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A RECURSIVE FORMULA FOR THE JONES POLYNOMIAL OF 2-BRIDGE LINKS AND APPLICATIONS

Eunju Lee, Sang Youl Lee, and Myoungsoo Seo

Abstract. In this paper, we give a recursive formula for the Jones polynomial of a 2-bridge knot or link with Conway normal form $C(-2n_1, 2n_2, -2n_3, \ldots, (-1)^r 2n_r)$ in terms of $n_1, n_2, \ldots, n_r$. As applications, we also give a recursive formula for the Jones polynomial of a 3-periodic link $L^{(3)}$ with rational quotient $L = C(2, n_1, -2, n_2, \ldots, n_r, (-1)^r 2)$ for any nonzero integers $n_1, n_2, \ldots, n_r$ and give a formula for the span of the Jones polynomial of $L^{(3)}$ in terms of $n_1, n_2, \ldots, n_r$ with $n_i \neq \pm 1$ for all $i = 1, 2, \ldots, r$.

1. Introduction

The Jones polynomial of an oriented link in S^3 was first introduced in [4]. Kauffman [8] and Murasugi [15] have used the Jones polynomial in verifying Tait conjecture which states that a reduced alternating diagram has minimal crossing number. Let D be a connected, prime diagram of an oriented link L. Then the span of the Jones polynomial of L is less than or equal to the number of crossings of D and the equality holds if and only if D is reduced alternating [8, 15, 23].

In 1956, a characterization of 2-bridge knots and links was introduced by Schubert [21]. In [1], Conway introduced another presentation, now called *Conway normal form*, of 2-bridge knots and links. Several people have studied the Jones polynomials of 2-bridge knots and links [5, 6, 13, 14, 18, 19, 22]. In 1987, Lichorish and Millett [13] gave an algorithm to calculate the Homfly polynomials of 2-bridge knots and links with matrix manipulations. In 2002, Nakabo [19] also presented an explicit formula of the Homfly polynomials of 2-bridge knots and links. Lu and Zhong [14] computed the Kauffman polynomials of 2-bridge knots and links using the Kauffman skein theory and linear algebra techniques. Note that the Jones polynomial can be obtained from the Homfly and Kauffman polynomials by substituting variables.

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On the other hand, Hilden, Lozano, and Montesinos-Amilibia [2] introduced a special kind of Conway normal form of a 2-bridge link with two components and studied the excellent component of the character variety of periodic knots in S^3 with rational quotients. In [16], Murasugi described several relationships between the Jones polynomials of a periodic link and its factor link. It is remarkable that the set of periodic links with rational quotients is a special family of periodic links which contains all 2-bridge knots and links, all torus knots and links and some pretzel knots and links, etc. It is known that every 2-bridge knot or link is a 2-periodic link with rational quotient and every 2-periodic link with rational quotient is a 2-bridge knot or link [3]. The second and third authors [10] re-examined Hilden, Lozano, and Montesinos-Amilibia's presentation to study the Alexander polynomials of 2-bridge links with Conway normal form $C(2, n_1, -2, n_2, \ldots, n_r, (-1)^r 2)$ and q-periodic links in S^3 with rational quotients $C(2, n_1, -2, n_2, \ldots, n_r, (-1)^r 2)$ in terms of $n_1, n_2, \ldots, n_r$ and its period q. Thereafter, some properties for the family of periodic links with rational quotients are studied [3, 9, 10, 11, 12].

In this paper, we first give a recursive formula for the Jones polynomial of a 2-periodic link with rational quotient, which is actually a recursive formula for the Jones polynomial of a 2-bridge knot or link. Generalizing this formula, we also obtain a recursive formula for the Jones polynomial of a 3-periodic link with rational quotient and a formula for the span of the Jones polynomial of this kind of 3-periodic link.

This paper is organized as follows. In Section 2, we review presentations of 2-bridge knots and links and periodic links with rational quotients. Section 3 contains the definition of bracket polynomial and formulas for periodic links with rational quotients. In Section 4, for arbitrary given nonzero integers $n_1, n_2, \ldots, n_r$, we give a recursive formula for the Jones polynomial of a 2-bridge knot or link with Conway normal form $C(-2n_1, 2n_2, -2n_3, \ldots, (-1)^r 2n_r)$ in terms of $n_1, n_2, \ldots, n_r$. In Section 5, we give a recursive formula for the Jones polynomial of a 3-periodic link $L^{(3)}$ with rational quotient $L = C(2, n_1, -2, n_2, \ldots, n_r, (-1)^r 2)$ for arbitrary given nonzero integers $n_1, n_2, \ldots, n_r$ and give a formula for the span of the Jones polynomial of $L^{(3)}$ in terms of $n_1, n_2, \ldots, n_r$ with $n_i \neq \pm 1$ for all $i = 1, 2, \ldots, r$. The formula for the span gives a lower bound for the minimal crossing number of the 3-periodic link $L^{(3)}$.

2. Periodic links with rational quotients

To each pair (α, β) of two co-prime integers subject to the condition that β is odd and $0 < |\beta| < \alpha$, Schubert [21] associated an oriented diagram on the 2-sphere S^2 of an oriented 2-bridge knot(α odd) or link(α even) L in S^3, now called the *Schubert normal form* of L and denoted by $S(\alpha, \beta)$, and showed that any (oriented) 2-bridge knots and links in S^3 can be represented in this way. Two such pairs of integers (α, β) and (α', β') define an equivalent oriented (resp. unoriented) knot or link if and only if

$$\alpha = \alpha' \text{ and } \beta^{\pm 1} \equiv \beta' \bmod 2\alpha \text{ (resp. mod } \alpha),$$

where β^{-1} denotes the integer with the properties $0 < \beta^{-1} < 2\alpha$ and $\beta\beta^{-1} \equiv 1 \bmod 2\alpha$.

Let $[a_1, a_2, \ldots, a_n]$ denote a continued fraction expansion of α/β:

$$[a_1, a_2, \ldots, a_n] \equiv a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \frac{1}{a_n}}} = \frac{\alpha}{\beta}.$$

Then $L = S(\alpha, \beta)$ has also a diagram $C(a_1, a_2, \ldots, a_n)$, called *Conway normal form* of L, as shown in Figure 1, depending on whether n is even or odd [1]. The integral tangles in Figure 1, which are rectangles labeled a_i, are the 2-braids with $|a_i|$ crossings as shown in Figure 2. It is well known that $L = S(\alpha, \beta)$ admits a diagram $C(2b_1, 2b_2, \ldots, 2b_m)$, which is equivalent to $C(a_1, a_2, \ldots, a_n)$ [7].

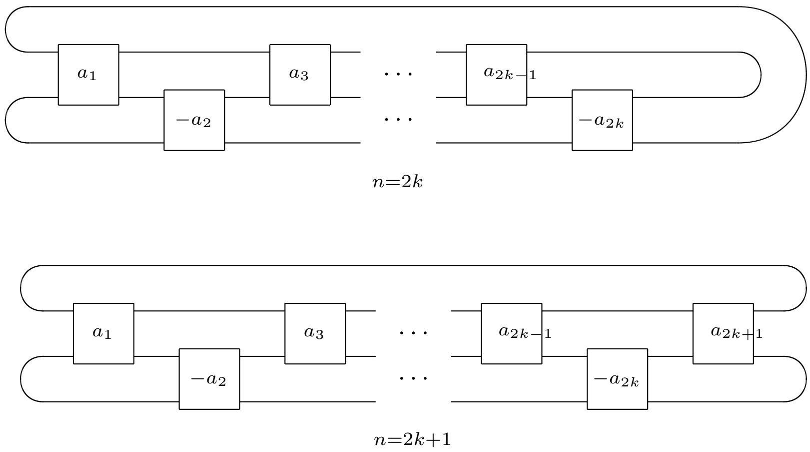


FIGURE 1

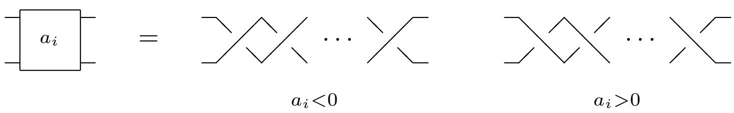


FIGURE 2

It is known [2, 10] that the 2-bridge link $L = S(\alpha, \beta)(\alpha$ even) can also be represented by Conway diagram of the form $C(2, n_1, -2, n_2, \ldots, n_r, (-1)^r 2)$ as shown in Figure 3. We choose an orientation of the 2-bridge link $C(2, n_1, -2, n_2, \ldots, n_r, (-1)^r 2)$ as shown in Figure 3. Then it is easy to see that the diagram shown in Figure 3 can be deformed to the diagrams in Figure 4 by using Reidemeister moves. Throughout this paper, an oriented 2-bridge link L in S^3 represented by the Conway normal form $C(2, n_1, -2, n_2, \ldots, n_r, (-1)^r 2)$ is denoted by $L = \overrightarrow{C}[[n_1, n_2, \ldots, n_r]]$.

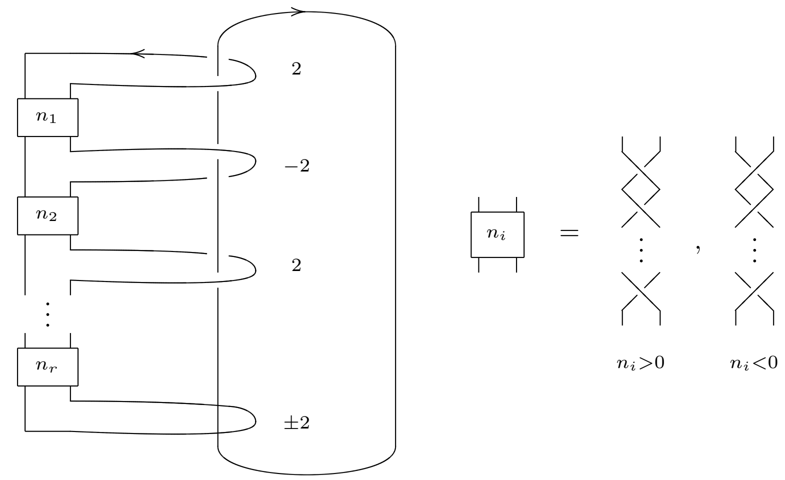


Figure 3

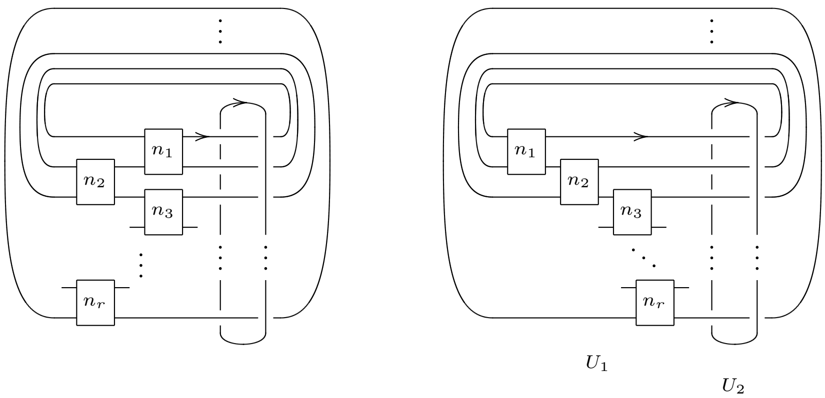


Figure 4

A link L in S^3 is called a *p-periodic link* ($p \geq 2$ an integer) if there exists an orientation preserving auto-homeomorphism h of S^3 such that $h(L) = L$, h is of order p and the set $\mathrm{Fix}(h)$ of fixed points of h is a circle disjoint from L. In this case, the link $L/\langle h\rangle \cup \mathrm{Fix}(h)$ in the orbit space $S^3/\langle h\rangle \cong S^3$ is called the *quotient link* of L. Let K be an oriented link in S^3 and U an oriented trivial knot with $K \cap U = \emptyset$. For any integer $p \geq 2$, let $\phi_U^p : \Sigma^3 \to S^3$ be a p-fold branched cyclic covering branched along U. Then Σ^3 is homeomorphic to the 3-sphere S^3, and $(\phi_U^p)^{-1}(K)$ is a p-periodic link in Σ^3 with $L = K \cup U$ as its quotient link. We give an orientation to $(\phi_U^p)^{-1}(K)$ induced by the orientation of K. Note that any periodic knot or link in S^3 arises in this manner.

Definition ([10])**.** A link $\tilde{L}$ in S^3 is called a *p-periodic link with rational quotient* if it is a p-periodic link whose quotient link is a 2-bridge link, or equivalently, if there exists a 2-bridge link $L = U_1 \cup U_2$ in S^3 such that $\tilde{L}$ is equivalent to the preimage $(\phi_{U_2}^p)^{-1}(U_1)$ of the component U_1 of L by a p-fold cyclic covering $\phi_{U_2}^p : \Sigma^3 \to S^3$ branched along the component U_2 of L.

Note that each component U_1 and U_2 of L is a trivial knot and they can be interchanged each other by an orientation preserving homeomorphism of S^3 [17]. This implies that $(\phi_{U_2}^p)^{-1}(U_1)$ is equivalent to $(\phi_{U_1}^p)^{-1}(U_2)$. Now let $L = \overrightarrow{C}[[n_1, n_2, \ldots, n_r]] = U_1 \cup U_2$ be an oriented 2-bridge link as shown Figure 4. Then the diagram $D^{(p)}$ shown in Figure 5 is a canonical oriented p-periodic diagram of the oriented p-periodic link $(\phi_{U_2}^p)^{-1}(U_1)$ with rational quotient $L = \overrightarrow{C}[[n_1, n_2, \ldots, n_r]]$. In what follows, we shall denote the oriented p-periodic link $(\phi_{U_2}^p)^{-1}(U_1)$ by $L^{(p)}$ or $\overrightarrow{C}[[n_1, n_2, \ldots, n_r]]^{(p)}$ for our convenience. Then any p-periodic link with rational quotient can be represented by $\overrightarrow{C}[[n_1, n_2, \ldots, n_r]]^{(p)}$ for some nonzero integers $n_1, n_2, \ldots, n_r$ [3, 10].

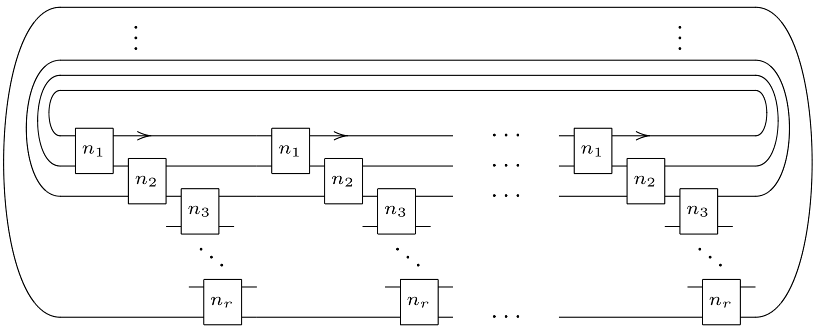


FIGURE 5. The canonical p-periodic diagram $D^{(p)}$ of $L^{(p)}$

3. Bracket polynomial of periodic links

The *bracket polynomial* of an unoriented link diagram D, denoted by $\langle D\rangle$, is a Laurent polynomial in a single variable A defined by the following three axioms:

1. If $\bigcirc$ denotes the standard diagram of the unknot, then

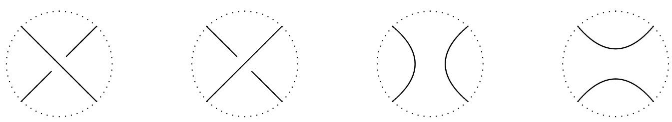
$$\langle \bigcirc \rangle = 1. \tag{1}$$

2. If $\delta = -A^{-2} - A^2$ and $D \sqcup \bigcirc$ denotes the diagram D together with the standard diagram of the unknot, disjoint from D, then

$$\langle D \sqcup \bigcirc \rangle = \delta\langle D\rangle. \tag{2}$$

3. Suppose that D_+, D_0 and D_∞ are the diagrams that are exactly the same except at a neighborhood of one crossing point in which the diagrams differ as shown in Figure 6. Then

$$\langle D_+\rangle = A\langle D_\infty\rangle + A^{-1}\langle D_0\rangle. \tag{3}$$



D_+ D_- D_0 D_∞

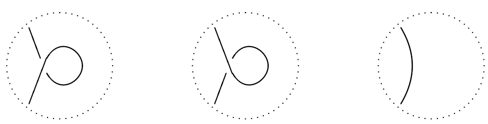
FIGURE 6

From (3), we also obtain the equation

$$\langle D_-\rangle = A\langle D_0\rangle + A^{-1}\langle D_\infty\rangle. \tag{4}$$

It is easy to see that $\langle D\rangle$ is an invariant under Reidemeister moves II and III, but not an invariant under Reidemeister move I. If φ_+, φ_- and φ_0 are the diagrams that are exactly the same except at a neighborhood of one crossing point in which the diagrams differ as shown in Figure 7, then we have

$$\langle \varphi_+\rangle = (-A)^3\langle \varphi_0\rangle,\ \langle \varphi_-\rangle = (-A)^{-3}\langle \varphi_0\rangle. \tag{5}$$



φ_+ φ_- φ_0

FIGURE 7

For a link L with its diagram D, the Jones polynomial $V_L(t)$ of L is defined as

$$V_L(t) = (-A)^{-3w(D)}\langle D\rangle$$

by setting $A^{-4} = t$ [8].

Lemma 3.1. *For each integer n, let $T(n)$ be a diagram with n-half twists and fixed outside as described in Figure 8. Then for any integer $n \geq 1$, we have*

$$\langle T(n)\rangle = A^{-n}\langle T(0)\rangle + A^{-n+2}\sum_{i=0}^{n-1}(-A^4)^i\langle T(\infty)\rangle \tag{6}$$

and

$$\langle T(-n)\rangle = A^{n}\langle T(0)\rangle + A^{n-2}\sum_{i=0}^{n-1}(-A^{-4})^i\langle T(\infty)\rangle. \tag{7}$$

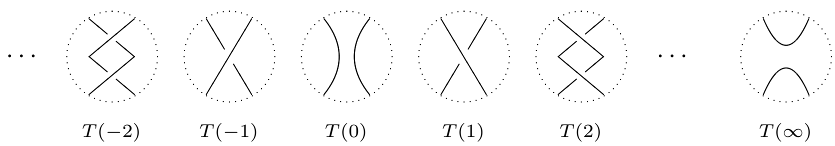


FIGURE 8

Proof. First we will prove that the equation (6) holds. If $n = 1$, then $\langle T(1)\rangle = A^{-1}\langle T(0)\rangle + A\langle T(\infty)\rangle$ by (3). For a positive integer $n > 1$, we assume that

$$\langle T(n)\rangle = A^{-n}\langle T(0)\rangle + A^{-n+2}\sum_{i=0}^{n-1}(-A^4)^i\langle T(\infty)\rangle.$$

Then it follows that

$$\begin{aligned}
\langle T(n+1)\rangle &= A^{-1}\langle T(n)\rangle + A(-A)^{3n}\langle T(\infty)\rangle \\
&= A^{-1}\left(A^{-n}\langle T(0)\rangle + A^{-n+2}\sum_{i=0}^{n-1}(-A^4)^i\langle T(\infty)\rangle\right) \\
&\quad + A^{-(n+1)+2}(-A^4)^n\langle T(\infty)\rangle \\
&= A^{-(n+1)}\langle T(0)\rangle + A^{-(n+1)+2}\sum_{i=0}^{n}(-A^4)^i\langle T(\infty)\rangle.
\end{aligned}$$

By a similar argument, we obtain the equation (7). $\square$

For any nonzero integer n, we define Laurent polynomials α_n and β_n by

$$\alpha_n = A^{-n}, \quad \beta_n = \begin{cases} A^{-n+2}\displaystyle\sum_{i=0}^{n-1}(-A^4)^i & \text{if } n \geq 1, \\ A^{-n-2}\displaystyle\sum_{i=0}^{-n-1}(-A^{-4})^i & \text{if } n \leq -1. \end{cases} \tag{8}$$

Then we have easily:

Lemma 3.2. *For given nonzero integers n and p, we have that*

$$\beta_n\delta + p\alpha_n = A^{-n}((-A^4)^n + (p-1)).$$

For any nonzero integers $n_1, n_2, \ldots, n_r$ $(r \geq 1)$ and a positive integer $p \geq 2$, let $L^{(p)}$ be the p-periodic link in S^3 with rational quotient $L = \overrightarrow{C}[[n_1, n_2, \ldots, n_r]]$. We consider the p-periodic diagram $D^{(p)}$ of $L^{(p)}$ as shown in Figure 9. In Figure 9, each $T_{i,j}$ denotes a 2-tangle with n_i-half twists as in Figure 3. If n_i is positive (respectively, negative), each crossing of $T_{i,j}$ is positive (respectively, negative). Since the writhe, denoted by $w(D^{(p)})$, of $D^{(p)}$ is the sum of crossing signs of crossings in $D^{(p)}$, we get

$$w(D^{(p)}) = p\sum_{i=1}^{r} n_i.$$

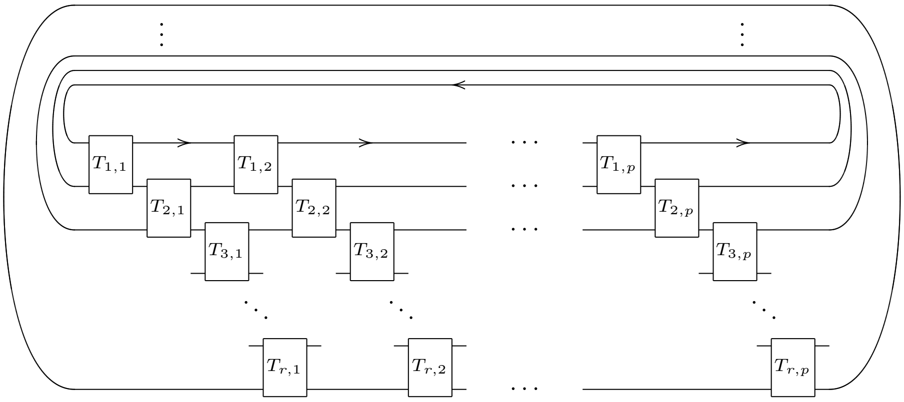


FIGURE 9

Put $\mathcal{T}_i = \{T_{i,1}, T_{i,2}, \ldots, T_{i,p}\}$ for each $i = 1, 2, \ldots, r$. We call a function $s : \mathcal{T}_i \to \{0, \infty\}$, where 0 denotes $T(0)$ and ∞ denotes $T(\infty)$ a *weight function* of $\mathcal{T}_i$. For each $i = 1, 2, \ldots, r$, let $\mathcal{S}_i$ denote the set of all weight functions of $\mathcal{T}_i$. For a weight function $s \in \mathcal{S}_r$, let $D^{(p)}(s)$ be the diagram obtained from $D^{(p)}$ by replacing each tangle $T_{r,k}$ in $\mathcal{T}_r$ by a $s(T_{r,k})$-tangle and we denote by $\phi(s)$ the number of the tangles in $s^{-1}(0)$. By applying Lemma 3.1 to each tangle $T_{r,k}$ in $\mathcal{T}_r$, we have:

Proposition 3.3. *For given nonzero integers $n_1, n_2, \ldots, n_r$ $(r \geq 1)$ and a positive integer $p \geq 2$, let $L^{(p)}$ be the p-periodic link with rational quotient $L = \overrightarrow{C}[[n_1, n_2, \ldots, n_r]]$ and $D^{(p)}$ its p-periodic diagram as shown in Figure 9. Then*

$$\langle D^{(p)} \rangle = \sum_{s \in \mathcal{S}_r} \alpha_{n_r}^{\phi(s)} \beta_{n_r}^{p-\phi(s)} \langle D^{(p)}(s) \rangle.$$

For an r-tuple $(s_1, s_2, \ldots, s_r)$ of weight functions with $s_i \in \mathcal{S}_i (i = 1, 2, \ldots, r)$, let $D^{(p)}(s_1, \ldots, s_r)$ be the diagram obtained from $D^{(P)}$ by replacing each tangle $T_{i,j}$ with the $s_i(T_{i,j})$-tangle and we denote by $\phi(s_k)$ the number of tangles in $s_k^{-1}(0)$ for each $k = 1, 2, \ldots, r$. By applying Lemma 3.1 to each tangle $T_{i,j}$ in $D^{(p)}$, we also have:

Proposition 3.4. *For given nonzero integers $n_1, n_2, \ldots, n_r$ $(r \geq 1)$ and a positive integer $p \geq 2$, let $L^{(p)}$ be the p-periodic link with rational quotient $L = \overrightarrow{C}[[n_1, n_2, \ldots, n_r]]$ and $D^{(p)}$ its p-periodic diagram as shown in Figure 9. Then*

$$\langle D^{(p)} \rangle = \sum_{(s_1, \ldots, s_r) \in \mathcal{S}_1 \times \cdots \times \mathcal{S}_r} \left(\prod_{k=1}^{r} \alpha_{n_k}^{\phi(s_k)} \beta_{n_k}^{p-\phi(s_k)} \right) \delta^{|D^{(p)}(s_1, \ldots, s_r)|-1},$$

where $\delta = -A^2 - A^{-2}$ and $|D^{(p)}(s_1, \ldots, s_r)|$ is the number of disjoint simple closed curves in $D^{(p)}(s_1, \ldots, s_r)$.

Remark 3.5. Each diagram $D^{(p)}(s_1, \ldots, s_r)$ is a disjoint union of simple closed curves. If we can find a formula for the number of disjoint simple closed curves in each $D^{(p)}(s_1, \ldots, s_r)$ in terms of $n_1, n_2, \ldots, n_r$ and p, then the Laurent polynomial $\langle D^{(p)} \rangle$ can be expressed by means of the integers $n_1, n_2, \ldots, n_r$ and p. However, it looks very difficult to make such a formula. The authors know of none.

4. Recursive formula for the Jones polynomial of 2-bridge links

It is well known that any 2-bridge knot or link admits a diagram with Conway normal form $C(2a_1, 2a_2, \ldots, 2a_r)$ for some integers $a_1, a_2, \ldots, a_r$ [7]. In [3], Jang, the second and third authors proved that the 2-periodic link $L^{(2)}$ with rational quotient $L = \overrightarrow{C}[[n_1, n_2, \ldots, n_k]]$ is a 2-bridge knot or link with Conway normal form $C(-2n_1, 2n_2, -2n_3, \ldots, (-1)^r 2n_r)$. In this section we give a recursive formula for the Jones polynomial of a 2-periodic link with rational quotient and give a formula for the span of the Jones polynomial. Consequently, we get a recursive formula for the Jones polynomial of 2-bridge knot or link with Conway normal form $C(2a_1, 2a_2, \ldots, 2a_r)$ in terms of $a_1, a_2, \ldots, a_r$.

Lemma 4.1. *Let $n_1, n_2, \ldots, n_r$ be given nonzero integers. For each $k = 1, 2, \ldots, r$, let $D_k^{(2)}$ be the canonical 2-periodic diagram of the 2-periodic link* *with rational quotient* $L_k = \overrightarrow{C}[[n_1, n_2, \ldots, n_k]]$. *Let* $D_0^{(2)}$ *denote the standard diagram of the unknot. Then we have the following recursive formula*:

$$
\begin{aligned}
&\langle D_0^{(2)} \rangle = 1,\\
(9) \qquad &\langle D_1^{(2)} \rangle = \beta_{n_1}^2 \delta + 2\alpha_{n_1}\beta_{n_1} + \alpha_{n_1}^2 \delta,\\
&\langle D_k^{(2)} \rangle = (\beta_{n_k}^2 \delta + 2\alpha_{n_k}\beta_{n_k})\langle D_{k-1}^{(2)} \rangle + \alpha_{n_k}^2 A^{6n_{k-1}} \langle D_{k-2}^{(2)} \rangle.
\end{aligned}
$$

Proof. For a weight function $s \in \mathcal{S}_k$, let $D_k^{(2)}(s(T_{k,1}), s(T_{k,2}))$ be the diagram obtained from $D_k^{(2)}$ by replacing each tangle $T_{k,i}$ by an $s(T_{k,i})$-tangle $(i = 1, 2)$.

If $k = 1$, then $D_1^{(2)}(0, 0)$, $D_1^{(2)}(0, \infty)$, $D_1^{(2)}(\infty, 0)$ and $D_1^{(2)}(\infty, \infty)$ consist of simple closed curves without crossings. We observe that $D_1^{(2)}(0, 0)$ and $D_1^{(2)}(\infty, \infty)$ have two components and $D_1^{(2)}(0, \infty)$ and $D_1^{(2)}(\infty, 0)$ have one component. By Proposition 3.4, we have

$$\langle D_1^{(2)} \rangle = \beta_{n_1}^2 \delta + 2\alpha_{n_1}\beta_{n_1} + \alpha_{n_1}^2 \delta.$$

Now we assume that the recursive formula (9) holds for $n_1, n_2, \ldots, n_{k-1}$ with $k \geq 2$. Then $D_k^{(2)}(0, 0)$, $D_k^{(2)}(0, \infty)$, $D_k^{(2)}(\infty, 0)$ and $D_k^{(2)}(\infty, \infty)$ are isotopic to the diagrams as shown in Figure 10. Thus $D_k^{(2)}(0, \infty)$ and $D_k^{(2)}(\infty, 0)$ are

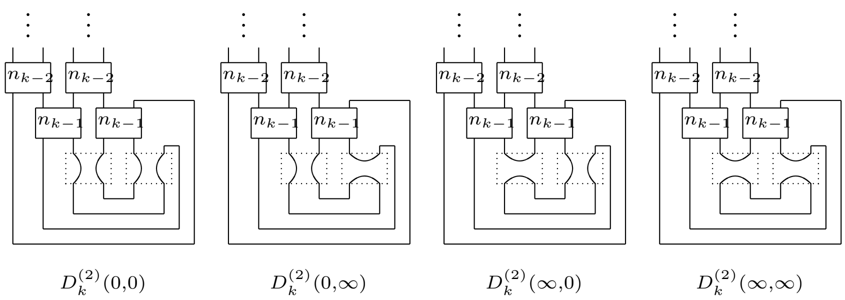


FIGURE 10

isotopic to the diagram $D_{k-1}^{(2)}$, and $D_k^{(2)}(\infty, \infty)$ is isotopic to the diagram $D_{k-1}^{(2)} \sqcup \bigcirc$. Moreover $D_k^{(2)}(0, 0)$ is obtained from $D_{k-2}^{(2)}$ by applying the Reidemeister move I. By (2), (5) and Proposition 3.3, we have

$$
\begin{aligned}
\langle D_k^{(2)} \rangle &= \beta_k^2 \langle D_k^{(2)}(\infty, \infty) \rangle + \beta_k \alpha_k \langle D_k^{(2)}(\infty, 0) \rangle\\
&\quad + \alpha_k \beta_k \langle D_k^{(2)}(0, \infty) \rangle + \alpha_k^2 \langle D_k^{(2)}(0, 0) \rangle\\
&= (\beta_k^2 \delta + 2\alpha_k\beta_k)\langle D_{k-1}^{(2)} \rangle + \alpha_k^2 A^{6n_{k-1}} \langle D_{k-2}^{(2)} \rangle.
\end{aligned}
$$

This completes the proof. $\square$

For any nonzero integer n, let $\mathcal{A}_n(t)$ be a Laurent polynomial in $\mathbb{Z}[t^{\pm\frac{1}{2}}]$ defined by

$$\mathcal{A}_n(t) = \begin{cases} t^{-\frac{1}{2}} \sum_{i=0}^{n-1} (-t)^{-i} & \text{if } n \geq 1, \\ t^{\frac{1}{2}} \sum_{i=0}^{-n-1} (-t)^{i} & \text{if } n \leq -1. \end{cases} \tag{10}$$

We note that $\beta_n|_{A=t^{-\frac{1}{4}}} = t^{\frac{n}{4}} \mathcal{A}_n(t)$.

Theorem 4.2. *Let $n_1, n_2, \ldots, n_r$ be given nonzero integers. For each $k = 1, 2, \ldots, r$, let $L_k^{(2)}$ be the 2-periodic link with rational quotient $L_k = \overrightarrow{C}[[n_1, n_2, \ldots, n_k]]$ and let $L_0^{(2)}$ the trivial knot. Let $V_k(t)$ be the Jones polynomial of $L_k^{(2)}$ for each $k = 0, 1, 2, \ldots, r$. Then we have the following recursive formula*:

$$V_0(t) = 1, \tag{11}$$

$$V_1(t) = t^{2n_1} \left(\mathcal{A}_{2n_1}(t) - t^{\frac{1}{2}} - t^{-\frac{1}{2}} \right), \tag{12}$$

$$V_k(t) = t^{2n_k} \mathcal{A}_{2n_k}(t) V_{k-1}(t) + t^{2n_k} V_{k-2}(t). \tag{13}$$

Proof. For each $k = 1, 2, \ldots, r$, let $D_k^{(2)}$ be the canonical 2-periodic diagram of the 2-periodic link $L_k^{(2)}$. Then

$$V_k(t) = (-A)^{-3w(D_k^{(2)})} \langle D_k^{(2)} \rangle \Big|_{A=t^{-\frac{1}{4}}}.$$

For each $k = 1, 2, \ldots, r$, put $f_k(A) = (-A)^{-3w(D_k^{(2)})} \langle D_k^{(2)} \rangle$. Then $V_k(t) = f_k(A)|_{A=t^{-\frac{1}{4}}}$. We note that $w(D_k^{(2)}) = 2\sum_{i=1}^{k} n_i$ and, by Lemma 3.2, $\beta_{n_i}\delta + 2\alpha_{n_i} = A^{-n_i}((-A^4)^{n_i} + 1)$.

Since $L_0^{(2)}$ is the trivial knot, $V_0(t) = 1$. If $n_1 \geq 1$, then

$$\begin{aligned} f_1(A) &= (-A)^{-6n_1} \left(\beta_{n_1}(\beta_{n_1}\delta + 2\alpha_{n_1}) + \alpha_{n_1}^2 \delta \right) \\ &= A^{-6n_1} \left(A^{-2n_1+2}((-A^4)^{n_1} + 1) \sum_{i=0}^{n_1-1} (-A^4)^i + A^{-2n_1}(-A^2 - A^{-2}) \right) \\ &= A^{-8n_1} \left(A^2 \sum_{i=0}^{2n_1-1} (-A^4)^i + (-A^2 - A^{-2}) \right) \end{aligned}$$

and hence

$$V_1(t) = t^{2n_1} \left(t^{-\frac{1}{2}} \sum_{i=0}^{2n_1-1} (-t)^{-i} + (-t^{-\frac{1}{2}} - t^{\frac{1}{2}}) \right) = t^{2n_1} \left(\mathcal{A}_{2n_1}(t) - t^{\frac{1}{2}} - t^{-\frac{1}{2}} \right).$$

If $n_1 \leq -1$, then $f_1(A) = A^{-8n_1}\left(A^{-2}\sum_{i=0}^{-2n_1-1}(-A^{-4})^i + (-A^2 - A^{-2})\right)$ and hence

$$V_1(t) = t^{2n_1}\left(t^{\frac{1}{2}}\sum_{i=0}^{-2n_1-1}(-t)^i + (-t^{-\frac{1}{2}} - t^{\frac{1}{2}})\right) = t^{2n_1}\left(\mathcal{A}_{2n_1}(t) - t^{\frac{1}{2}} - t^{-\frac{1}{2}}\right).$$

For $k \geq 2$, from Lemma 4.1, we obtain

$$f_k(A) = (-A)^{-6n_k}(\beta_{n_k}^2\delta + 2\alpha_{n_k}\beta_{n_k})f_{k-1}(A) + (-A)^{-6n_k}\alpha_{n_k}^2 f_{k-2}(A).$$

Immediately we have $(-A)^{-6n_k}\alpha_{n_k}^2|_{A=t^{-\frac{1}{4}}} = t^{2n_k}$. If $n_k \geq 1$, then it follows that

$$(-A)^{-6n_k}(\beta_{n_k}^2\delta + 2\alpha_{n_k}\beta_{n_k}) = A^{-8n_k+2}\sum_{i=0}^{2n_k-1}(-A^4)^i$$

and hence

$$(-A)^{-6n_k}(\beta_{n_k}^2\delta + 2\alpha_{n_k}\beta_{n_k})|_{A=t^{-\frac{1}{4}}} = t^{2n_k}\mathcal{A}_{2n_k}(t).$$

If $n_k \leq -1$, then

$$(-A)^{-6n_k}(\beta_{n_k}^2\delta + 2\alpha_{n_k}\beta_{n_k}) = A^{-8n_k-2}\sum_{i=0}^{-2n_k-1}(-A^{-4})^i$$

and hence

$$(-A)^{-6n_k}(\beta_{n_k}^2\delta + 2\alpha_{n_k}\beta_{n_k})|_{A=t^{-\frac{1}{4}}} = t^{2n_k}\mathcal{A}_{2n_k}(t).$$

Therefore we have

$$\begin{aligned} V_k(t) &= f_k(A)|_{A=t^{-\frac{1}{4}}} \\ &= A^{-6n_k}(\beta_{n_k}^2\delta + 2\alpha_{n_k}\beta_{n_k})|_{A=t^{-\frac{1}{4}}}V_{k-1}(t) + A^{-6n_k}\alpha_{n_k}^2|_{A=t^{-\frac{1}{4}}}V_{k-2}(t) \\ &= t^{2n_k}\mathcal{A}_{2n_k}(t)V_{k-1}(t) + t^{2n_k}V_{k-2}(t). \end{aligned}$$

This completes the proof. □

Example 4.3. Let L be the 2-bridge knot with Conway normal form $C(-2, -4)$. It is the mirror image of the knot 5_2 in Rolfsen's table [20]. By the discussion in the beginning of this section, L is the 2-periodic knot with rational quotient $\overrightarrow{C}[[1,2]]$. Let $n_1 = 1$ and $n_2 = 2$. Then we have that $\mathcal{A}_2(t) = t^{-\frac{1}{2}} - t^{-\frac{3}{2}}$ and $\mathcal{A}_4(t) = t^{-\frac{1}{2}} - t^{-\frac{3}{2}} + t^{-\frac{5}{2}} - t^{-\frac{7}{2}}$. From Theorem 4.2, it follows that $V_0(t) = 1$, $V_1(t) = -t^{\frac{1}{2}} - t^{\frac{5}{2}}$ and $V_2(t) = -t^6 + t^5 - t^4 + 2t^3 - t^2 + t$. Hence the Jones polynomial of L is

$$V_L(t) = -t^6 + t^5 - t^4 + 2t^3 - t^2 + t.$$

For the Jones polynomial $V_L(t)$ of a link L, we denote the *maximum* (resp. *minimum*) *degree* of $V_L(t)$ by $\max\deg V_L(t)$ (resp. $\min\deg V_L(t)$). We also denote the *span* of $V_L(t)$ by span $V_L(t)$, i.e., span $V_L(t) = \max\deg V_L(t) - \min\deg V_L(t)$.

Lemma 4.4. *For given nonzero integers $n_1, n_2, \dots, n_r$, let $V_k(t)$ be the Jones polynomial of the 2-periodic link $L_k^{(2)}$ with rational quotient $L = \overrightarrow{C}[[n_1, n_2, \dots, n_k]]$ for each $k = 1, 2, \dots, r$. Put $\epsilon_k = \frac{|n_k|}{n_k}$ $(k = 1, 2, \dots, r)$. For each $k = 1, 2, \dots, r$, we have that*

$$\max\deg V_k(t) = \frac{1-k}{2} + \sum_{i=1}^{k}(n_i + |n_i|) + \frac{\epsilon_1}{2} + \frac{1}{4}\sum_{j=1}^{k-1}(1-\epsilon_j)(1+\epsilon_{j+1}) \tag{14}$$

and

$$\min\deg V_k(t) = \frac{k-1}{2} + \sum_{i=1}^{k}(n_i - |n_i|) + \frac{\epsilon_1}{2} - \frac{1}{4}\sum_{j=1}^{k-1}(1+\epsilon_j)(1-\epsilon_{j+1}). \tag{15}$$

Proof. Let n be any nonzero integer. From (10), we have that

$$\max\deg \mathcal{A}_{2n}(t) = -\frac{1}{2} + |n| - n, \quad \min\deg \mathcal{A}_{2n}(t) = \frac{1}{2} - |n| - n. \tag{16}$$

We will use the recursive formula in Theorem 4.2 and induction on k.

If $n_1 \geq 1$, then $\max\deg \mathcal{A}_{2n_1}(t) = -\frac{1}{2}$ and hence $\max\deg V_1(t) = 2n_1 + \frac{1}{2}$. If $n_1 \leq -1$, then $\max\deg \mathcal{A}_{2n_1}(t) = -\frac{1}{2} - 2n_1$ and hence $\max\deg V_1(t) = 2n_1 - \frac{1}{2} - 2n_1 = -\frac{1}{2}$. Therefore we have

$$\max\deg V_1(t) = (n_1 + |n_1|) + \frac{\epsilon_1}{2}.$$

We assume that the formula (14) holds for k.

Case (i) : If $n_{k+1} \leq -1$ or $n_k \geq 1$, then

$$\begin{aligned}
&\max\deg \mathcal{A}_{2n_{k+1}}(t)V_k(t)\\
=\ &\max\deg \mathcal{A}_{2n_{k+1}}(t) + \max\deg V_k(t)\\
=\ &-\frac{1}{2} + |n_{k+1}| - n_{k+1}\\
&+\frac{1-k}{2} + \sum_{i=1}^{k}(n_i + |n_i|) + \frac{\epsilon_1}{2} + \frac{1}{4}\sum_{j=1}^{k-1}(1-\epsilon_j)(1+\epsilon_{j+1})\\
\geq\ &\frac{2-k}{2} + \sum_{i=1}^{k-1}(n_i + |n_i|) + \frac{\epsilon_1}{2} + \frac{1}{4}\sum_{j=1}^{k-2}(1-\epsilon_j)(1+\epsilon_{j+1})\\
&-1 + |n_{k+1}| - n_{k+1} + (n_k + |n_k|)
\end{aligned}$$

and

$$\max\deg V_{k-1}(t) = \frac{2-k}{2} + \sum_{i=1}^{k-1}(n_i + |n_i|) + \frac{\epsilon_1}{2} + \frac{1}{4}\sum_{j=1}^{k-2}(1-\epsilon_j)(1+\epsilon_{j+1}).$$

Since $-1+|n_{k+1}|-n_{k+1}+(n_k+|n_k|)\geq 1$,

$$\max\deg \mathcal{A}_{2n_{k+1}}(t)V_k(t) > \max\deg V_{k-1}(t).$$

Thus by (13), we have that

$$\begin{aligned}
&\max\deg V_{k+1}(t)\\
=&\ 2n_{k+1}+\max\deg \mathcal{A}_{2n_{k+1}}(t)V_k(t)\\
=&\ 2n_{k+1}-\frac{1}{2}+|n_{k+1}|-n_{k+1}\\
&+\frac{1-k}{2}+\sum_{i=1}^{k}(n_i+|n_i|)+\frac{\epsilon_1}{2}+\frac{1}{4}\sum_{j=1}^{k-1}(1-\epsilon_j)(1+\epsilon_{j+1})\\
=&\ \frac{1-(k+1)}{2}+\sum_{i=1}^{k+1}(n_i+|n_i|)+\frac{\epsilon_1}{2}+\frac{1}{4}\sum_{j=1}^{k}(1-\epsilon_j)(1+\epsilon_{j+1}).
\end{aligned}$$

Case (ii) : If $n_{k+1}\geq 1$ and $n_k\leq -1$, then

$$\begin{aligned}
&\max\deg \mathcal{A}_{2n_{k+1}}(t)V_k(t)\\
=&\ \max\deg \mathcal{A}_{2n_{k+1}}(t)+\max\deg V_k(t)\\
=&\ -\frac{1}{2}+\frac{1-k}{2}+\sum_{i=1}^{k-1}(n_i+|n_i|)+\frac{\epsilon_1}{2}+\frac{1}{4}\sum_{j=1}^{k-2}(1-\epsilon_j)(1+\epsilon_{j+1})
\end{aligned}$$

and

$$\max\deg V_{k-1}(t)=\frac{2-k}{2}+\sum_{i=1}^{k-1}(n_i+|n_i|)+\frac{\epsilon_1}{2}+\frac{1}{4}\sum_{j=1}^{k-2}(1-\epsilon_j)(1+\epsilon_{j+1}).$$

Hence $\max\deg \mathcal{A}_{2n_{k+1}}(t)V_k(t)+1=\max\deg V_{k-1}(t)$. Thus we have

$$\begin{aligned}
&\max\deg V_{k+1}(t)\\
=&\ 2n_{k+1}+\max\deg V_{k-1}(t)\\
=&\ 2n_{k+1}+\frac{2-k}{2}+\sum_{i=1}^{k-1}(n_i+|n_i|)+\frac{\epsilon_1}{2}+\frac{1}{4}\sum_{j=1}^{k-2}(1-\epsilon_j)(1+\epsilon_{j+1})\\
=&\ \frac{1-(k+1)}{2}+\sum_{i=1}^{k+1}(n_i+|n_i|)+\frac{\epsilon_1}{2}+\frac{1}{4}\sum_{j=1}^{k}(1-\epsilon_j)(1+\epsilon_{j+1}).
\end{aligned}$$

By a similar argument, we also have the formula (15). □

For given nonzero integers $n_1,n_2,\ldots,n_r$, we define an integer $\kappa(n_1,n_2,\ldots,n_r)$ (or briefly $\kappa(n_i;r)$) as the number of elements in the set$\{(n_i,n_{i+1})\mid n_in_{i+1}>0, 1\leq i\leq r-1\}$. For example, $\kappa(2,3,2,-1)=2, \kappa(1,2,3,4)=3$ and $\kappa(-1,1,-2,4)=0$. We note that $0\leq\kappa(n_i;r)\leq r-1$.

Theorem 4.5. *For given nonzero integers $n_1, n_2, \ldots, n_r$, let $L^{(2)}$ be the 2-periodic link with rational quotient $L = \overrightarrow{C}[[n_1, n_2, \ldots, n_r]]$. Then the span of the Jones polynomial $V_{L^{(2)}}(t)$ of $L^{(2)}$ is given by*

$$\operatorname{span} V_{L^{(2)}}(t) = 2\sum_{i=1}^{r} |n_i| - \kappa(n_i; r).$$

Proof. From Lemma 4.4, we have

$$\begin{aligned}
&\operatorname{span} V_{L^{(2)}}(t) = \max\deg V_{L^{(2)}}(t) - \min\deg V_{L^{(2)}}(t) \\
&= \frac{1-r}{2} + \sum_{i=1}^{r}(n_i + |n_i|) + \frac{\epsilon_1}{2} + \frac{1}{4}\sum_{j=1}^{r-1}(1-\epsilon_j)(1+\epsilon_{j+1}) \\
&\quad - \frac{r-1}{2} - \sum_{i=1}^{r}(n_i - |n_i|) - \frac{\epsilon_1}{2} + \frac{1}{4}\sum_{j=1}^{r-1}(1+\epsilon_j)(1-\epsilon_{j+1}) \\
&= (1-r) + 2\sum_{i=1}^{r}|n_i| + \frac{1}{4}\sum_{j=1}^{r-1}[(1-\epsilon_j)(1+\epsilon_{j+1}) + (1+\epsilon_j)(1-\epsilon_{j+1})] \\
&= 2\sum_{i=1}^{r}|n_i| - \left\{(r-1) - \sum_{j=1}^{r-1}\frac{1-\epsilon_j\epsilon_{j+1}}{2}\right\}.
\end{aligned}$$

Because $(1-\epsilon_j\epsilon_{j+1})/2$ is 1 if n_j and n_{j+1} have different signs and 0 otherwise, $\sum_{j=1}^{r-1}(1-\epsilon_j\epsilon_{j+1})/2$ counts the number of pairs (n_j, n_{j+1}) with $n_j n_{j+1} < 0$. Therefore,

$$\kappa(n_i; r) = (r-1) - \sum_{j=1}^{r-1}\frac{1-\epsilon_j\epsilon_{j+1}}{2},$$

hence we have

$$\operatorname{span} V_{L^{(2)}}(t) = 2\sum_{i=1}^{r}|n_i| - \kappa(n_i; r).$$

This completes the proof. □

Corollary 4.6. *For given nonzero integers $a_1, a_2, \ldots, a_r$, let L be the 2-bridge knot or link with Conway normal form $C(2a_1, 2a_2, \ldots, 2a_r)$. Then the crossing number of L is given by*

$$c(L) = 2\sum_{i=1}^{r}|a_i| - \kappa(-a_1, a_2, -a_3, \ldots, (-1)^r a_r). \tag{17}$$

Proof. From [3, Theorem 2.1], L is a 2-periodic link with rational quotient $\overrightarrow{C}[[-a_1, a_2, \ldots, (-1)^r a_r]]$ (for more detail, see Remark 4.7 (2)). By Theorem 4.5, we obtain that

$$\operatorname{span} V_L(t) = 2\sum_{i=1}^{r}|a_i| - \kappa(-a_1, a_2, -a_3, \ldots, (-1)^r a_r).$$

Since any 2-bridge link is alternating, the span of its Jones polynomial is equal to its crossing number. Hence the crossing number of L is given by

$$c(L) = 2\sum_{i=1}^{r} |a_i| - \kappa(-a_1, a_2, -a_3, \ldots, (-1)^r a_r).$$

This completes the proof. $\square$

Remark 4.7. (1) It should be noticed that the result in Corollary 4.6 is not new. It is well known that every 2-bridge knot or link L has the standard Conway normal form $C(b_1, b_2, \ldots, b_n)$ such that all $b_1, b_2, \ldots, b_n$ are either positive or negative and $C(b_1, b_2, \ldots, b_n)$ is a reduced alternating diagram for L. Hence $c(L) = |b_1| + |b_2| + \cdots + |b_n|$. It is also known that L admits a Conway normal form $C(2a_1, 2a_2, \ldots, 2a_r)$ for some nonzero integers $a_1, a_2, \ldots, a_r$ [7], adopted in Corollary 4.6. The authors do not know whether the formula (17) of Corollary 4.6 can be directly derived from the standard Conway normal form $C(b_1, b_2, \ldots, b_n)$ or not.

(2) Let L be a link of two components and let L_1 be the same link as L but with the opposite orientation on only one component of L. Note that L and L_1 are may be different. But the crossing numbers of L and L_1 are the same. Since every 2-bridge link is invertible, there are at most two oriented 2-bridge links with the same unoriented diagram. Without loss of generality, in the proof of Corollary 4.6, we can consider that L is a 2-periodic link with rational quotient $\overrightarrow{C}[[-a_1, a_2, \ldots, (-1)^r a_r]]$.

5. Recursive formula for the Jones polynomial of 3-periodic links

In this section, we give a recursive formula for the Jones polynomial of a 3-periodic link with rational quotient. We also calculate the span of the Jones polynomial under certain conditions.

Lemma 5.1. *Let $n_1, n_2, \ldots, n_r$ be given nonzero integers. For each $k = 1, 2, \ldots, r$, let $D_k^{(3)}$ be the canonical 3-periodic diagram of the 3-periodic link with rational quotient $L_k = \overrightarrow{C}[[n_1, n_2, \ldots, n_k]]$. Let $D_0^{(2)}$ denote the standard diagram of the unknot. Then we have the following recursive formula*:

$$\begin{aligned}
&\langle D_0^{(3)}\rangle = 1,\\
&\gamma_1 = \delta,\\
(18)\quad &\langle D_1^{(3)}\rangle = 3\alpha_{n_1}\beta_{n_1}^2 + (3\alpha_{n_1}^2\beta_{n_1} + \beta_{n_1}^3)\delta + \alpha_{n_1}^3\delta^2,\\
&\gamma_k = (-A)^{3n_{k-1}}\left(\alpha_{n_{k-1}}^2\gamma_{n_{k-1}} + (\beta_{n_{k-1}}^2\delta + 2\alpha_{n_{k-1}}\beta_{n_{k-1}})\langle D_{k-2}^{(3)}\rangle\right),\\
&\langle D_k^{(3)}\rangle = (\beta_{n_k}^3\delta + 3\alpha_{n_k}\beta_{n_k}^2)\langle D_{k-1}^{(3)}\rangle + 3\alpha_{n_k}^2\beta_{n_k}\gamma_k\\
&\qquad + (-A)^{9n_{k-1}}\alpha_{n_k}^3\langle D_{k-2}^{(3)}\rangle, \qquad k = 2, 3, \ldots, r.
\end{aligned}$$

Proof. For a weight function $s \in \mathcal{S}_k$, let $D_k^{(3)}(s(T_{k,1}), s(T_{k,2}), s(T_{k,3}))$ be the diagram obtained from $D_k^{(3)}$ by replacing each tangle $T_{k,i}$ by a $s(T_{k,i})$-tangle $(i = 1, 2, 3)$.

If $k = 1$, then $D_1^{(3)}(0, \infty, \infty)$, $D_1^{(3)}(\infty, 0, \infty)$ and $D_1^{(3)}(\infty, \infty, 0)$ consist of a simple closed curve. We also observe that $D_1^{(3)}(0, 0, \infty)$, $D_1^{(3)}(0, \infty, 0)$, $D_1^{(3)}(\infty, 0, 0)$ and $D_1^{(3)}(\infty,\ \infty,\ \infty)$ consist of two simple closed curves, and $D_1^{(3)}(0,\ 0,\ 0)$ consists of three simple closed curves. By Proposition 3.4, we have

$$\langle D_1^{(3)} \rangle = 3\alpha_{n_1}\beta_{n_1}^2 + (3\alpha_{n_1}^2\beta_{n_1} + \beta_{n_1}^3)\delta + \alpha_{n_1}^3\delta^2.$$

Now we assume that the recursive formula (18) holds for $n_1, n_2, \ldots, n_{k-1}$ with $k \geq 2$. Then $D_k^{(3)}(\infty, \infty, \infty), D_k^{(3)}(0, \infty, \infty), D_k^{(3)}(\infty, 0, \infty), D_k^{(3)}(\infty, \infty, 0)$, $D_k^{(3)}(0, 0, \infty)$, $D_k^{(3)}(0, \infty, 0)$, $D_k^{(3)}(\infty, 0, 0)$ and $D_k^{(3)}(0, 0, 0)$ are isotopic to the diagrams as shown in Figure 11. Thus $D_k^{(3)}(0, \infty, \infty)$, $D_k^{(3)}(\infty, 0, \infty)$ and $D_k^{(3)}(\infty, \infty, 0)$ are isotopic to the diagram $D_{k-1}^{(3)}$, and $D_k^{(3)}(\infty, \infty, \infty)$ is isotopic to the diagram $D_{k-1}^{(3)} \sqcup \bigcirc$. Moreover $D_k^{(3)}(0, 0, 0)$ is obtained from $D_{k-2}^{(3)}$ by applying the Reidemeister move I. Since $D_k^{(3)}$ is a periodic diagram, $D_k^{(3)}(0, 0, \infty)$, $D_k^{(3)}(0, \infty, 0)$ and $D_k^{(3)}(\infty, 0, 0)$ are isotopic to each other. Let $\overline{D_k^{(3)}(\infty, 0, 0)}$ and $\overline{\overline{D_k^{(3)}(\infty, 0, 0)}}$ be the diagrams in Figure 12(a) and (b), respectively. They are obtained from $D_{k-1}^{(3)}(\infty, 0, 0)$ and $D_{k-2}^{(3)}$ respectively by applying the Reidemeister move I. For each $k = 1, 2, \ldots, r$, we define a Laurent polynomials γ_k by

$$\gamma_k = \langle D_k^{(3)}(\infty, 0, 0) \rangle.$$

Since $D_1^{(3)}(\infty, 0, 0)$ is of two components, $\gamma_1 = \delta$. By applying Lemma 3.1 to $T_{k-1,1}$ and $T_{k-1,2}$ in $D_k^{(3)}(\infty, 0, 0)$, we get

$$\begin{aligned}
\gamma_k &= \alpha_{n_{k-1}}^2 \langle \overline{D_k^{(3)}(\infty, 0, 0)} \rangle + (\beta_{n_{k-1}}^2\delta + 2\alpha_{n_{k-1}}\beta_{n_{k-1}}) \langle \overline{\overline{D_k^{(3)}(\infty, 0, 0)}} \rangle \\
&= (-A)^{3n_{k-1}} \left(\alpha_{n_{k-1}}^2 \gamma_{n_{k-1}} + (\beta_{n_{k-1}}^2\delta + 2\alpha_{n_{k-1}}\beta_{n_{k-1}}) \langle D_{k-2}^{(3)} \rangle \right).
\end{aligned}$$

Hence we have

$$\begin{aligned}
\langle D_k^{(3)} \rangle &= \beta_{n_k}^3 \langle D_k^{(3)}(\infty, \infty, \infty) \rangle + \beta_{n_k}^2\alpha_{n_k} \langle D_k^{(3)}(0, \infty, \infty) \rangle \\
&\quad + \beta_{n_k}^2\alpha_{n_k} \langle D_k^{(3)}(\infty, 0, \infty) \rangle + \beta_{n_k}^2\alpha_{n_k} \langle D_k^{(3)}(\infty, \infty, 0) \rangle \\
&\quad + \beta_{n_k}\alpha_{n_k}^2 \langle D_k^{(3)}(0, 0, \infty) \rangle + \beta_{n_k}\alpha_{n_k}^2 \langle D_k^{(3)}(0, \infty, 0) \rangle \\
&\quad + \beta_{n_k}\alpha_{n_k}^2 \langle D_k^{(3)}(\infty, 0, 0) \rangle + \alpha_{n_k}^3 \langle D_k^{(3)}(0, 0, 0) \rangle \\
&= (\beta_{n_k}^3\delta + 3\alpha_{n_k}\beta_{n_k}^2) \langle D_{k-1}^{(3)} \rangle + 3\alpha_{n_k}^2\beta_{n_k}\gamma_k + (-A)^{9n_{k-1}}\alpha_{n_k}^3 \langle D_{k-2}^{(3)} \rangle.
\end{aligned}$$

This completes the proof. $\square$

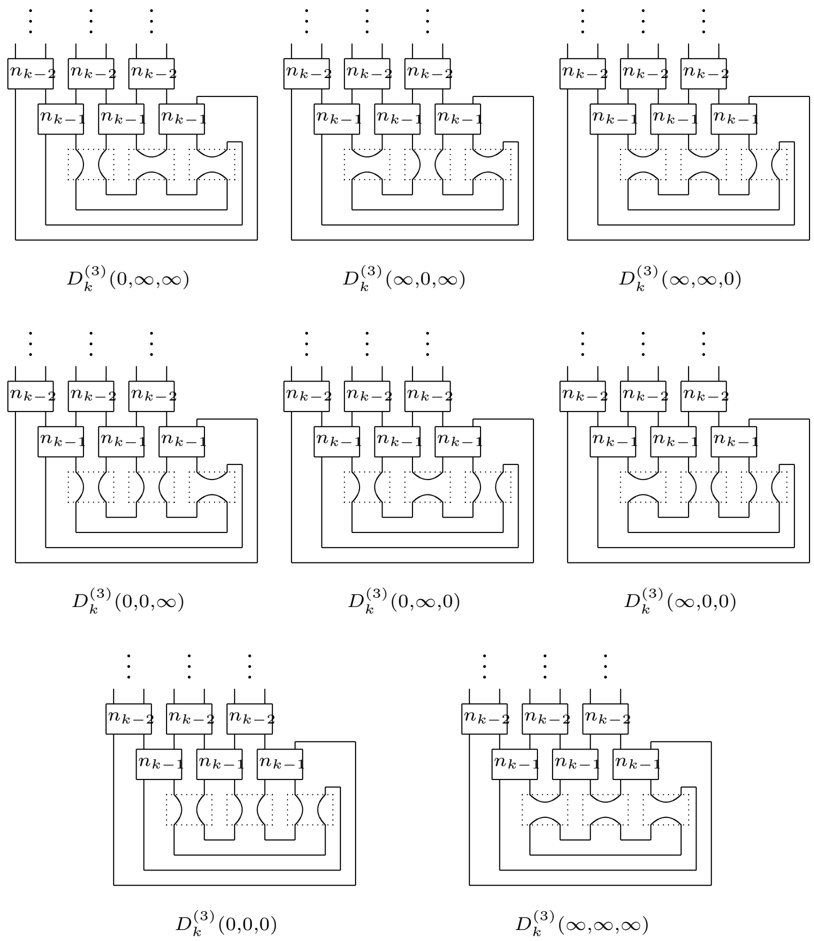
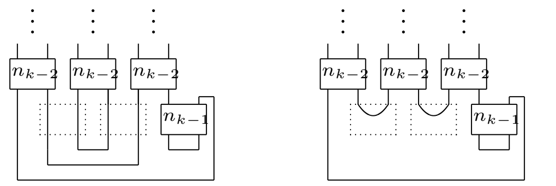


FIGURE 11

For any nonzero integer n, let $\mathcal{B}_n(t)$ be a Laurent polynomial in $\mathbb{Z}[t^{\pm\frac{1}{2}}]$ defined by

$$\mathcal{B}_n(t) = \begin{cases} t^{-1}((-t)^{-n}+2)\left(\displaystyle\sum_{i=0}^{n-1}(-t)^{-i}\right)^2 & \text{if } n \geq 1, \\ t((-t)^{-n}+2)\left(\displaystyle\sum_{i=0}^{-n-1}(-t)^{i}\right)^2 & \text{if } n \leq -1. \end{cases} \tag{19}$$

We note that $\mathcal{B}_n(t) = ((-t)^{-n}+2)\mathcal{A}_n(t)^2$.



$\overline{D_k^{(3)}(\infty,0,0)}$ (a) $\overline{\overline{D_k^{(3)}(\infty,0,0)}}$ (b)

FIGURE 12

Theorem 5.2. *Let $n_1, n_2, \ldots, n_r$ be given nonzero integers. Let $L_k^{(3)}$ be 3-periodic link with rational quotient $L_k = \overrightarrow{C}[[n_1, n_2, \ldots, n_k]]$ and let $L_0^{(3)}$ the trivial knot. Let $V_k(t)$ be the Jones polynomial of $L_k^{(3)}$ for each $k = 0, 1, 2, \ldots, r$. Then we have the following recursive formula:*

$$V_0(t) = 1, \tag{20}$$

$$V_1(t) = (-t)^{3n_1}\left(\mathcal{B}_{n_1}(t) + 3(-t)^{-n_1} + t^{-1} - 1 + t\right), \tag{21}$$

$$V_k(t) = (-t)^{3n_k}\left(\mathcal{B}_{n_k}(t)V_{k-1}(t) + 3\mathcal{A}_{n_k}(t)\lambda_k(t) + V_{k-2}(t)\right), \tag{22}$$

where

$$\lambda_1(t) = -t^{-\frac{1}{2}} - t^{\frac{1}{2}}, \tag{23}$$

$$\lambda_k(t) = t^{2n_{k-1}}\left(\lambda_{k-1}(t) + \mathcal{A}_{2n_{k-1}}(t)V_{k-2}(t)\right). \tag{24}$$

Proof. For each $k = 1, 2, \ldots, r$, let $D_k^{(3)}$ be the canonical 3-periodic diagram of the 3-periodic link $L_k^{(3)}$ with rational quotient $L = \overrightarrow{C}[[n_1, n_2, \ldots, n_k]]$. Then

$$V_k(t) = (-A)^{-3w(D_k^{(3)})}\left\langle D_k^{(3)}\right\rangle\Big|_{A=t^{-\frac{1}{4}}}.$$

For each $k = 1, 2, \ldots, r$, put $f_k(A) = (-A)^{-3w(D_k^{(3)})}\langle D_k^{(3)}\rangle$. Then $V_k(t) = f_k(A)\big|_{A=t^{-\frac{1}{4}}}$. We note that $w(D_k^{(3)}) = 3\sum_{i=1}^{k} n_i$ and that, by Lemma 3.2, $\beta_{n_i}\delta + 3\alpha_{n_i} = A^{-n_i}((-A^4)^{n_i} + 2)$.

Since $L_0^{(3)}$ is the trivial knot, $V_0(t) = 1$. If $n_1 \geq 1$, then

$$\begin{aligned}
f_1(A) &= (-A)^{-9n_1}(\beta_{n_1}^2(3\alpha_{n_1} + \beta_{n_1}\delta) + \alpha_{n_1}^2\delta(3\beta_{n_1} + \alpha_{n_1}\delta)) \\
&= (-1)^{n_1}A^{-12n_1+4}\left(\sum_{i=0}^{n_1-1}(-A^4)^i\right)^2((-A^4)^{n_1} + 2) \\
&\quad + (-1)^{n_1}A^{-12n_1+4}(-1 - A^{-4})\left(3\sum_{i=0}^{n_1-1}(-A^4)^i + (-1 - A^{-4})\right)
\end{aligned}$$

and hence

$$\begin{aligned}V_1(t) &= (-1)^{n_1} t^{3n_1-1} \left(\sum_{i=0}^{n_1-1} (-t)^{-i} \right)^2 ((-t)^{-n_1} + 2) \\ &\quad + (-1)^{n_1} t^{3n_1-1} (-1-t) \left(3 \sum_{i=0}^{n_1-1} (-t)^{-i} + (-1-t) \right) \\ &= (-t)^{3n_1} \left(\mathcal{B}_{n_1}(t) + 3(-t)^{-n_1} + t^{-1} - 1 + t \right).\end{aligned}$$

If $n_1 \leq -1$, then

$$\begin{aligned}f_1(A) &= (-1)^{n_1} A^{-12n_1-4} \left(\sum_{i=0}^{-n_1-1} (-A^{-4})^i \right)^2 ((-A^4)^{n_1} + 2) \\ &\quad + (-1)^{n_1} A^{-12n_1+4} (-1-A^{-4}) \left(3A^{-4} \sum_{i=0}^{-n_1-1} (-A^{-4})^i + (-1-A^{-4}) \right)\end{aligned}$$

and hence

$$\begin{aligned}V_1(t) &= (-1)^{n_1} t^{3n_1+1} \left(\sum_{i=0}^{-n_1-1} (-t)^i \right)^2 ((-t)^{-n_1} + 2) \\ &\quad + (-1)^{n_1} t^{3n_1-1} (-1-t) \left(3t \sum_{i=0}^{-n_1-1} (-t)^i + (-1-t) \right) \\ &= (-t)^{3n_1} \left(\mathcal{B}_{n_1}(t) + 3(-t)^{-n_1} + t^{-1} - 1 + t \right).\end{aligned}$$

Put $\lambda_1(t) = \delta \,|_{A=t^{-\frac{1}{4}}}$ and $\lambda_k(t) = (-A)^{-9\sum_{i=1}^{k-1} n_i} \gamma_k|_{A=t^{-\frac{1}{4}}}$ for each $k = 2, 3, \ldots, r$. Then $\lambda_1(t) = -t^{-\frac{1}{2}} - t^{\frac{1}{2}}$ and

$$\begin{aligned}\lambda_k(t) &= (-A)^{-9\sum_{i=1}^{k-1} n_i} (-A)^{3n_{k-1}} \alpha_{n_{k-1}}^2 \gamma_{n_{k-1}}|_{A=t^{-\frac{1}{4}}} \\ &\quad + (-A)^{-9\sum_{i=1}^{k-1} n_i} (-A)^{3n_{k-1}} \beta_{n_{k-1}} (\beta_{n_{k-1}} \delta + 2\alpha_{n_{k-1}}) \langle D_{k-2}^{(3)} \rangle|_{A=t^{-\frac{1}{4}}} \\ &= A^{-8n_{k-1}} (-A)^{-9\sum_{i=1}^{k-2} n_i} \gamma_{n_{k-1}}|_{A=t^{-\frac{1}{4}}} \\ &\quad + A^{-7n_{k-1}} ((-A^4)^{n_{k-1}} + 1) \beta_{n_{k-1}} (-A)^{-9\sum_{i=1}^{k-2} n_i} \langle D_{k-2}^{(3)} \rangle|_{A=t^{-\frac{1}{4}}} \\ &= t^{2n_{k-1}} \lambda_{k-1}(t) + t^{2n_{k-1}} ((-t)^{-n_{k-1}} + 1) \mathcal{A}_{n_{k-1}}(t) V_{k-2}(t) \\ &= t^{2n_{k-1}} \lambda_{k-1}(t) + t^{2n_{k-1}} \mathcal{A}_{2n_{k-1}}(t) V_{k-2}(t).\end{aligned}$$

For $k \geq 2$, by Lemma 5.1, we obtain

$$\begin{aligned}
f_k(A) &= (-A)^{-9\sum_{i=1}^{k} n_i}(\beta_{n_k}^3\delta + 3\alpha_{n_k}\beta_{n_k}^2)\langle D_{k-1}^{(3)}\rangle \\
&\quad + (-A)^{-9\sum_{i=1}^{k} n_i} 3\alpha_{n_k}^2\beta_{n_k}\gamma_k + (-A)^{-9\sum_{i=1}^{k} n_i}(-A)^{9n_{k-1}}\alpha_{n_k}^3\langle D_{k-2}^{(3)}\rangle \\
&= (-A)^{-9n_k}(\beta_{n_k}^3\delta + 3\alpha_{n_k}\beta_{n_k}^2)f_{k-1}(A) \\
&\quad + (-A)^{-9\sum_{i=1}^{k} n_i} 3\alpha_{n_k}^2\beta_{n_k}\gamma_k + (-A)^{-9n_k}\alpha_{n_k}^3 f_{k-2}(A).
\end{aligned}$$

Note that $(-A)^{-9n_k}\alpha_{n_k}^3|_{A=t^{-\frac{1}{4}}} = (-1)^{n_k}t^{3n_k}$ and $(-A)^{-9n_k}\alpha_{n_k}^2\beta_{n_k}|_{A=t^{-\frac{1}{4}}} = (-1)^{n_k}t^{3n_k}\mathcal{A}_{n_k}(t)$. If $n_k \geq 1$, then by Lemma 3.2 we have

$$(-A)^{-9n_k}(\beta_{n_k}^3\delta + 3\alpha_{n_k}\beta_{n_k}^2) = (-1)^{n_k}A^{-12n_k+4}((-A^4)^{n_k}+2)\left(\sum_{i=0}^{n_k-1}(-A^4)^i\right)^2$$

and hence

$$\begin{aligned}
&(-A)^{-9n_k}(\beta_{n_k}^3\delta + 3\alpha_{n_k}\beta_{n_k}^2)|_{A=t^{-\frac{1}{4}}} \\
&= (-1)^{n_k}t^{3n_k-1}((-t)^{-n_k}+2)\left(\sum_{i=0}^{n_k-1}(-t)^{-i}\right)^2 \\
&= (-t)^{3n_k}\mathcal{B}_{n_k}(t).
\end{aligned}$$

If $n_k \leq -1$, then by Lemma 3.2 we also have

$$(-A)^{-9n_k}(\beta_{n_k}^3\delta + 3\alpha_{n_k}\beta_{n_k}^2) = (-1)^{n_k}A^{-12n_k-4}((-A^4)^{n_k}+2)\left(\sum_{i=0}^{-n_k-1}(-A^{-4})^i\right)^2$$

and hence

$$\begin{aligned}
&(-A)^{-9n_k}(\beta_{n_k}^3\delta + 3\alpha_{n_k}\beta_{n_k}^2)|_{A=t^{-\frac{1}{4}}} \\
&= (-1)^{n_k}t^{3n_k+1}((-t)^{-n_k}+2)\left(\sum_{i=0}^{-n_k-1}(-t)^{i}\right)^2 \\
&= (-t)^{3n_k}\mathcal{B}_{n_k}(t).
\end{aligned}$$

Therefore we have

$$\begin{aligned}
V_k(t) &= f_k(A)|_{A=t^{-\frac{1}{4}}} \\
&= (-A)^{-9n_k}(\beta_{n_k}^3\delta + 3\alpha_{n_k}\beta_{n_k}^2)f_{k-1}(A)|_{A=t^{-\frac{1}{4}}} \\
&\quad + (-A)^{-9\sum_{i=1}^{k} n_i} 3\alpha_{n_k}^2\beta_{n_k}\gamma_k|_{A=t^{-\frac{1}{4}}} + (-A)^{-9n_k}\alpha_{n_k}^3 f_{k-2}(A)|_{A=t^{-\frac{1}{4}}} \\
&= (-t)^{3n_k}\mathcal{B}_{n_k}(t)V_{k-1}(t) + 3(-t)^{3n_k}\mathcal{A}_{n_k}(t)\lambda_k(t) + (-t)^{3n_k}V_{k-2}(t).
\end{aligned}$$

This completes the proof. □

Example 5.3. Let L be the 3-periodic knot with rational quotient $\overrightarrow{C}[[1,-1,1]]$. Then L is the knot 9_{40} in Rolfsen's table [20]. Let $n_1=1$, $n_2=-1$ and $n_3=1$. Then we have that $\mathcal{A}_1(t)=t^{-\frac{1}{2}}$, $\mathcal{A}_{-1}(t)=t^{\frac{1}{2}}$, $\mathcal{A}_2(t)=t^{-\frac{1}{2}}-t^{-\frac{3}{2}}$, $\mathcal{A}_{-2}(t)=t^{\frac{1}{2}}-t^{\frac{3}{2}}$, $\mathcal{B}_1(t)=2t^{-1}-t^{-2}$ and $\mathcal{B}_{-1}(t)=2t-t^2$. From Theorem 5.2, we get that $V_0(t)=1$, $V_1(t)=-t^4+t^3+t$, $\lambda_2=-t^{\frac{5}{2}}-t^{\frac{1}{2}}$, $V_2(t)=-t^3+3t^2-2t+4-2t^{-1}+3t^{-2}-t^{-3}$, $\lambda_3=t^{\frac{7}{2}}-2t^{\frac{5}{2}}+t^{\frac{3}{2}}-2t^{\frac{1}{2}}+t^{-\frac{1}{2}}-t^{-\frac{3}{2}}$ and $V_3(t)=t^7-4t^6+8t^5-11t^4+13t^3-13t^2+11t-8+5t^{-1}-t^{-2}$. Hence the Jones polynomial of L is

$$V_L(t)=t^7-4t^6+8t^5-11t^4+13t^3-13t^2+11t-8+5t^{-1}-t^{-2}.$$

Let n be any nonzero integer. From (19), we note that

$$\max\deg\mathcal{B}_n(t)=-1+\frac{3}{2}(|n|-n),\quad \min\deg\mathcal{B}_n(t)=1-\frac{3}{2}(|n|+n). \tag{25}$$

From (10), we also note that

$$\max\deg\mathcal{A}_n(t)=-\frac{1}{2}+\frac{1}{2}(|n|-n),\quad \min\deg\mathcal{A}_n(t)=\frac{1}{2}-\frac{1}{2}(|n|+n). \tag{26}$$

Lemma 5.4. *For given nonzero integers $n_1,n_2,\ldots,n_r$, let $V_k(t)$ be the Jones polynomial of the* 3*-periodic link $L_k^{(3)}$ with rational quotient $L_k=\overrightarrow{C}[[n_1,n_2,\ldots,n_k]]$. Suppose that $n_i\neq 1$ for all $i=1,2,\ldots,r$. In the recursive formula in Theorem 5.2, we have the following properties for each $k=2,3,\ldots,r$:*

(1) *If $n_k>1$ and $n_{k-1}\leq -1$, then* $\max\deg\mathcal{B}_{n_k}(t)V_{k-1}(t)<\max\deg V_{k-2}(t)$ *and* $\max\deg\mathcal{A}_{n_k}(t)\lambda_k(t)<\max\deg V_{k-2}(t)$.

(2) *If $n_k\leq -1$ or $n_{k-1}>1$, then* $\max\deg V_{k-2}(t)<\max\deg\mathcal{B}_{n_k}(t)V_{k-1}(t)$ *and* $\max\deg\mathcal{A}_{n_k}(t)\lambda_k(t)<\max\deg\mathcal{B}_{n_k}(t)V_{k-1}(t)$.

Proof. We will use induction on k. If $n_1>1$, then $\max\deg\mathcal{B}_{n_1}(t)=-1$. From (21), we have $\max\deg V_1(t)=3n_1+1$. If $n_1\leq -1$, then $\max\deg\mathcal{B}_{n_1}(t)=-1-3n_1$. From (21), we get $\max\deg V_1(t)=-1$. Therefore we have

$$\max\deg V_1(t)=\frac{3}{2}(|n_1|+n_1)+\epsilon_1.$$

From (20) and (24), we note that $\max\deg\lambda_2(t)=(|n_1|+n_1)+\frac{1}{2}\epsilon_1$ and $\max\deg V_0(t)=0$. If $n_2>1$ and $n_1\leq -1$, then $\max\deg\mathcal{B}_{n_2}(t)V_1(t)=-2$ and $\max\deg\mathcal{A}_{n_2}(t)\lambda_2(t)=-1$. Hence we have that if $n_2>1$ and $n_1\leq -1$, then

$$\max\deg\mathcal{B}_{n_2}(t)V_1(t)<\max\deg V_0(t)$$

and

$$\max\deg\mathcal{A}_{n_2}(t)\lambda_2(t)<\max\deg V_0(t).$$

If $n_2\leq -1$, then

$$\max\deg\mathcal{B}_{n_2}(t)V_1(t)=-1-3n_2+\frac{3}{2}(|n_1|+n_1)+\epsilon_1$$

and

$$\max\deg\mathcal{A}_{n_2}(t)\lambda_2(t)=-\frac{1}{2}-n_2+(|n_1|+n_1)+\frac{1}{2}\epsilon_1.$$

If $n_1 > 1$, then

$$\max\deg \mathcal{B}_{n_2}(t)V_1(t) = \frac{3}{2}(|n_2| - n_2) + 3n_1$$

and

$$\max\deg \mathcal{A}_{n_2}(t)\lambda_2(t) = \frac{1}{2}(|n_2| - n_2) + 2n_1.$$

Hence we have that if $n_2 \leq -1$ or $n_1 > 1$, then

$$\max\deg V_0(t) < \max\deg \mathcal{B}_{n_2}(t)V_1(t)$$

and

$$\max\deg \mathcal{A}_{n_2}(t)\lambda_2(t) < \max\deg \mathcal{B}_{n_2}(t)V_1(t).$$

Now we assume that the statements hold for $\leq k$. From now on we will prove that the statements hold for $k+1$.

Case (i) : Suppose that $n_{k+1} > 1$ and $n_k \leq -1$. By the induction hypothesis and (22), we have $\max\deg V_k(t) = 3n_k + \max\deg \mathcal{B}_{n_k}(t)V_{k-1}(t)$. Hence we have

$$\begin{aligned}
\max\deg \mathcal{B}_{n_{k+1}}(t)V_k(t) &= -1 + \max\deg V_k(t) \\
&= -1 + 3n_k + \max\deg \mathcal{B}_{n_k}(t)V_{k-1}(t) \\
&= -1 + 3n_k + (-1 - 3n_k) + \max\deg V_{k-1}(t) \\
&= -2 + \max\deg V_{k-1}(t) \\
&< \max\deg V_{k-1}(t). \qquad (27)
\end{aligned}$$

From (24), it is true that either $\max\deg \lambda_{k+1}(t) \leq 2n_k + \max\deg \lambda_k(t)$ or $\max\deg \lambda_{k+1}(t) \leq 2n_k + \max\deg \mathcal{A}_{2n_k}(t)V_{k-1}(t)$. If $\max\deg \lambda_{k+1}(t) \leq 2n_k + \max\deg \lambda_k(t)$, then, by the induction hypothesis, we get

$$\begin{aligned}
\max\deg \lambda_{k+1}(t) &\leq 2n_k + \max\deg \lambda_k(t) \\
&< 2n_k - \max\deg \mathcal{A}_{n_k}(t) + \max\deg \mathcal{B}_{n_k}(t)V_{k-1}(t) \\
&= -\frac{1}{2} + \max\deg V_{k-1}(t).
\end{aligned}$$

If $\max\deg \lambda_{k+1}(t) \leq 2n_k + \max\deg \mathcal{A}_{2n_k}(t)V_{k-1}(t)$, then we have

$$\max\deg \lambda_{k+1}(t) \leq -\frac{1}{2} + \max\deg V_{k-1}(t).$$

Hence we know

$$\begin{aligned}
\max\deg \mathcal{A}_{n_{k+1}}(t)\lambda_{k+1}(t) &= -\frac{1}{2} + \max\deg \lambda_{k+1}(t) \\
&\leq -1 + \max\deg V_{k-1}(t) \\
&< \max\deg V_{k-1}(t). \qquad (28)
\end{aligned}$$

By (27) and (28), it follows that the statement (1) holds.

Case (ii) : Suppose that $n_{k+1} \leq -1$ or $n_k > 1$. By the induction hypothesis and (22), we have that

$$\begin{aligned} \max\deg V_k(t) &\geq 3n_k + \max\deg \mathcal{B}_{n_k}(t) + \max\deg V_{k-1}(t) \\ &= -1 + \frac{3}{2}(|n_k| + n_k) + \max\deg V_{k-1}(t) \end{aligned} \tag{29}$$

and

$$\begin{aligned} \max\deg \lambda_k(t) &\leq \max\deg V_k(t) - 3n_k - \max\deg \mathcal{A}_{n_k}(t) \\ &= \max\deg V_k(t) - 3n_k + \frac{1}{2} - \frac{1}{2}(|n_k| - n_k). \end{aligned} \tag{30}$$

From (29), we get

$$\begin{aligned} & \max\deg \mathcal{B}_{n_{k+1}}(t)V_k(t) \\ = & -1 + \frac{3}{2}(|n_{k+1}| - n_{k+1}) + \max\deg V_k(t) \\ \geq & -1 + \frac{3}{2}(|n_{k+1}| - n_{k+1}) - 1 + \frac{3}{2}(|n_k| + n_k) + \max\deg V_{k-1}(t) \\ = & -2 + \frac{3}{2}(|n_{k+1}| - n_{k+1}) + \frac{3}{2}(|n_k| + n_k) + \max\deg V_{k-1}(t) \\ > & \max\deg V_{k-1}(t). \end{aligned} \tag{31}$$

From (24), we know that either

$$\max\deg \lambda_{k+1}(t) \leq 2n_k + \max\deg \lambda_k(t)$$

or

$$\max\deg \lambda_{k+1}(t) \leq 2n_k + \max\deg \mathcal{A}_{2n_k}(t) + \max\deg V_{k-1}(t).$$

If $\max\deg \lambda_{k+1}(t) \leq 2n_k + \max\deg \lambda_k(t)$, then from (30), we calculate

$$\begin{aligned} \max\deg \lambda_{k+1}(t) &\leq 2n_k + \max\deg \lambda_k(t) \\ &\leq 2n_k + \max\deg V_k(t) - 3n_k + \frac{1}{2} - \frac{1}{2}(|n_k| - n_k) \\ &= \max\deg V_k(t) + \frac{1}{2} - \frac{1}{2}(|n_k| + n_k) \\ &= \max\deg \mathcal{B}_{n_{k+1}}(t)V_k(t) - \max\deg \mathcal{A}_{n_{k+1}}(t) \\ &\quad + \left(1 - \frac{1}{2}(|n_k| + n_k) - (|n_{k+1}| - n_{k+1})\right). \end{aligned}$$

If $\max\deg \lambda_{k+1}(t) \leq 2n_k + \max\deg \mathcal{A}_{2n_k}(t) + \max\deg V_{k-1}(t)$, then from (29), we have

$$\begin{aligned}\max\deg \lambda_{k+1}(t) &\leq 2n_k + \max\deg \mathcal{A}_{2n_k}(t) + \max\deg V_{k-1}(t)\\ &\leq 2n_k - \frac{1}{2} + (|n_k| - n_k) + 1 - \frac{3}{2}(|n_k| + n_k) + \max\deg V_k(t)\\ &= \frac{1}{2} - \frac{1}{2}(|n_k| + n_k) + \max\deg V_k(t)\\ &= \max\deg \mathcal{B}_{n_{k+1}}(t)V_k(t) - \max\deg \mathcal{A}_{n_{k+1}}(t)\\ &\quad + \left(1 - \frac{1}{2}(|n_k| + n_k) - (|n_{k+1}| - n_{k+1})\right).\end{aligned}$$

Since $n_{k+1} \leq -1$ or $n_k > 1$, we get that $1 - \frac{1}{2}(|n_k| + n_k) - (|n_{k+1}| - n_{k+1}) < 0$ and hence

$$\max\deg \mathcal{A}_{n_{k+1}}(t)\lambda_{k+1}(t) < \max\deg \mathcal{B}_{n_{k+1}}(t)V_k(t). \tag{32}$$

By (31) and (32), it follows that the statement (2) holds. This completes the proof. □

Lemma 5.5. *For given nonzero integers $n_1, n_2, \ldots, n_r$, let $V_k(t)$ be the Jones polynomial of the* 3*-periodic link $L_k^{(3)}$ with rational quotient $L_k = \overrightarrow{C}[[n_1, n_2, \ldots, n_k]]$. Suppose that $n_i \neq -1$ for all $i = 1, 2, \ldots, r$. In the recursive formula in Theorem 5.2, we have the following properties for each $k = 2, 3, \ldots, r$:*

(1) *If $n_k < -1$ and $n_{k-1} \geq 1$, then* $\min\deg \mathcal{B}_{n_k}(t)V_{k-1}(t) > \min\deg V_{k-2}(t)$ *and* $\min\deg \mathcal{A}_{n_k}(t)\lambda_k(t) > \min\deg V_{k-2}(t)$.

(2) *If $n_k \geq 1$ or $n_{k-1} < -1$, then* $\min\deg V_{k-2}(t) > \min\deg \mathcal{B}_{n_k}(t)V_{k-1}(t)$ *and* $\min\deg \mathcal{A}_{n_k}(t)\lambda_k(t) > \min\deg \mathcal{B}_{n_k}(t)V_{k-1}(t)$.

Proof. Let $m_i = -n_i$ for all $i = 1, 2, \ldots, r$ and let $\tilde{V}_k(t)$ be the Jones polynomial of the 3-periodic link $\tilde{L}_k^{(3)}$ with rational quotient $\tilde{L}_k = \overrightarrow{C}[[m_1, m_2, \ldots, m_k]]$. Let $\tilde{\lambda}_k(t)$ be the Laurent polynomial recursively defined by

$$\tilde{\lambda}_1(t) = -t^{-\frac{1}{2}} - t^{\frac{1}{2}},\ \tilde{\lambda}_k(t) = t^{2m_{k-1}}(\tilde{\lambda}_{k-1}(t) + \mathcal{A}_{2m_{k-1}}(t)\tilde{V}_{k-2}(t)).$$

By (25) and (26),

$$\max\deg \mathcal{B}_{m_k}(t) = -\min\deg \mathcal{B}_{n_k}(t),\ \max\deg \mathcal{A}_{m_k}(t) = -\min\deg \mathcal{A}_{n_k}(t).$$

We observe that $\tilde{L}_k^{(3)}$ is the mirror image of $L_k^{(3)}$ and hence $\tilde{V}_k(t) = V_k(t^{-1})$. Therefore

$$\max\deg \tilde{V}_k(t) = -\min\deg V_k(t),\ \max\deg \tilde{\lambda}_k(t) = -\min\deg \lambda_k(t).$$

By Lemma 5.4,

(1) if $m_k > 1$ and $m_{k-1} \leq -1$, then $\max\deg \mathcal{B}_{m_k}(t)\tilde{V}_{k-1}(t) < \max\deg \tilde{V}_{k-2}(t)$ and $\max\deg \mathcal{A}_{m_k}(t)\tilde{\lambda}_k(t) < \max\deg \tilde{V}_{k-2}(t)$,

(2) if $m_k \leq -1$ or $m_{k-1} > 1$, then $\max\deg \tilde{V}_{k-2}(t) < \max\deg \mathcal{B}_{m_k}(t)\tilde{V}_{k-1}(t)$ and $\max\deg \mathcal{A}_{m_k}(t)\tilde{\lambda}_k(t) < \max\deg \mathcal{B}_{m_k}(t)\tilde{V}_{k-1}(t)$.

This completes the proof. □

Theorem 5.6. *For given nonzero integers $n_1, n_2, \ldots, n_r$, let $V_k(t)$ be the Jones polynomial of the 3-periodic link $L_k^{(3)}$ with rational quotient $L_k = \overrightarrow{C}[[n_1, n_2, \ldots, n_k]]$. Suppose that $n_i \neq \pm 1$ for all $i = 1, 2, \ldots, r$. Then*

$$\max\deg V_k(t) = (1-k) + \frac{3}{2}\sum_{i=1}^{k}(n_i + |n_i|) + \epsilon_1 + \frac{1}{2}\sum_{j=1}^{k-1}(1-\epsilon_j)(1+\epsilon_{j+1})$$

and

$$\min\deg V_k(t) = (k-1) + \frac{3}{2}\sum_{i=1}^{k}(n_i - |n_i|) + \epsilon_1 - \frac{1}{2}\sum_{j=1}^{k-1}(1+\epsilon_j)(1-\epsilon_{j+1}).$$

Proof. In the proof of Lemma 5.4, we have

$$\max\deg V_1(t) = \frac{3}{2}(|n_1| + n_1) + \epsilon_1.$$

If $n_2 > 1$ and $n_1 < -1$, then by (1) in Lemma 5.4 we obtain

$$\max\deg V_2(t) = 3n_2 + \max\deg V_0(t) = \frac{3}{2}(n_2 + |n_2|).$$

If $n_2 < -1$ or $n_1 > 1$, then by (2) in Lemma 5.4 we get

$$\begin{aligned}\max\deg V_2(t) &= 3n_2 + \max\deg \mathcal{B}_{n_2}(t)V_1(t)\\ &= 3n_2 + \left(-1 + \frac{3}{2}(|n_2| - n_2) + \frac{3}{2}(|n_1| + n_1) + \epsilon_1\right)\\ &= -1 + \frac{3}{2}(|n_2| + n_2) + \frac{3}{2}(|n_1| + n_1) + \epsilon_1.\end{aligned}$$

Therefore we have

$$\max\deg V_2(t) = -1 + \frac{3}{2}\sum_{i=1}^{2}(n_i + |n_i|) + \epsilon_1 + \frac{1}{2}(1-\epsilon_1)(1+\epsilon_2).$$

If $n_{k+1} > 1$ and $n_k < -1$, then by (1) in Lemma 5.4 we get

$$\begin{aligned}\max\deg V_{k+1}(t) &= 3n_{k+1} + \max\deg V_{k-1}(t)\\ &= 3n_{k+1} + (2-k) + \frac{3}{2}\sum_{i=1}^{k-1}(n_i + |n_i|) + \epsilon_1\\ &\quad + \frac{1}{2}\sum_{j=1}^{k-2}(1-\epsilon_j)(1+\epsilon_{j+1}).\end{aligned}$$

If $n_{k+1} < -1$ or $n_k > 1$, then by (2) in Lemma 5.4 we calculate

$$\begin{aligned}\max\deg V_{k+1}(t) &= 3n_{k+1} + \max\deg \mathcal{B}_{n_{k+1}}(t)V_k(t)\\ &= -1 + \frac{3}{2}(|n_{k+1}| + n_{k+1}) + (1-k)\\ &\quad + \frac{3}{2}\sum_{i=1}^{k}(n_i + |n_i|) + \epsilon_1 + \frac{1}{2}\sum_{j=1}^{k-1}(1-\epsilon_j)(1+\epsilon_{j+1}).\end{aligned}$$

Therefore we have

$$\max\deg V_{k+1}(t) = -k + \frac{3}{2}\sum_{i=1}^{k+1}(n_i + |n_i|) + \epsilon_1 + \frac{1}{2}\sum_{j=1}^{k}(1-\epsilon_j)(1+\epsilon_{j+1}).$$

By a similar argument, we also have

$$\min\deg V_{k+1}(t) = k + \frac{3}{2}\sum_{i=1}^{k+1}(n_i - |n_i|) + \epsilon_1 - \frac{1}{2}\sum_{j=1}^{k}(1+\epsilon_j)(1-\epsilon_{j+1}).$$

This completes the proof. □

Theorem 5.7. *For given nonzero integers $n_1, n_2, \dots, n_r$, let $L^{(3)}$ be 3-periodic link with rational quotient $L = \overrightarrow{C}[[n_1, n_2, \dots, n_r]]$. If $|n_i| \geq 2$ for all $i = 1, 2, \dots, r$, then the span of the Jones polynomial $V_{L^{(3)}}(t)$ of $L^{(3)}$ is given by*

$$\text{span } V_{L^{(3)}}(t) = 3\sum_{i=1}^{r}|n_i| - 2\kappa(n_i; r).$$

Proof. From Theorem 5.6, we have

$$\begin{aligned}\text{span } V_{L^{(3)}}(t) &= \max\deg V_r(t) - \min\deg V_r(t)\\ &= (1-r) + \frac{3}{2}\sum_{i=1}^{r}(n_i + |n_i|) + \epsilon_1 + \frac{1}{2}\sum_{j=1}^{r-1}(1-\epsilon_j)(1+\epsilon_{j+1})\\ &\quad - (r-1) - \frac{3}{2}\sum_{i=1}^{r}(n_i - |n_i|) - \epsilon_1 + \frac{1}{2}\sum_{j=1}^{r-1}(1+\epsilon_j)(1-\epsilon_{j+1})\\ &= 3\sum_{i=1}^{r}|n_i| - 2\kappa(n_i; r)\end{aligned}$$

because

$$\kappa(n_i; r) = r - 1 - \frac{1}{4}\sum_{j=1}^{r-1}[(1-\epsilon_j)(1+\epsilon_{j+1}) + (1+\epsilon_j)(1-\epsilon_{j+1})]$$

as we have seen in the proof of Theorem 4.5. This completes the proof. □

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