECKS FOR BURST ERROR DETECTION AND CORRECTION IN EUCLIDEAN CODES

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Abstract. There are three standard weight functions on a linear code viz. Hamming weight, Lee weight, and Euclidean weight. Euclidean weight function is useful in connection with the lattice constructions [2] where the minimum norm of vectors in the lattice is related to the minimum Euclidean weight of the code. In this paper, we obtain an upper bound over the number of parity check digits for Euclidean weight codes detecting and correcting burst errors.

1. Introduction

There are three standard weight functions (or equivalently distance/metric functions) on a linear code viz. Hamming weight [1, 4, 14], Lee weight [7, 10, 11, 12, 13] and Euclidean weight [2, 6]. The choice of a metric for a given communication system plays an important role as the channel model should match the metric d to be employed for developing a suitable code, and hence for a communication system to operate reliably. Thus given a modulation scheme, one metric may be better suited than another. Euclidean weight (or Euclidean square distance) is useful in connection with the latice construction where the minimum norm of vectors in the lattice is related to the minimum Euclidean weight of the code [2]. Also, it is well known that during the process of transmission errors occur predominantly in the from of bursts. However, it does not generally happen that all the digits inside any burst length gets corrupted. In other words, the weight of the burst errors are not large. Codes developed to detect and correct burst errors with respect to Hamming and Lee weight functions have been studied by many authors [3, 5, 10, 12] and [15]. In this paper, we first obtain an upper bound for Euclidean codes detecting burst errors and then we obtain an upper bound for codes correcting such type of errors.

In what follows, we consider the following:

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Let $\mathbb{Z}_q$ be the ring of integers modulo q. Let V_q^n be the set of all n-(tuples) over $\mathbb{Z}_q$. Then V_q^n is a module over $\mathbb{Z}_q$. Let V be a submodule of the module V_q^n over $\mathbb{Z}_q$. For q prime, $\mathbb{Z}_q$ becomes a field and correspondingly V_q^n and V become the vector space and subspace respectively over the field $\mathbb{Z}_q$. Also, we define the Euclidean value $|a|^2$ of an element $a \in \mathbb{Z}_q$ by

$$|a|^2 = \begin{cases} a^2 & \text{if} \quad 0 \leq \ a \leq q/2, \\ (q-a)^2 & \text{if} \quad q/2 < a \leq q-1, \end{cases}$$

or in other words

$$|a|^2 = \min(a^2, (q-a)^2),$$

then, for a given vector $u = (a_0, a_1, \ldots, a_{n-1}), a_i \in \mathbb{Z}_q$, the Euclidean weight $w_E(u)$ of u is given by

$$w_E(u) = \sum_{i=0}^{n-1} |a_i|^2.$$

Note that in determining the Euclidean weight of vector, a nonzero entry a has a contribution $|a|^2$ which is obtained by two different entries a and $q-a$ provided $\{q$ is odd $\}$ or $\{q$ is even and $a \neq q/2\}$. That is,

$$|a|^2 = |q-a|^2 \quad \text{if} \quad \begin{cases} q \quad \text{is odd} \\ \text{or} \\ q \text{ is even and} \quad a \neq q/2. \end{cases}$$

If q is even and $a = q/2$ or if $a = 0$, then $|a|^2$ is obtained in only one way viz. $|a|^2 = a^2$.

Thus for the Euclidean weight, there may be one or two entries from $\mathbb{Z}_q$ having the same Euclidean value $|a|^2$ and we call these entries as repetitive equivalent Euclidean values of a. The number of repetitive equivalent Euclidean values of a will be denoted by e_a, where

$$e_a = \begin{cases} 1 & \text{if } \{\ q \quad \text{is even and } a = q/2\} \text{ or } \{a = 0\} \\ 2 & \text{if } \{\ q \text{ is odd and } a \neq 0\} \text{ or } \{q \text{ is even, } a \neq 0 \text{ and } a \neq q/2\}. \end{cases}$$

The Euclidean square distance between the two vectors $u=(a_0, a_1, \ldots, a_{n-1})$ and $v = (b_0, b_1, \ldots, b_{n-1})$ is defined as the Euclidean weight of their difference, i.e.,

$$d_E^2(u, v) = w_E(u - v).$$

The minimum Euclidean square distance of a code is the smallest Euclidean square distance between all its distinct pair of code words. Also, minimum Euclidean square distance (d_E^2) and minimum Euclidean weight of a code coincide.

We note that Euclidean weight (or equivalent Euclidean square distance) of a vector over $\mathbb{Z}_q$ can assume values which can be expressed as sum of squares of positive integers and the integers are chosen from the set $\{0, 1, 2, 3, \ldots, [q/2]\}$.

We shall denote $[x]$ as the largest integer less than or equal to x.

2. An upper bound for Euclidean weight codes detecting burst errors

In this section, we obtain an upper bound on the number of parity check digits for codes detecting burst errors of length b or less with Euclidean weight w_E or less $(1 \leq w_E \leq b[q/2]^2)$.

To prove the result, we first prove the following lemma:

Lemma 2.1. *If $V_t^{(n)}$ denotes the number of all n-tuples of Euclidean weight t or less over $\mathbb{Z}_q$, then $V_t^{(n)}$ is given by*

$$V_t^{(n)} = \sum_{\eta=0}^{t} A_\eta^{(n)}, \tag{1}$$

where

$$A_\eta^{(n)} = \begin{cases} \displaystyle\sum_{N=1}^{[q/2]} \sum_{r_0, r_1 \ldots, r_N} \frac{n!}{r_0! r_1! \cdots r_N!} e_1^{r_1} e_2^{r_2} \cdots e_N^{r_N} & \text{for } \eta > 0 \\ 1 & \text{for } \eta = 0, \end{cases} \tag{2}$$

and $r_i's$ are integers such that

$$\begin{aligned} &r_0 + r_1 + r_2 + \cdots + r_N = n, \qquad r_N \geq 1, \qquad r_i \geq 0 \quad (i \neq N), \\ &1^2 r_1 + 2^2 r_2 + \cdots + N^2 r_N = \eta. \end{aligned} \tag{3}$$

Proof. Clearly, $A_\eta^{(n)} = 1$ for $\eta = 0$ is obvious. So, assume $\eta \geq 1$. We consider partitions of the integer η $(1 \leq \eta \leq t)$, the largest entry in which has an equivalent Euclidean value N^2 $(1 \leq N \leq [q/2]$). If r_i is the number of times, entries with equivalent Euclidean value $|i|^2$ occurs in the partition, then the number of vectors of length n that can be formed by filling n positions from the integers $0, 1, \ldots, N$ is given by

$$\frac{n!}{r_0! r_1! \cdots r_N!} e_1^{r_1} e_2^{r_2} \cdots e_N^{r_N}. \tag{4}$$

Condition (3) follows immediately as the total number of entries is n and the sum of Euclidean values of the entries is η. Now summing (4) for all possible values of $r_i's$ and of N $(1 \leq N \leq [q/2])$, we get (2). Finally, $V_t^{(n)}$ is obtained by summing $A_\eta^{(n)}$ for all possible values of η which range from 0 to t. Hence the proof is completed. □

We now give a definition:

Definition 2.1. A linear combination of n-vectors $u_1, u_2, \ldots, u_n$ given by

$$\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n,$$

where $\alpha_i \in \mathbb{Z}_q, u_i \in \mathbb{Z}_q^n$ $(1 \leq i \leq n)$ is called a linear combination of Euclidean weight w_E if

$$\text{Euclidean weight}(\alpha_1, \alpha_2, \ldots, \alpha_n) = w_E.$$

Theorem 2.1. *Given positive integers b and w_E $(1 \leq w_E \leq b[q/2]^2)$, a sufficient condition that there exists an (n, k) linear code over $\mathbb{Z}_q$ (q prime) that has no burst of length b or less with Euclidean weight w_E or less as a code word is*

$$q^{n-k} > 1 + \sum_{l=1}^{[q/2]} e_l V_{w_E - l^2}^{(b-1)}. \tag{5}$$

Proof. The existence of such a code will be proved by constructing a suitable $(n-k) \times n$ parity check matrix H for the desired code. We select any non zero $(n-k)$-tuple as the first column of H. Subsequent columns are added to H in such a way that after having selected $j-1$ columns $h_1, h_2, \ldots, h_{j-1}$ suitably, a nonzero $(n-k)$-tuple is chosen as the j^{th} column h_j such that

$$\lambda h_j \neq \lambda_{i_1} h_{i_1} + \lambda_{i_2} h_{i_2} + \cdots + \lambda_{i_p} h_{i_p}, \tag{6}$$

where

$$\lambda, \lambda_{i_j} \in \mathbb{Z}_q \ (j = 1 \text{ to } p),\ |\lambda_{i_1}|^2 + |\lambda_{i_2}|^2 + \cdots + |\lambda_{i_p}|^2 + |\lambda|^2 \leq w_E$$

and

$$\{h_{i_1}, h_{i_2}, \ldots, h_{i_p}\} \subseteq \{h_{j-1}, h_{j-2}, \ldots, h_{j-b+1}\},$$

i.e., h_j is chosen in such a way that no linear combination of Euclidean weight w_E or less from the immediately preceding $b-1$ columns and column h_j is zero i.e., no linear combination of Euclidean weight w_E or less from the columns $h_{j-b+1}, h_{j-b+2}, \ldots, h_{j-1}, h_j$ is zero. Such a condition will ensure that a burst of length b or less with Euclidean weight w_E or less cannot be a code vector in the code whose parity check matrix is H.

The number of possible linear combinations in equation (6) including the pattern of all zeros is given by

$$1 + e_1 V_{w_E - 1^2}^{(b-1)} + e_2 V_{w_E - 2^2}^{(b-1)} + \cdots + e_{[q/2]} V_{w_E - [q/2]^2}^{(b-1)} = 1 + \sum_{l=1}^{[q/2]} e_l V_{w_E - l^2}^{(b-1)}.$$

Therefore, a column h_j can be added to H provided that this number is less than the total number of $(n-k)$-tuples which is q^{n-k}.

At worst, all these linear combinations might yield a distant sum, therefore, h_j can always be added to H provided that

$$q^{n-k} > 1 + \sum_{l=1}^{[q/2]} e_l V_{w_E - l^2}^{(b-1)}. \tag{7}$$

It is important to note that the relation in equation (7) is independent of j. Therefore, we can go on adding the columns as long as we wish but for code of length j, we shall stop after choosing j columns. So for $j = n$, we shall add up to n columns.

This completes the proof of the theorem. □

Example 2.1. Consider the following 3×4 parity check matrix of a (4,1) linear Euclidean weight code over $\mathbb{Z}_5$:

$$H = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}_{3\times 4}$$

This matrix has been constructed by the synthesis procedure outlined in the proof of Theorem 2.1 by taking $b = 3$ and $w_E = 4$. The code whose parity check matrix is H satisfies the sufficient condition of Theorem 2.1 as shown below:

$$\begin{aligned} 1 + \sum_{l=1}^{[q/2]} e_l V_{w_E - l^2}^{(b-1)} &= 1 + \sum_{l=1}^{2} e_l V_{4-l^2}^{(2)} \\ &= 1 + e_1 V_3^{(2)} + e_2 V_0^{(2)} \\ &= 1 + (2 \times 9) + (2 \times 1) \\ &\quad (\text{since } V_3^{(2)} = 9 \text{ and } V_0^{(2)} = 1) \\ &= 21. \end{aligned}$$

Also, $q^{n-k} = 5^3 = 125$.

Therefore,

$$q^{n-k} = 125 > 21 = 1 + \sum_{l=1}^{[q/2]} e_l V_{w_E - l^2}^{(b-1)}.$$

By Theorem 2.1 we get that the code detects all burst errors of length 3 or less with Euclidean weight 4 or less. We verify this fact as follows:

The generator matrix G of the code corresponding to the above given parity check matrix H is given by

$$G = [3 \quad 3 \quad 2 \quad 1]_{1\times 4}.$$

The code words of this code are

$$\begin{aligned} v_0 &= 0000, \text{ Euclidean weight}(v_0) = 0 \\ v_1 &= 3321, \text{ Euclidean weight}(v_1) = 13 \\ v_2 &= 1142, \text{ Euclidean weight}(v_2) = 7 \\ v_3 &= 4413, \text{ Euclidean weight}(v_3) = 7 \\ v_4 &= 2234, \text{ Euclidean weight}(v_4) = 13. \end{aligned}$$

Thus, all the code words of the code whose parity check matrix is H are not bursts of length 3 or less with Euclidean weight 4 or less over $\mathbb{Z}_5$, i.e., code detects all bursts of length 3 or less with Euclidean weight 4 or less over $\mathbb{Z}_5$.

3. An upper bound for Euclidean weight codes correcting burst errors

In this section, we obtain an upper bound on the number of parity check digits for codes correcting burst errors of length b or less with Euclidean weight w_E or less $(1 \leq w_E \leq b[q/2]^2)$.

Theorem 3.1. *A sufficient condition for the existence of an (n,k) linear Euclidean weight code over $\mathbb{Z}_q$ (q prime) which corrects all bursts of length b or less $(n > 2b)$ with Euclidean weight w_E or less $(1 \leq w_E \leq b[q/2]^2)$ is given by*

(8)
$$\begin{aligned} q^{n-k} > 1 &+ \left[\sum_{\lambda=1}^{[q/2]} e_\lambda V^{(b-1)}_{w_E-\lambda^2}\right]\left[\sum_{i=1}^{b}(n-b-i+1)(V^{(i)}_{w_E}-1)\right] \\ &+ \sum_{\lambda=1}^{[q/2]} e_\lambda \left(V^{(b-1)}_{2w_E-\lambda^2} - V^{(b-1)}_{1-\lambda^2}\right) \\ &+ \sum_{\lambda=1}^{[q/2]} e_\lambda \left(\sum_{k=1}^{(b-1)} \sum_{\lambda_1=1}^{\min\{[q/2],[\sqrt{w_E-1}]\}} \sum_{r_{1\lambda_1},r_{2\lambda_1},r_{3\lambda_1}} e_{\lambda_1} A^{(b-k-1)}_{r_{1\lambda_1}} A^{(k)}_{r_{2\lambda_1}} A^{(b-k-1)}_{r_{3\lambda_1}}\right), \end{aligned}$$

where

(9)
$$\begin{aligned} 2-\lambda^2 &\leq \lambda_1^2 + r_{1\lambda_1} + r_{2\lambda_1} + r_{3\lambda_1} \leq 2w_E - \lambda^2, \\ 1 &\leq \lambda_1^2 + r_{1\lambda_1} \leq w_E - 1, \\ 1 &\leq r_{2\lambda_1} \leq 2w_E - 1 - \lambda^2, \\ 0 &\leq r_{3\lambda_1} \leq w_E - \lambda^2, \\ r_{2\lambda_1} &+ r_{3\lambda_1} \geq 1 - \lambda^2, \\ \lambda_1^2 &+ r_{1\lambda_1} + r_{2\lambda_1} \geq 1. \end{aligned}$$

Proof. The existence of such a code will be proved by constructing a suitable $(n-k)\times n$ parity check matrix H for the desired code. We select any nonzero $(n-k)$-tuple as the first column of parity check matrix H. Subsequent columns are added to H in such a way that after having selected $j-1$ columns $h_1, h_2, \ldots, h_{j-1}$ suitably, a nonzero $(n-k)$-tuple is chosen as the j^{th} column h_j such that αh_j $(1 \leq \alpha \leq q-1)$ is not a linear combination of any number of columns of Euclidean weight $w_E - |\alpha|^2$ or less from the immediately preceding $b-1$ columns $h_{j-b+1}, h_{j-b+2}, \ldots, h_{j-1}$ together with any number of columns with Euclidean weight w_E or less among any b consecutive columns out of all the $j-1$ columns selected so far. In other words, column h_j can be added to H provided that

(10) $$\alpha h_j \neq (\alpha_{i_1}h_{i_1} + \alpha_{i_2}h_{i_2} + \cdots + \alpha_{i_r}h_{i_r}) + (\beta_{j_1}h_{j_1} + \beta_{j_2}h_{j_2} + \cdots + \beta_{j_m}h_{j_m}),$$

where $1 \leq \alpha \leq q-1$ (or, equivalently $1 \leq |\alpha|^2 \leq [q/2]^2$), $\alpha_{i_1}h_{i_1} + \alpha_{i_2}h_{i_2} + \cdots + \alpha_{i_r}h_{i_r}$ is any linear combination of Euclidean weight less than or equal to

$w_E - |\alpha|^2$ of the columns from $h_{j-b+1}, h_{j-b+2}, \ldots, h_{j-1}$ and $\beta_{j_1}h_{j_1} + \beta_{j_2}h_{j_2} + \cdots + \beta_{j_m}h_{j_m}$ is any linear combination of Euclidean weight w_E or less from a set of b consecutive columns among all $j-1$ columns.

To compute the number of all possible linear combinations occurring in (10) for all possible choices of α_{i_j}'s and β_{j_k}'s, we analyze the situation in three different cases.

Case 1. When β_{j_k}'s in Equation (10) are taken from the first $j-b$ columns.

It is clear that

$$1 \le \sum_{k=1}^{m} |\beta_{j_k}|^2 \le w_E \tag{11}$$

and

$$0 \le \sum_{j=1}^{r} |\alpha_{i_j}|^2 \le w_E - |\alpha|^2, \tag{12}$$

where $1 \le |\alpha|^2 \le [q/2]^2$.

The number of β_{j_k}'s satisfying equation (11) is

$$\sum_{i=1}^{b} (j-b-i+1)(V_{w_E}^{(i)} - V_0^{(i)}) = \sum_{i=1}^{b} (j-b-i+1)(V_{w_E}^{(i)} - 1). \tag{13}$$

The number of α_{i_j}'s satisfying inequality (12) is

$$\sum_{\lambda=1}^{[q/2]} e_\lambda V_{w_E - \lambda^2}^{(b-1)}. \tag{14}$$

Case 2. When β_{j_k}'s are taken from the immediately preceding $b-1$ columns.

In this case, the number of linear combinations occurring in equation (10) or, in other words, number of additional ways in which α_{i_j}'s and β_{j_k}'s can be selected is given by

$$\sum_{\lambda=1}^{[q/2]} e_\lambda \left(V_{2w_E - \lambda^2}^{(b-1)} - V_{1-\lambda^2}^{(b-1)} \right). \tag{15}$$

Case 3. When β_{j_k}'s are neither completely confined to the first $j-b$ columns nor to the last $b-1$ columns.

In this case, h_{j_k}'s are selected from $h_{j-2b+2}, h_{j-2b+3}, \ldots, h_{j-1}$ in such a way that not all are taken either from $h_{j-2b+2}, h_{j-2b+3}, \ldots, h_{j-b}$ or from h_{j-b+1}, $h_{j-b+2}, \ldots, h_{j-1}$. Let us suppose that the burst starts from the $(j-2b+k+1)^{th}$ position which may continue upto $(j-b+k)^{th}$ position $(1 \le k \le b-1)$. Let the Euclidean value of the element occurring at the starting position $(j - 2b + k + 1)$ of the burst be λ_1^2. Further, let us have linear combinations, of Euclidean weight $r_{1\lambda_1}$, of columns from the $(j-2b+k+1)^{th}, \ldots, (j-b)^{th}$ columns; linear combinations of Euclidean weight $r_{2\lambda_1}$, of columns from the $(j-b+1)^{th}, \ldots, (j-b+k)^{th}$ columns; and linear combinations of Euclidean weight $r_{3\lambda_1}$, of columns from the $(j-b+k+1)^{th}, \ldots, (j-1)^{th}$ columns. Then, in this case, the total number of choices of the linear combinations of equation (10) turns out to be

$$\sum_{\lambda=1}^{[q/2]} e_\lambda \left(\sum_{k=1}^{(b-1)} \sum_{\lambda_1=1}^{\min\{[q/2],[\sqrt{w_E-1}\,]\}} \sum_{r_{1\lambda_1}, r_{2\lambda_1}, r_{3\lambda_1}} e_{\lambda_1} A_{r_{1\lambda_1}}^{(b-k-1)} A_{r_{2\lambda_1}}^{(k)} A_{r_{3\lambda_1}}^{(b-k-1)} \right), \tag{16}$$

where

$$\begin{aligned} 2-\lambda^2 &\leq \lambda_1^2 + r_{1\lambda_1} + r_{2\lambda_1} + r_{3\lambda_1} \leq 2w_E - \lambda^2, \\ 1 &\leq \lambda_1^2 + r_{1\lambda_1} \leq w_E - 1, \\ 1 &\leq r_{2\lambda_1} \leq 2w_E - 1 - \lambda^2, \\ 0 &\leq r_{3\lambda_1} \leq w_E - \lambda^2, \\ r_{2\lambda_1} + r_{3\lambda_1} &\geq 1 - \lambda^2, \\ \lambda_1^2 + r_{1\lambda_1} + r_{2\lambda_1} &\geq 1. \end{aligned} \tag{17}$$

Thus total number of possible distinct linear combinations arising out of all the three cases including the patterns of all zeros is given by

$$1+\left[\sum_{\lambda=1}^{[q/2]} e_\lambda V_{w_E-\lambda^2}^{(b-1)}\right]\left[\sum_{i=1}^{b}(j-b-i+1)(V_{w_E}^{(i)}-1)\right]+\sum_{\lambda=1}^{[q/2]} e_\lambda \left(V_{2w_E-\lambda^2}^{(b-1)} - V_{1-\lambda^2}^{(b-1)}\right)$$
$$+\sum_{\lambda=1}^{[q/2]} e_\lambda \left(\sum_{k=1}^{(b-1)} \sum_{\lambda_1=1}^{\min\{[q/2],[\sqrt{w_E-1}]\}} \sum_{r_{1\lambda_1}, r_{2\lambda_1}, r_{3\lambda_1}} e_{\lambda_1} A_{r_{1\lambda_1}}^{(b-k-1)} A_{r_{2\lambda_1}}^{(k)} A_{r_{3\lambda_1}}^{(b-k-1)} \right) =: L,$$

where $r_{1\lambda_1}, r_{2\lambda_1}, r_{3\lambda_1}$ satisfy the constraints given in inequalities (17).

Therefore, a column h_j can be added to the parity check matrix H provided that

$$q^{n-k} > L. \tag{18}$$

But for an (n,k) linear code to exist, the inequality (18) should hold for $j=n$ and we get (8).

This completes the proof of the theorem. □

Example 3.1. Take $b=2, w_E=2, q=5, n=5, k=1$. We show the existence of a (5,1) linear code over $\mathbb{Z}_5$ satisfying the sufficient condition (8) and correcting all burst errors of length 2 or less with Euclidean weight 2 or less.

We now compute the value of R.H.S. of inequality (8) for above mentioned values of parameters, i.e., for $b=2, w_E=2, q=5, n=5, k=1$.

$$\text{(R.H.S.) of (8)} = 1 + \sum_{\lambda=1}^{2} e_\lambda \left[V_{2-\lambda^2}^{(1)} \left(3(V_2^{(1)}-1) + 2(V_2^{(2)}-1) \right) + \left(V_{4-\lambda^2}^{(1)} - V_{1-\lambda^2}^{(1)} \right) \right. \tag{19}$$

$$+\sum_{k=1}^{1}\sum_{\lambda_1=1}^{1}\sum_{r_{1\lambda_1},r_{2\lambda_1},r_{3\lambda_1}} e_{\lambda_1}A^{(1-k)}_{r_{1\lambda_1}}A^{(k)}_{r_{2\lambda_1}}A^{(1-k)}_{r_{3\lambda_1}}\Bigg]$$

$$= 1+\sum_{\lambda=1}^{2}e_\lambda M_\lambda,$$

where

$$\begin{aligned} M_\lambda &= V^{(1)}_{2-\lambda^2}\Bigg(3(V^{(1)}_2-1)+2(V^{(2)}_2-1)\Bigg)+\Bigg(V^{(1)}_{4-\lambda^2}-V^{(1)}_{1-\lambda^2}\Bigg) \\ &+\sum_{k=1}^{1}\sum_{\lambda_1=1}^{1}\sum_{r_{1\lambda_1},r_{2\lambda_1},r_{3\lambda_1}} e_{\lambda_1}A^{(1-k)}_{r_{1\lambda_1}}A^{(k)}_{r_{2\lambda_1}}A^{(1-k)}_{r_{3\lambda_1}}, \end{aligned}$$

and

$$\begin{aligned} &2-\lambda^2 \le \lambda_1^2+r_{1\lambda_1}+r_{2\lambda_1}+r_{3\lambda_1}\le 4-\lambda^2,\\ &1\le \lambda_1^2+r_{1\lambda_1}\le 1,\\ &1\le r_{2\lambda_1}\le 3-\lambda^2,\\ &0\le r_{3\lambda_1}\le 2-\lambda^2,\\ &r_{2\lambda_1}+r_{3\lambda_1}\ge 1-\lambda^2,\\ &\lambda_1^2+r_{1\lambda_1}+r_{2\lambda_1}\ge 1. \end{aligned}$$

Since λ varies from 1 to 2 in equation (19), therefore, we compute M_λ corresponding to each value of λ in the following two cases:

Case 1. When $\lambda = 1$.

In this case, $r_{1\lambda_1}, r_{2\lambda_1}, r_{3\lambda_1}$ have following three sets of feasible solutions:

(i) $r_{1\lambda_1}=0, \quad r_{2\lambda_1}=1, \quad r_{3\lambda_1}=1$

(ii) $r_{1\lambda_1}=0, \quad r_{2\lambda_1}=1, \quad r_{3\lambda_1}=0$

(iii) $r_{1\lambda_1}=0, \quad r_{2\lambda_1}=2, \quad r_{3\lambda_1}=0.$

Therefore, the value of M_λ in (19) for $\lambda=1$ is given by

$$\begin{aligned} M_\lambda\Bigg|_{\lambda=1} &= V^{(1)}_1\Bigg(3(V^{(1)}_2-1)+2(V^{(2)}_2-1)\Bigg)+\Bigg(V^{(1)}_3-V^{(1)}_0\Bigg) \\ &+2\Bigg(A^{(0)}_0A^{(1)}_1A^{(0)}_1+A^{(0)}_0A^{(1)}_1A^{(0)}_0+A^{(0)}_0A^{(1)}_2A^{(0)}_0\Bigg) \\ &= 3(3\times 2+2\times 8)+(3-1)+2(0+0+0) \\ &= 68. \end{aligned} \tag{20}$$

Case 2. When $\lambda = 2$.

For this case, there is no feasible solution for $r_{1\lambda_1}, r_{2\lambda_1}, r_{3\lambda_1}$. Therefore, the value of M_λ in (19) for $\lambda=2$ is taken to be zero, i.e.,

$$M_\lambda\Bigg|_{\lambda=2} = 0. \tag{21}$$

Substituting the values of M_λ for $\lambda = 1, 2$ from (20) and (21) respectively in equation (19), we get

$$\begin{aligned}(19) &= 1 + e_1 \times 68 + e_2 \times 0 \\ &= 1 + 2 \times 68 + 2 \times 0 \\ &= 137.\end{aligned}$$

Also, L.H.S. of (8) for a (5,1) linear code over $\mathbb{Z}_5 = 5^{n-k} = 5^4 = 625$.

Since $625 > 137$, therefore, the sufficient condition (8) is satisfied for a (5,1) linear code over $\mathbb{Z}_5$ for $b = 2, w_E = 2$.

Now, consider the following 4×5 parity check matrix of a (5,1) linear code over $\mathbb{Z}_5$:

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}_{4\times 5}$$

This matrix has been constructed by the synthesis procedure outlined in the proof of Theorem 3.1 by taking $b = 2$ and $w_E = 2$. The null space of this matrix is the desired code which corrects all burst errors of length 2 or less with Euclidean weight 2 or less. It can be seen from the following table that syndromes of all the correctable error patterns, i.e., all burst errors of length 2 or less with Euclidean weight 2 or less are distinct.

Table.

Error Pattern	Syndrome
(1 0 0 0 0)	(1 0 0 0)
(4 0 0 0 0)	(4 0 0 0)
(0 1 0 0 0)	(0 1 0 0)
(0 4 0 0 0)	(0 4 0 0)
(0 0 1 0 0)	(0 0 1 0)
(0 0 4 0 0)	(0 0 4 0)
(0 0 0 1 0)	(0 0 0 1)
(0 0 0 4 0)	(0 0 0 4)
(0 0 0 0 1)	(3 0 2 1)
(0 0 0 0 4)	(2 0 3 4)
(1 1 0 0 0)	(1 1 0 0)
(1 4 0 0 0)	(1 4 0 0)
(4 1 0 0 0)	(4 1 0 0)
(4 4 0 0 0)	(4 4 0 0)
(0 1 1 0 0)	(0 1 1 0)
(0 1 4 0 0)	(0 1 4 0)
(0 4 1 0 0)	(0 4 1 0)
(0 4 4 0 0)	(0 4 4 0)
(0 0 1 1 0)	(0 0 1 1)
(0 0 1 4 0)	(0 0 1 4)
(0 0 4 1 0)	(0 0 4 1)
(0 0 4 4 0)	(0 0 4 4)
(0 0 0 1 1)	(3 0 2 2)
(0 0 0 1 4)	(2 0 3 0)
(0 0 0 4 1)	(3 0 2 0)
(0 0 0 4 4)	(2 0 3 3)

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