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RIGIDITY OF PROPER HOLOMORPHIC MAPS FROM $\mathbf{B}^{n+1}$ TO $\mathbf{B}^{3n-1}$

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Abstract. Let $\mathbf{B}^{n+1}$ be the unit ball in the complex vector space $\mathbb{C}^{n+1}$ with the standard Hermitian metric. Let $\Sigma^n = \partial\mathbf{B}^{n+1} = S^{2n+1}$ be the boundary sphere with the induced CR structure. Let $f : \Sigma^n \hookrightarrow \Sigma^N$ be a local CR immersion. If $N < 3n-1$, the asymptotic vectors of the CR second fundamental form of f at each point form a subspace of the CR(horizontal) tangent space of Σ^n of codimension at most 1. We study the higher order derivatives of this relation, and we show that a linearly full local CR immersion $f : \Sigma^n \hookrightarrow \Sigma^N$, $N \leq 3n-2$, can only occur when $N = n, 2n$, or $2n+1$. As a consequence, it gives an extension of the classification of the rational proper holomorphic maps from $\mathbf{B}^{n+1}$ to $\mathbf{B}^{2n+2}$ by Hamada to the classification of the rational proper holomorphic maps from $\mathbf{B}^{n+1}$ to $\mathbf{B}^{3n-1}$.

Introduction

Let $\mathbf{B}^{n+1}$ be the unit ball in the complex vector space $\mathbb{C}^{n+1}$ with the standard Hermitian metric. Hamada gave a classification of the rational proper holomorphic maps from $\mathbf{B}^{n+1}$ to $\mathbf{B}^{2n+2}$, $n \geq 3$ ([8]). The purpose of this paper is to show that the classification by Hamada continues to hold when we increase the dimension of the target unit ball from $2n+2$ to $3n-1$. Let $z = (z^0, z^i)$, $1 \leq i \leq n$, be the standard coordinates of $\mathbb{C}^{n+1}$.

Theorem. *Let $F : \boldsymbol{B}^{n+1} \to \boldsymbol{B}^{3n-1}$ be a proper holomorphic map which is C^3 up to the boundary, $n \geq 3$. Up to automorphisms of the unit balls, F belongs to one of the following three classes of polynomial maps.*

A. *Linear embedding $F : \boldsymbol{B}^{n+1} \to \boldsymbol{B}^{n+1} \subset \boldsymbol{B}^{3n-1}$.*

B. *Whitney map $F : \boldsymbol{B}^{n+1} \to \boldsymbol{B}^{2n+1} \subset \boldsymbol{B}^{3n-1}$ defined by*

$$F(z) = (z^i, z^i z^0, (z^0)^2).$$

C. *$F : \boldsymbol{B}^{n+1} \to \boldsymbol{B}^{2n+2} \subset \boldsymbol{B}^{3n-1}$ is defined for some $\phi, 0 < \phi < \frac{\pi}{2}$, by*

$$F(z) = (z^i, \cos(\phi)\, z^0, \sin(\phi)\, z^i z^0, \sin(\phi)\, (z^0)^2).$$

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The study of proper holomorphic maps between the unit balls is a subject with a long and fruitful history which goes back to Poincare, and Alexander [5, 7]. Recently, Huang and Ji showed that a rational proper holomorphic map from $\mathbf{B}^{n+1}$ to $\mathbf{B}^{2n+1}$, $n \geq 2$, belongs to either the class of linear embedding $\mathbf{A}$, or the class of Whitney map $\mathbf{B}$ [11]. Hamada showed that a rational proper holomorphic map from $\mathbf{B}^{n+1}$ to $\mathbf{B}^{2n+2}$, $n \geq 3$, belongs to one of three classes of maps $\mathbf{A}$, $\mathbf{B}$, or $\mathbf{C}$ [8]. The one parameter family of maps $\mathbf{C}$ was introduced by D'Angelo [4].

Let $\Sigma^n = \partial \mathbf{B}^{n+1} = S^{2n+1}$ be the boundary sphere with the induced CR structure. A proper holomorphic map $F : \mathbf{B}^{n+1} \to \mathbf{B}^{3n-1}$ which is C^3 up to the boundary is rational [12, Corollary 1.4], and in particular it is real analytic. For a proof of Theorem, it thus suffices to show that a local CR immersion $f : \Sigma^n \hookrightarrow \Sigma^{3n-2}$ takes values in a lower dimensional sphere $f : \Sigma^n \hookrightarrow \Sigma^{2n+1} \subset \Sigma^{3n-2}$ for $n \geq 3$, thereby reducing the classification problem to the case already treated by Hamada. Our analysis shows that a linearly full local CR immersion $f : \Sigma^n \hookrightarrow \Sigma^N$, $N \leq 3n-2$, can only occur when $N = n$, $2n$, or $2n+1$.

The asymptotic vectors of the CR second fundamental form of a local CR immersion $f : \Sigma^n \hookrightarrow \Sigma^N$ form a subspace, rather than a cone, of the CR (horizontal) tangent space of Σ^n. *Rank* of the second fundamental form at a given point is defined as the codimension of the asymptotic subspace. When $N < 3n-1$, an algebraic analysis of CR Gauß equation shows that the rank is at most 1 [14]. Our idea is to study the structure of the higher order derivatives of this relation to obtain the desired rigidity theorem.

The differential analysis of the derivatives of CR second fundamental form here is reminiscent of Cartan's analysis of local isometric embedding of Hyperbolic space $\mathbb{H}^n$ in Euclidean space $\mathbb{E}^{2n-1}$ via exteriorly orthogonal symmetric bilinear forms [2]. Overdetermined nature of CR geometry forces the structure equations to close up rather than becoming involutive.

The analysis suggests that this type of *gap phenomena* may persist for local CR immersions $f : \Sigma^n \to \Sigma^{\mu(n+1)}$ with second fundamental form of rank 1 for $\mu \leq n-1$ [13]. We make a relevant remark at the end of Section 2.

Huang introduced the notion of geometric rank of a CR map between spheres [10]. A computation shows that if a CR immersion has second fundamental form of rank 1 everywhere, then it has geometric rank 1. Geometric rank is likely bounded above by the rank of the second fundamental form at a generic point.

After the present paper was circulated, we were informed of the foundational work [13] by Huang, Ji, and Xu. A deeper and extensive investigation of CR maps between spheres with bounded geometric rank, and its applications to the study of holomorphic maps between unit balls is presented in [13], which includes the rigidity theorem of the present paper among many other valuable results. The central idea of our proof would be that we exploit the overdetermined nature of tangential Cauchy-Riemann equation directly from the outset, and the analysis of the associated structure equations becomes amenable to the standard machinery of overdetermined systems.

1. CR submanifold

In this section, we set up the basic structure equations for CR submanifolds in spheres. We then apply an inequality due to Iwatani to determine a normal form for the CR second fundamental form of a CR flat submanifold. For general references for CR geometry, [3, 5, 7]. For a reference for CR submanifolds in spheres, [6].

Let $\mathbb{C}^{N+1,1}$ be the complex vector space with coordinates $z = (z^0, z^A, z^{N+1})$, $1 \leq A \leq N$, and a Hermitian scalar product

$$\langle z, \bar{z} \rangle = z^A \bar{z}^A + \mathrm{i}(z^0 \bar{z}^{N+1} - z^{N+1} \bar{z}^0).$$

Let Σ^N be the set of equivalence classes up to scale of null vectors with respect to this product. Let $\mathrm{SU}(N+1,1)$ be the group of unimodular linear transformations that leave the form $\langle z, \bar{z} \rangle$ invariant. $\mathrm{SU}(N+1,1)$ acts transitively on Σ^N, and

$$p : \mathrm{SU}(N+1,1) \to \Sigma^N = \mathrm{SU}(N+1,1)/P$$

for an appropriate subgroup P [3].

Explicitly, consider an element $Z = (Z_0, Z_A, Z_{N+1}) \in \mathrm{SU}(N+1,1)$ as an ordered set of $(N+2)$-column vectors in $\mathbb{C}^{N+1,1}$ such that $\det(Z) = 1$, and that

$$\langle Z_A, \bar{Z}_B \rangle = \delta_{AB}, \quad \langle Z_0, \bar{Z}_{N+1} \rangle = -\langle Z_{N+1}, \bar{Z}_0 \rangle = \mathrm{i}, \tag{1}$$

while all other scalar products are zero. We define $p(Z) = [Z_0]$, where $[Z_0]$ is the equivalence class of null vectors represented by Z_0. The left invariant Maurer-Cartan form π of $\mathrm{SU}(N+1,1)$ is defined by the equation

$$d\,Z = Z\pi,$$

which is in coordinates

$$d(Z_0, Z_A, Z_{N+1}) = (Z_0, Z_B, Z_{N+1}) \begin{pmatrix} \pi^0_0 & \pi^0_A & \pi^0_{N+1} \\ \pi^B_0 & \pi^B_A & \pi^B_{N+1} \\ \pi^{N+1}_0 & \pi^{N+1}_A & \pi^{N+1}_{N+1} \end{pmatrix}. \tag{2}$$

Coefficients of π are subject to the relations obtained from differentiating (1), which are

$$\begin{aligned} \pi^0_0 + \bar{\pi}^{N+1}_{N+1} &= 0 \\ \pi^{N+1}_0 &= \bar{\pi}^{N+1}_0, \quad \pi^0_{N+1} = \bar{\pi}^0_{N+1} \\ \pi^{N+1}_A &= -\mathrm{i}\bar{\pi}^A_0, \quad \pi^A_{N+1} = \mathrm{i}\bar{\pi}^0_A \\ \pi^A_B + \bar{\pi}^B_A &= 0 \\ \mathrm{tr}\pi &= 0, \end{aligned}$$

and π satisfies the Cartan structure equation

$$-d\pi = \pi \wedge \pi. \tag{3}$$

It is well known that the $\mathrm{SU}(N+1,1)$-invariant CR structure on $\Sigma^N \subset \mathbb{C}P^{N+1}$ as a real hypersurface is biholomorphically equivalent to the standard CR structure on $S^{2N+1} = \partial \mathbf{B}^{N+1}$, where $\mathbf{B}^{N+1} \subset \mathbb{C}^{N+1}$ is the unit ball. The structure equation (2) shows that for a local section $s : \Sigma^N \to \mathrm{SU}(N+1,1)$, the contact hyperplane fields $\mathcal{H}$ on Σ^N is defined by $\mathcal{H} = (s^*\pi_0^{N+1})^{\perp}$, and the set of (1,0)-forms are given by $\{s^*\pi_0^A\}$.

Definition 1.1. Let M be a manifold of dimension $2n+1$. An immersion $f : M \hookrightarrow \Sigma^N$ is a *CR immersion* if $f_*T_pM \cap \mathcal{H}_{f(p)}$ is an n-dimensional complex subspace of $\mathcal{H}_{f(p)} = \mathcal{H}|_{f(p)}$ for each $p \in M$. A *CR submanifold* of Σ^N is a submanifold defined by a CR immersion.

Note that the hyperplane field $f_*^{-1}(f_*(TM) \cap \mathcal{H})$ necessarily defines a contact structure on M, and M has an induced nondegenerate CR structure of hypersurface type. When M itself is equipped with a CR structure, an immersion $f : M \to \Sigma^N$ is CR if f induces a CR structure which is equivalent to the given one.

We employ the method of moving frames to describe the geometry of CR immersions from a CR sphere to a CR sphere. In the following, we closely follow the ideas developed in [6], particularly [6, Section 5].

Let $f : M^{2n+1} \hookrightarrow \Sigma^N$ be a CR submanifold. Consider the 0-*adapted* bundle $F_0 = f^*\mathrm{SU}(N+1,1) \to M$. From the general theory of moving frames [6], there exists an 1-*adapted* subbundle $F_1 \subset F_0$ on which

$$\pi_0^\alpha = 0 \quad \text{for } n+1 \leq \alpha \leq N = n+m.$$

Differentiating this by the structure equations (3), we get

$$\pi_i^\alpha \wedge \pi_0^i + \pi_{N+1}^\alpha \wedge \pi_0^{N+1} = 0.$$

By Cartan's lemma,

$$\pi_i^\alpha \equiv H_{ij}^\alpha \pi_0^j \qquad \mod\ \pi_0^{N+1} \tag{4}$$

for coefficients $H_{ij}^\alpha = H_{ji}^\alpha$. The coefficients $\{H_{ij}^\alpha\}$ represent the CR second fundamental form of f [6].

When $M = \Sigma^n$, it is CR flat, and the CR analogue of Gauß equation implies a set of quadratic equations for $\{H_{ij}^\alpha\}$ [6, Proposition 5.2]. When the codimension $N-n$ is sufficiently small as stated in Theorem, this set of equations allows us to put $\{H_{ij}^\alpha\}$ in a simple normal form (5). It is based on the following inequality due to Iwatani on the dimension of the asymptotic subspace of the second fundamental form of a Bochner-Kähler submanifold [14, 1]. Note that the CR Gauß equation implies the vanishing of only the Bochner curvature component of the image of CR second fundamental form under Gauß equation [6].

Let $V = \mathbb{C}^n$, and $W = \mathbb{C}^m$ be the complex vector spaces equipped with the standard Hermitian metric. Let $\{z^i\}$, $1 \leq i \leq n$, be a unitary $(1,0)$-basis for V^*, and let $\{w_\alpha\}$, $1 \leq \alpha \leq m$, be a unitary basis for W. Let $S^{p,q}$ denote the space of polynomials of type (p,q) on V.

Theorem 1.1 ([10, 14]). *Suppose $H = H^\alpha_{ij} z^i z^j \otimes w_a \in S^{2,0} \otimes W$ satisfies*

$$\gamma(H,H) = H^\alpha_{ij} \bar{H}^\alpha_{kl} z^i z^j \otimes \bar{z}^k \bar{z}^l = (z^k \bar{z}^k) h \in S^{2,2}, \quad h \in S^{1,1}.$$

Equivalently, suppose $\gamma(H,H)$ is Bochner-flat [1]. *Then the set of asymptotic vectors $\{v \in V \mid H(v,v) = 0\}$ is a subspace of V. Let $n-k$ be the dimension of the asymptotic subspace of H. Then,*

$$m \geq \frac{1}{2} k \,(2n - \, k+1),$$

where k is called the rank *of H.*

Remark. The original proof of Theorem 1.1 by Iwatani is algebraic [14]. Huang recently gave an alternative analytic proof [10].

We wish to apply Theorem 1.1 to normalize the coefficients of the second fundamental form (4). For a CR immersion $f : \Sigma^n \hookrightarrow \Sigma^N$, let us identify $V = f_* T_p \Sigma^n \cap \mathcal{H}_{f(p)}$, and $W = V^\perp \subset \mathcal{H}$ for $p \in \Sigma^n$. Let H be the second fundamental form of f represented by $\{H^\alpha_{ij}\}$. When $\dim W = N - n < 2n-1$, Theorem 1.1 implies that the rank of H is at most 1.

Since H has rank at most 1, up to a unitary transformation on V, we may arrange so that

$$H^\alpha_{ij} = H^\alpha_i \delta_{jn} + H^\alpha_j \delta_{in}$$

for coefficients H^α_i.

Set $\nu_i = H^\alpha_i w_\alpha \in W$. A computation shows that $\gamma(H,H)$ is Bochner-flat when $\langle \nu_i, \nu_j \rangle = 0$ for $i \neq j$, and $\langle \nu_i, \nu_i \rangle = \langle \nu_j, \nu_j \rangle$ for all i, j. Up to a unitary transformation on W, we may set

$$\nu_i = \lambda w_i$$

for a coefficient λ.

Suppose $N - n < n$. Then H has rank 0, and f is easily seen to be a part of the linear embedding $f : \Sigma^n \hookrightarrow \Sigma^n \subset \Sigma^N$.

Suppose $n \leq N - n < 2n - 1$. Then H has rank at most 1, and from the argument above, we have in (4),

$$\begin{aligned}
(5) \qquad \pi^{n+i}_q &\equiv \lambda \delta_{iq} \pi^n_0 \mod \pi^{N+1}_0 \quad \text{for } q < n, \ i \leq n, \\
\pi^{n+i}_n &\equiv \lambda(1 + \delta_{in})\, \pi^i_0 \mod \pi^{N+1}_0 \quad \text{for } i \leq n, \\
\pi^a_i &\equiv 0 \mod \pi^{N+1}_0 \quad \text{for } a > 2n.
\end{aligned}$$

In the next section, we study the higher order derivatives of the normalized structure equations (5).

2. Proof of Theorem

In this section, we analyze the structure of higher order jets of CR immersions from spheres to spheres by differentiating (5). The overdetermined nature of CR geometry closes up the structure equations in finite steps. A proof of Theorem follows by giving a geometric interpretation of the resulting structure equations.

To facilitate the computation, we shall agree on the index range

$$\begin{aligned} 1 \le p,\ q,\ s,\ t \le n-1, &\quad p' = n+p, \\ 1 \le i,\ j,\ k,\ l \le n, &\quad i' = n+i, \\ n'+1 \le a,\ b \le m = n'+r. & \end{aligned}$$

The induced Maurer-Cartan form (2) on $f^*\mathrm{SU}(N{+}1{,}1)$ decomposes according to these indices as follows.

$$\pi = \begin{pmatrix} \pi_0^0 & \pi_q^0 & \pi_n^0 & \pi_{q'}^0 & \pi_{n'}^0 & \pi_b^0 & \pi_{N+1}^0 \\ \pi_0^p & \pi_q^p & \pi_n^p & \pi_{q'}^p & \pi_{n'}^p & \pi_b^p & \pi_{N+1}^p \\ \pi_0^n & \pi_q^n & \pi_n^n & \pi_{q'}^n & \pi_{n'}^n & \pi_b^n & \pi_{N+1}^n \\ \cdot & \pi_q^{p'} & \pi_n^{p'} & \pi_{q'}^{p'} & \pi_{n'}^{p'} & \pi_b^{p'} & \pi_{N+1}^{p'} \\ \cdot & \pi_q^{n'} & \pi_n^{n'} & \pi_{q'}^{n'} & \pi_{n'}^{n'} & \pi_b^{n'} & \pi_{N+1}^{n'} \\ \cdot & \pi_q^a & \pi_n^a & \pi_{q'}^a & \pi_{n'}^a & \pi_b^a & \pi_{N+1}^a \\ \pi_0^{N+1} & \pi_q^{N+1} & \pi_n^{N+1} & \cdot & \cdot & \cdot & \pi_{N+1}^{N+1} \end{pmatrix}, \tag{6}$$

where '$\cdot$' denotes 0. We denote $\pi_0^i = \eta^i$, $\pi_0^{N+1} = \theta$, and let $-d\theta \equiv \mathrm{i}\eta^k \wedge \bar{\eta}^k = \mathrm{i}\varpi$ mod θ for the sake of notation.

Equation (5) can now be written as

$$\begin{pmatrix} \pi_q^{p'} & \pi_n^{p'} \\ \pi_q^{n'} & \pi_n^{n'} \\ \pi_q^a & \pi_n^a \end{pmatrix} \equiv \begin{pmatrix} \lambda\delta_{pq}\eta^n & \lambda\eta^p \\ \cdot & 2\lambda\eta^n \\ \cdot & \cdot \end{pmatrix} \quad \text{mod } \theta.$$

The case $\lambda \equiv 0 (H \equiv 0)$ has already been identified with a part of the linear embedding. Suppose $\lambda \neq 0$, and H has rank 1. We may scale $\lambda = 1$ using the group action by $\mathrm{Re}\pi_0^0$, which is equivalent to a reduction to a subbundle $F_2 \subset F_1$ defined by the equation $\{\lambda = 1\}$. We obtain the following normalized structure equation for a nonlinear local CR immersion $f : \Sigma^n \hookrightarrow \Sigma^N$ with second fundamental form of rank 1,

$$\begin{pmatrix} \pi_q^{p'} & \pi_n^{p'} \\ \pi_q^{n'} & \pi_n^{n'} \\ \pi_q^a & \pi_n^a \end{pmatrix} = \begin{pmatrix} \delta_{pq}\eta^n & \eta^p \\ \cdot & 2\eta^n \\ \cdot & \cdot \end{pmatrix} + \begin{pmatrix} h_q^{p'} & h_n^{p'} \\ h_q^{n'} & h_n^{n'} \\ h_q^a & h_n^a \end{pmatrix} \theta \tag{7}$$

for coefficients $h_j^{i'}, h_j^a$.

Theorem is obtained by successive application of Maurer-Cartan equation (3) to this structure equation and its consequences. Instead of giving the details of computations, which are long but straightforward, we present the steps of the computations that show the idea of the higher order differential analysis. We assume $n \geq 4$ for simplicity for the rest of this section, as $n = 3$ case can be treated with minor modifications. The expression "differentiate X mod Y" would mean "differentiate X, and considering mod Y".

Step 1. Differentiate $\pi_s^{n'} = h_s^{n'}\theta \mod \theta$, we get

$$\mathrm{i}\, h_s^{n'}\varpi \equiv (\pi_{s'}^{n'} - 2\pi_s^n) \wedge \eta^n + \pi_{N+1}^{n'} \wedge (-\mathrm{i}\bar{\eta}^s) \qquad \mod \theta.$$

Since $n-1 \geq 2$, this implies $h_s^{n'} = 0$, and by Cartan's lemma

$$\begin{pmatrix} \pi_{s'}^{n'} - 2\pi_s^n \\ \pi_{N+1}^{n'} \end{pmatrix} \equiv \begin{pmatrix} 2c_s & u \\ u & 0 \end{pmatrix} \begin{pmatrix} \eta^n \\ -\mathrm{i}\bar{\eta}^s \end{pmatrix} \qquad \mod \theta$$

for coefficients c_s, u.

Differentiate $\pi_s^a = h_s^a\theta \mod \theta$, we get

$$\mathrm{i}\, h_s^a\varpi \equiv \pi_{s'}^a \wedge \eta^n + \pi_{N+1}^a \wedge (-\mathrm{i}\bar{\eta}^s) \qquad \mod \theta.$$

Since $n-1 \geq 2$, this implies $h_s^a = 0$, and by Cartan's lemma

$$\begin{pmatrix} \pi_{s'}^a \\ \pi_{N+1}^a \end{pmatrix} \equiv \begin{pmatrix} 2C_s^a & u^a \\ u^a & 0 \end{pmatrix} \begin{pmatrix} \eta^n \\ -\mathrm{i}\bar{\eta}^s \end{pmatrix} \qquad \mod \theta$$

for coefficients C_s^a, u^a.

Differentiate $\pi_n^a = h_n^a\theta \mod \theta$, we get

$$\mathrm{i}\, h_n^a\varpi \equiv \pi_{p'}^a \wedge \eta^p + \pi_{n'}^a \wedge 2\eta^n + \pi_{N+1}^a \wedge (-\mathrm{i}\bar{\eta}^n) \qquad \mod \theta.$$

This implies $h_n^a = u^a$, and

$$\pi_{n'}^a \equiv C_p^a\eta^p + C_n^a\eta^n - \mathrm{i}u^a\bar{\eta}^n \quad \mod \theta$$

for coefficients C_n^a.

Step 2. Differentiate $\pi_s^{t'} = h_s^{t'}\theta \mod \theta$ for $t \neq s$, we get

$$\mathrm{i}\, h_s^{t'}\varpi \equiv (\pi_{s'}^{t'} - \pi_s^t) \wedge \eta^n - \pi_s^n \wedge \eta^t + \pi_{N+1}^{t'} \wedge (-\mathrm{i}\bar{\eta}^s) \quad \mod \theta.$$

For $n-1 \geq 3$, this implies $h_s^{t'} = 0$ for $t \neq s$, and by Cartan's lemma

$$\begin{pmatrix} \pi_{s'}^{t'} - \pi_s^t \\ -\pi_s^n \\ \pi_{N+1}^{t'} \end{pmatrix} \equiv \begin{pmatrix} 0 & b_s & -\mathrm{i}\bar{b}_t \\ b_s & 0 & e \\ -\mathrm{i}\bar{b}_t & e & 0 \end{pmatrix} \begin{pmatrix} \eta^n \\ \eta^t \\ -\mathrm{i}\bar{\eta}^s \end{pmatrix} \quad \mod \theta$$

for coefficients b_s, e. Since $\pi_{s'}^{t'} - \pi_s^t$ is skew Hermitian, it cannot have any η^n-term.

Step 3. Differentiate $\pi_t^{t'} = \eta^n + h_t^{t'}\theta \mod \theta$, we get

$$h_t^{t'}\varpi \equiv \Delta_t \wedge \eta^n + (b_p\eta^n - \mathrm{i}e\bar{\eta}^p) \wedge \eta^p + (-b_t\eta^t + \bar{b}_t\bar{\eta}^t) \wedge \eta^n \quad \mod \theta,$$

where $\Delta_t = \pi_{t'}^{t'} - \pi_t^t + \pi_0^0 - \pi_n^n$. This implies $h_t^{t'} = e$, and

$$\Delta_t = a_t\eta^n - \mathrm{i}e\bar{\eta}^n + (b_t\eta^t - \bar{b}_t\bar{\eta}^t) + \sum_p b_p\eta^p - A_t\theta$$

for coefficients a_t, A_t.

Step 4. Since $h^{t'}_s = \delta_{st}\, e$ and from (7), we may add $e\theta$ to η^n to translate $e = 0$, which we assume from now on. We also translate $h^{t'}_n = 0$ similarly by adding $h^{t'}_n\theta$ to η^t. Differentiating $\pi^{t'}_n = \eta^t \mod \theta$ with these relations and collecting terms, we get $b_t = c_t = 0$, and

$$0 \equiv a_t\, \eta^n \wedge \eta^t - 2\mathrm{i}\bar{u}\, \eta^t \wedge \eta^n \mod \theta.$$

This implies $a_t = -2\,\mathrm{i}\,\bar{u}$.

Step 5. Differentiate $\pi^{n'}_n = 2\eta^n + h^{n'}_n\theta \mod \theta$, and collecting terms, we get $h^{n'}_n = u$, and $0 \equiv \Delta_n \wedge \eta^n - \mathrm{i}u\eta^n \wedge \bar{\eta}^n \mod \theta$, where $\Delta_n = \pi^{n'}_{n'} - \pi^n_n + \pi^0_0 - \pi^n_n$. Hence

$$\Delta_n = -\mathrm{i}u\bar{\eta}^n + a_n\eta^n - A_n\theta$$

for coefficients a_n, A_n. But $\Delta_t - \Delta_n$ is purely imaginary, and comparing with *Step 3*, $a_n = -3\,\mathrm{i}\,\bar{u}$.

Step 6. Considering θ-terms in *Step 1, 2, 3, 4, 5*, and the fact that $\pi^\alpha_i \wedge \eta^i + \pi^\alpha_{N+1} \wedge \theta = 0$, we obtain the following refined structure equations. We omit the details of computations.

$$\begin{pmatrix} \pi^{p'}_q & \pi^{p'}_n \\ \pi^{n'}_q & \pi^{n'}_n \\ \pi^a_q & \pi^a_n \end{pmatrix} = \begin{pmatrix} \delta_{pq}\eta^n & \eta^p \\ 0 & 2\eta^n + u\theta \\ 0 & u^a\theta \end{pmatrix},$$

$$\begin{pmatrix} \pi^{p'}_{N+1} \\ \pi^{n'}_{N+1} \\ \pi^a_{N+1} \end{pmatrix} = \begin{pmatrix} 0 \\ u\eta^n \\ u^a\eta^n \end{pmatrix},$$

$$\begin{pmatrix} \pi^{n'}_{s'} \\ \pi^a_{s'} \end{pmatrix} = \begin{pmatrix} -\mathrm{i}u\bar{\eta}^s \\ -\mathrm{i}u^a\bar{\eta}^s + 2C^a_s\eta^n \end{pmatrix},$$

$$\pi^n_s = 0, \pi^{t'}_{s'} = \pi^t_s \text{ for } t \neq s,$$

$$\pi^a_{n'} = C^a_p\eta^p + C^a_n\eta^n - \mathrm{i}u^a\bar{\eta}^n + h^a_{n'}\theta,$$

$$\begin{pmatrix} \pi^t_{N+1} \\ \pi^n_{N+1} \end{pmatrix} = \begin{pmatrix} (A - \mathrm{i}(u\bar{u} + u^a\bar{u}^a))\eta^t - 2u^a\bar{C}^a_t\bar{\eta}^n + B_t\theta \\ A\eta^n + B_n\theta \end{pmatrix},$$

$$\begin{pmatrix} \Delta_t \\ \Delta_n \end{pmatrix} = \begin{pmatrix} -2\mathrm{i}\,\bar{u}\eta^n - A\theta \\ -3\mathrm{i}\bar{u}\eta^n - \mathrm{i}u\bar{\eta}^n - A_n\theta \end{pmatrix},$$

$$A_n + \bar{A}_n = A + \bar{A},$$

$$du = u(\pi^{N+1}_{N+1} - \pi^0_0 + \pi^n_n - \pi^{n'}_{n'}) + 2(A - A_n)\eta^n - u^a\pi^{n'}_a + u_0\theta,$$

$$du^a = u^a(\pi^{N+1}_{N+1} - \pi^0_0 + \pi^n_n) - u^b\pi^a_b - u\pi^a_{n'} + 2h^a_{n'}\eta^n + u^a_0\theta. \tag{8}$$

Proof of Theorem. From [12, Corollary 1.4], a proper holomorphic map $F : \mathbf{B}^{n+1} \to \mathbf{B}^{3n-1}$ which is C^3 up to the boundary is rational, and hence it is real analytic up to the boundary. Hamada showed that a rational proper holomorphic map from $\mathbf{B}^{n+1}$ to $\mathbf{B}^{2n+2}$, $n \geq 3$, belongs to one of the three classes of maps in Theorem [8]. It thus suffices to show that a analytic local CR immersion $f : \Sigma^n \hookrightarrow \Sigma^{3n-2}$ lies in a lower dimensional sphere $f : \Sigma^n \hookrightarrow \Sigma^{2n+1} \subset \Sigma^{3n-2}$ for $n \geq 3$.

Step 1. From the refined structure equations above, differentiate $\pi^{p'}_{N+1} = 0$, $\pi^{n'}_{N+1} = u\eta^n$, and $\pi^a_{N+1} = u^a\eta^n$. A short computation gives $B_p = 0, B_n = 0$, and

$$u_0 = -2uA,\ u^a_0 = -2u^a A. \tag{9}$$

Differentiating $\pi^n_q = 0$, and $\pi^{t'}_{s'} = \pi^t_s$ for $t \neq s$ with these relations, we get

$$A - \bar{A} = \mathrm{i}(u\bar{u} + u^a\bar{u}^a - 1),$$

$$\sum_{a=n'+1}^{n'+r} C^a_t \bar{C}^a_s = 0 \quad \text{for } t \neq s. \tag{10}$$

Step 2. Differentiate $\mathrm{i}\pi^{n'}_{q'} = u\bar{\eta}^q$, and collecting $\eta^n \wedge \bar{\eta}^q$-terms, we get

$$\sum_{a=n'+1}^{n'+r} C^a_q \bar{C}^a_q = 1 - u\bar{u} + \mathrm{i}(A - A_n),$$

which is independent of the index q. Since $r < n - 1$ from our assumption on the codimension, this and (10) force

$$C^a_q = 0, \tag{11}$$

and $A_n = A + \mathrm{i}(u\bar{u} - 1)$.

Step 3. Differentiate $\pi^a_{n'} = C^a_n\eta^n - \mathrm{i}u^a\bar{\eta}^n + h^a_{n'}\theta \mod \theta, \eta^n, \bar{\eta}^n$, we get

$$h^a_{n'} = -\mathrm{i}u^a\bar{u}. \tag{12}$$

Step 4. Differentiate $\pi^p_{N+1} = (\bar{A} - \mathrm{i})\eta^p$, $\pi^n_{N+1} = A\eta^n$, we get

$$dA = A(\pi^{N+1}_{N+1} - \pi^0_0) + \pi^0_{N+1} + 2(u\bar{\eta}^n - \bar{u}\eta^n) + (u\bar{u} + u^a\bar{u}^a - A^2)\theta.$$

Note $\Delta_t + \bar{\Delta}_t = \pi^0_0 + \bar{\pi}^0_0 = \pi^0_0 - \pi^{N+1}_{N+1} = 2\,\mathrm{i}(u\bar{\eta}^n - \bar{u}\eta^n) - (A + \bar{A})\theta$. Differentiating $\Delta_n = -3\mathrm{i}\bar{u}\eta^n - \mathrm{i}u\bar{\eta}^n - A_n\theta$ with these relations, and collecting $\eta^n \wedge \bar{\eta}^n$-terms, we get $\sum_{a=n'+1}^{n'+r} C^a_n \bar{C}^a_n = 0$, and hence

$$C^a_n = 0. \tag{13}$$

Case B. Suppose $u^a = 0$ for all a. From (11), (12), (13), $\pi^a_n = \pi^a_{s'} = \pi^a_{n'} = \pi^a_{N+1} = 0$. From the equation for Maurer-Cartan form π (2), this implies that the complex $(2n+2)$-plane $Z_0 \wedge Z_1 \wedge \cdots \wedge Z_{2n} \wedge Z_{N+1}$ is constant along the CR immersion f. Hence $f : \Sigma^n \hookrightarrow \Sigma^{2n} \subset \Sigma^{3n-2}$.

Case C. Suppose $\vec{u} = (u^{n'+1}, \ldots, u^{n'+r}) \neq 0$. Using a group action by π^a_b, we may rotate so that $\vec{u} = (u^{n'+1}, 0, \ldots, 0)$ with $u^{n'+1} \neq 0$. From (11), (12), (13), we have $\pi^a_n = \pi^a_{s'} = \pi^a_{n'} = \pi^a_{N+1} = 0$ for $a > n' + 1$. (8) shows that $\pi^a_{n'+1} = 0$ for $a > n'+1$. From the equation for Maurer-Cartan form π (2), this implies that the complex $(2n+3)-$plane $Z_0 \wedge Z_1 \wedge \cdots \wedge Z_{2n} \wedge Z_{2n+1} \wedge Z_{N+1}$ is constant along the CR immersion f. Hence $f : \Sigma^n \hookrightarrow \Sigma^{2n+1} \subset \Sigma^{3n-2}$. $\square$

At this stage, note that the only possibly independent coefficients in the structure equations are A, u, u^a, and that the expressions for their derivatives do not involve any new variables. The structure equations for local CR immersion $f : \Sigma^n \hookrightarrow \Sigma^{3n-2}$ thus close up at order 3. A long but direct computation shows that these equations are compatible, i.e., $d^2 = 0$ is a formal identity of the structure equation.

It is known that a CR immersion with the structure equations of **Case B** is locally equivalent to the boundary of Whitney map [15]. It is likely that a similar argument could be applied to show that a CR immersion with the structure equations of **Case C** is locally equivalent to the boundary of a type **C** proper holomorphic map in Theorem.

The computation suggests that the gap phenomena in Theorem may persist for CR immersions between spheres with second fundamental form of rank 1.

In this regard, consider the following generalization of Whitney map;

(14)

$$F : \mathbf{B}^{n+1} \to \mathbf{B}^{\mu(n+1)},$$

$$F(z^i, z^0) = (x_1 z^i, x_2 z^i z^0, \ldots, x_\mu z^i (z^0)^{\mu-1}, y_1 z^0, y_2 z^0 z^0, \ldots, y_\mu z^0 (z^0)^{\mu-1}),$$

where x_A, y_A are constants satisfying $x_1 = 1, x_\mu = y_\mu, x_A^2 = y_A^2 + x_{A+1}^2$ for $1 \leq A \leq \mu - 1$. It would be an interesting problem to understand how exhaustive the set of maps (14) is among the set of proper holomorphic maps with rank 1 second fundamental form at the boundary.

References

[1] R. Bryant, *Bochner-Kähler metrics*, J. Amer. Math. Soc. **14** (2001), no. 3, 623–715.

[2] E. Cartan, *Sur les variétés de courbure constante d'un espace euclidien ou non-euclidien*, Bull. Soc. Math. France **47** (1919), 125–160; Bull. Soc. Math. France **48** (1920), 132–208.

[3] S. S. Chern and J. K. Moser, *Real hypersurfaces in complex manifolds*, Acta Math. **133** (1974), 219–271.

[4] J. P. D'Angelo, *Proper holomorphic maps between balls of different dimensions*, Michigan Math. J. **35** (1988), no. 1, 83–90.

[5] ———, *Several Complex Variables and the Geometry of Real Hypersurfaces*, Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1993.

[6] P. Ebenfelt, X. Huang, and D. Zaitsev, *Rigidity of CR-immersions into spheres*, Comm. Anal. Geom. **12** (2004), no. 3, 631–670.

[7] F. Forstnerič, *Proper holomorphic maps from balls*, Duke Math. J. **53** (1986), no. 2, 427–441.

[8] H. Hamada, *Rational proper holomorphic maps from* $\mathbf{B}^n$ *into* $\mathbf{B}^{2n}$, Math. Ann. **331** (2005), no. 3, 693–711.

[9] C.-K. Han, *Complete differential system for the mappings of CR manifolds of nondegenerate Levi forms*, Math. Ann. **309** (1997), no. 3, 401–409.

[10] X. Huang, *On a semi-rigidity property for holomorphic maps*, Asian J. Math. **7** (2003), no. 4, 463–492.

[11] X. Huang and S. Ji, *Mapping* $\mathbf{B}^n$ *into* $\mathbf{B}^{2n-1}$, Invent. Math. **145** (2001), no. 2, 219–250.

[12] X. Huang, S. Ji, and D. Xu, *Several results for holomorphic mappings from* $\mathbf{B}^n$ *into* $\mathbf{B}^N$, Geometric analysis of PDE and several complex variables, 267–292, Contemp. Math., 368, Amer. Math. Soc., Providence, RI, 2005.

[13] ———, *A new gap phenomenon for proper holomorphic mappings from* B^n *into* B^N, Math. Res. Lett. **13** (2006), no. 4, 515–529.

[14] T. Iwatani, *Kähler submanifolds with vanishing Bochner curvature tensor*, Mem. Fac. Sci. Kyushu Univ. Ser. A **30** (1976), no. 2, 319–321.

[15] S. H. Wang, *1-rigidity for CR submanifolds in spheres*, arXiv. math.DG/0506134.

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