Honam Mathematical J. **31** (2009), No. 4, pp. 559–578

GENERALIZED MINIMAX THEOREMS IN GENERALIZED CONVEX SPACES

Hoonjoo Kim†

Abstract. In this work, we obtain intersection theorem, analytic alternative and von Neumann type minimax theorem in G-convex spaces. We also generalize Ky Fan minimax inequality to acyclic versions in G-convex spaces. The result is applied to formulate acyclic versions of other minimax results, a theorem of systems of inequalities and analytic alternative.

1. Introduction

There have appeared many generalizations of the concept of convex subset of a topological vector space. Generalized convex space or G-convex space introduced by [25, 26] is considered common and general one.

In G-convex spaces, we obtain intersection theorem, analytic alternative and von Neumann type minimax theorem. We also generalize Ky Fan minimax inequality to acyclic versions in G-convex spaces. The result is applied to formulate acyclic versions of other minimax results, a theorem of systems of inequalities and analytic alternative. These results generalize and improve the corresponding results in [1, 3–15, 17–19, 28–32].

A *multimap* (or simply, a *map*) $F : X \multimap Y$ is a function from a set X into the power set of Y; that is, a function with the *values* $F(x) \subset Y$ for

Received October 8, 2009. Accepted November 19, 2009.
2000 Mathematics Subject Classification : 49J35, 54C60, 46A55.
Key words and phrases: Minimax Theorem, Generalized Convex Space, Acyclic, Φ-map, f-quasiconcave, Transfer l.s.c., Intersection Theorem, Systems of Inequalities, Analytic Alternative.
† This paper was supported by the Daebul University Research Fund in 2009.

$x \in X$ and the *fibers* $F^{-}(y) := \{x \in X \,|\, y \in F(x)\}$ for $y \in Y$. For $A \subset X$, let $F(A) := \bigcup\{F(x) \,|\, x \in A\}$. Throughout this paper, we assume that multimaps have nonempty values otherwise explicitly stated or obvious from the context. The closure operation and graph of F are denoted by $\overline{\quad}$ and GrF, resp.

Let X be a set (in a vector space) and D a nonempty subset of X. Then (X, D) is called a *convex space* if convex hulls of any nonempty finite subsets of D is contained in X and X has a topology that induces the Euclidean topology on such convex hulls.

A *continuous selection* $f : X \to Y$ of a map $F : X \multimap Y$ is a continuous function such that $f(x) \in F(x)$ for all $x \in X$.

A function $f : X \to Y$ is *compactly l.s.c.* (resp., *compactly u.s.c.*) on Y, if f is lower (resp., upper) semicontinuous on each non-empty compact subset of Y.

Let $\langle D \rangle$ denote the set of all nonempty finite subsets of a set D.

For topological spaces X and Y, a map $F : X \multimap Y$ is *upper semicontinuous* (u.s.c.) if it has nonempty values and, for each closed set $B \subset Y$, $F^{-}(B)$ is closed in X.

Note that composites of u.s.c. maps are u.s.c. and that the image of a compact set under an u.s.c. map with compact values is compact.

A *generalized convex space* or a *G-convex space* $(X, D; \Gamma)$ consists of a topological space X, a nonempty set D, and a multimap $\Gamma : \langle D \rangle \multimap X$ such that for each $A \in \langle D \rangle$ with its cardinal $|A| = n + 1$, there exists a continuous function $\phi_A : \Delta_n \to \Gamma_A := \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset \Gamma_J := \Gamma(J)$. In certain cases, we may assume $\phi_A(\Delta_n) = \Gamma_A$.

Note that Δ_n is an n-simplex with vertices $v_0, v_1, \cdots, v_n$, and Δ_J the face of Δ_n corresponding to $J \in \langle A \rangle$; that is, if $A = \{a_0, a_1, \cdots, a_n\}$ and $J =$

$\{a_{i_0}, a_{i_1}, \cdots, a_{i_k}\} \subset A$, then $\Delta_J = \mathrm{co}\{v_{i_0}, v_{i_1}, \cdots, v_{i_k}\}$, where co denotes the convex hull.

We may write $(X;\Gamma) = (X, X;\Gamma)$.

In case to emphasize $X \supset D$, $(X, D;\Gamma)$ will be denoted by $(X \supset D;\Gamma)$.

For a G-convex space $(X \supset D;\Gamma)$, a subset $Y \subset X$ is said to be Γ-*convex* if for each $N \in \langle D\rangle$, $N \subset Y$ implies $\Gamma_N \subset Y$.

Examples of G-convex spaces can be found in Park [21–23] and references therein. Any convex space (X, D) becomes a G-convex space $(X, D;\Gamma)$ by putting $\Gamma_A = \mathrm{co}\, A$. A C-*space* (or an H-*space*) (X, F) is a G-convex space $(X;\Gamma)$. In fact, by putting $\Gamma_A = F(A)$ for each $A \in \langle X\rangle$ with $|A| = n+1$, there exists a continuous map $\phi_A : \Delta_n \to X$ such that for all $J \subset A$, $\phi_A(\Delta_J) \subset F(J)$ by Horvath [13, Theorem 1].

The other major examples of G-convex spaces are convex subsets of a t.v.s., Komiya's convex spaces [14] and Bielawski's simplicial convexities [5].

A class $\mathfrak{A}_c^\kappa$ is defined as follows:

$T \in \mathfrak{A}_c^\kappa(X, Y) \iff T$ is one such that, for each T and each nonempty compact subset K of X, there exists a map $\Gamma \in \mathfrak{A}_c(K, Y)$ satisfying $\Gamma x \subset Tx$ for all $x \in K$; where $\mathfrak{A}_c$ is consisting of finite composites of maps in $\mathfrak{A}$, and $\mathfrak{A}$ is a class of maps satisfying the following properties:

(i) $\mathfrak{A}$ contains the class $\mathbb{C}$ of (single-valued) continuous functions;

(ii) each $F \in \mathfrak{A}_c$ is u.s.c. and compact-valued; and

(iii) for any polytope P, each $F \in \mathfrak{A}_c(P, P)$ has a fixed point.

Here, a polytope P is a homeomorphic image of a standard simplex. For details, see [25, 26].

Let X be a Hausdorff space and $(Y, D;\Gamma)$ be a G-convex space. A multimap $T : X \multimap Y$ is called a Φ-*map* provided that there exists a multimap $S : X \multimap D$ satisfying

(a) for each $x \in X$, $M \in \langle S(x)\rangle$ implies $\Gamma_M \subset T(x)$; and

(b)$X = \bigcup\{\mathrm{Int}\ S^{-}(y) : y \in D\}$,

where $\mathrm{Int}S^{-}(y)$ denotes the interior of $S^{-}(y)$ in X. See Park [20].

For a G-convex space $(X;\Gamma)$, a real function $f : X \to \overline{\mathbb{R}}$ is said to be *quasiconvex* [resp. *quasiconcave*] if $\{x \in X \mid f(x) < \lambda\}$ [resp. $\{x \in X \mid f(x) > \lambda\}$] is Γ-convex for $\lambda \in \overline{\mathbb{R}}$. See Park [21].

2. Coincidence and Intersection Theorems

We begin with the following coincidence theorem [25, Theorem 1], [26, Theorem 1];

Theorem 1. *Let $(X, D;\Gamma)$ be a G-convex space, Y a Hausdorff space, $S : D \multimap Y$, $T : X \multimap Y$ maps, and $F \in \mathfrak{A}_c^{\kappa}(X, Y)$. Suppose that*

(1.1) *for each $x \in D$, $S(x)$ is compactly open in Y;*

(1.2) *for each $y \in F(X)$, $M \in \langle S^{-}(y)\rangle$ implies $\Gamma_M \subset T^{-}(y)$;*

(1.3) *there exists a nonempty compact subset K of Y such that $\overline{F(X)} \cap K \subset S(D)$; and*

(1.4) *either*

(i) *$Y \backslash K \subset S(M)$ for some $M \in \langle D\rangle$; or*

(ii) *for $D \subset X$ and each $N \in \langle D\rangle$, there exists a compact Γ-convex subset L_N of X containing N such that $F(L_N)\backslash K \subset S(L_N \cap D)$.*

Then there exists an $\bar{x} \in X$ such that $F(\bar{x}) \cap T(\bar{x}) \neq \emptyset$.

Note that, in [25, 26], Theorem 1 is applied to establish fundamental theorems in the KKM theory on generalized convex spaces, fixed point theorems and many others.

The following is a selection theorem which also shows that Φ is an example of maps in $\mathfrak{A}_c^{\kappa}$ Park [20, Theorem 1]:

Theorem 2. *Let X be a Hausdorff space, $(Y, D; \Gamma)$ be a G-convex space multimap and $T : X \multimap Y$ a Φ-map. Then for any nonempty compact subset K of X, $T|_K$ has a continuous selection $f : K \to Y$ such that $f(K) \subset \Gamma_A$ for some $A \in \langle D\rangle$. More precisely, there exist two functions $p : K \to \Delta_n$ and $\phi_A : \Delta_n \to \Gamma_A$ such that $f = \phi_A \circ p$ for some $A \in \langle D\rangle$ with $|A| = n + 1$.*

For any set $S \subset X \times Y$ and $x \in X$, $y \in Y$, let $S(x) = \{y \in Y : (x, y) \in S\}$ and $S(y) = \{x \in X : (x, y) \in S\}$.

From Theorem 1 and 2, we can deduce the following intersection theorem:

Theorem 3. *Let (X, Γ) and (Y, Γ') be Hausdorff G-convex spaces and $S_1, S_2, T_1, T_2 \subset X \times Y$ such that*

- (3.1) *for each $x \in X$, $M' \in \langle S_1(x)\rangle$ implies $\Gamma'_{M'} \subset T_1(x)$ and $X = \{\bigcup \mathrm{Int} S_1(y) : y \in Y\}$;*
- (3.2) *for each $y \in Y$, $M \in \langle S_2(y)\rangle$ implies $\Gamma_M \subset T_2(y)$ and $Y = \{\bigcup \mathrm{Int} S_2(x) : x \in X\}$; and*
- (3.3) *for each $N \in \langle X\rangle$, there exists a compact Γ-convex subset L_N of X containing N such that $\{y \in Y : y \notin \mathrm{Int} S_2(x)$ for all $x \in L_N\}$ is relatively compact in Y.*

Then $T_1 \cap T_2 \neq \emptyset$.

Proof. Let $F : X \multimap Y$ be a map whose graph is T_1. Then by Theorem 2, $F \in \mathfrak{A}_c^{\kappa}(X, Y)$. Let $T : X \multimap Y$ be a map whose graph is T_2 and $S : X \multimap Y$ a map defined by $S(x) = \mathrm{Int} S_2(x)$ for $x \in X$. Now we apply Theorem 1 (ii) with $X = D$.

Note that (1.1) holds trivially. For each $y \in Y$, by (3.2), $M \in \langle S^-(y)\rangle \subset \langle S_2(y)\rangle$ implies $\Gamma_M \subset T_2(y) = T^-(y)$; and hence (1.2) is satisfied. Moreover, by (3.2),

$$Y = \bigcup\{\mathrm{Int} S_2(x) : x \in X\} = \bigcup\{S(x) : x \in X\} = S(X)$$

and hence (1.3) is satisfied. Finally, let K be a compact subset of Y containing the set in (3.3). Then

$$\begin{aligned} F(L_N)\backslash K \subset \{y \in Y : y \notin K\} &\subset \{y \in Y : y \in \mathrm{Int} S_2(x) \text{ for some } x \in L_N\} \\ &\subset \{S(x) : x \in L_N\} \subset S(L_N), \end{aligned}$$

which implies (1.4). Therefore, by Theorem 1, F and T have a coincidence point, and hence the conclusion holds. $\square$

Particular forms.

1. Fan [8, Theorem 1′]: X and Y are compact convex subsets of Hausdorff topological vector spaces. Note that compactness of X is superfluous.

2. Liu [17, Theorem 2]: X is a compact convex subset of a Hausdorff topological vector space, Y is a convex space.

3. Ha [12, Theorem 1]: X and Y are convex subsets of Hausdorff topological vector spaces, and a slightly stronger form of the compactness condition is assumed.

4. Ben-El-Mechaiekh et al [4, Corollaire 3.4]: Same as for Fan [8].

5. Shih and Tan [28, Theorem $\hat{7}$]: Same as above.

6. Shih and Tan [29, Theorems 2 and 3]: A non-compact version of the preceding one.

7. Bielawski [5, (4.12) Proposition]: X and Y are compact spaces having certain simplicial convexities.

8. Ding [7, Theorem 5.1]: X and Y are C-spaces with a stronger condition than (3.3).

Note that 1–5 and 7 are the case for $S_1 = T_1, S_2 = T_2$ and 6 and 8 are the results about four sets as Theorem 3.

Remark. Int $S_i(z)$ in Theorem 3 can be replaced by any nonempty compactly open subset of $S_i(z)$ for $z \in X$ or $z \in Y$.

From Theorem 3, we obtain the following analytic alternative which is the basis of many minimax theorems:

Theorem 4. *Let* (X,Γ) *and* (Y,Γ') *be Hausdorff* G*-convex spaces and* $f,s,t,g : X\times Y \to \mathbb{R}$ *functions, and* $a\in\mathbb{R}$ *satisfying*

(4.1) $s\le t$;

(4.2) *for each* $x\in X$, $f(x,\cdot)$ *is compactly l.s.c. on* Y, *and* $A\in\langle\{y\in Y : g(x,y)<a\}\rangle$ *implies* $\Gamma'_A\subset\{y\in Y : t(x,y)<a\}$;

(4.3) *for each* $y\in Y$, $g(\cdot,y)$ *is compactly u.s.c. on* X, *and* $B\in\langle\{x\in X : f(x,y)>a\}\rangle$ *implies* $\Gamma_B\subset\{x\in X : s(x,y)>a\}$; *and*

(4.4) *for each* $N\in\langle X\rangle$, *there exists an* L_N *as in Theorem 1 such that* $\{y\in Y : f(x,y)\le a \text{ for all } x\in L_N\}$ *is compact in* Y.

Then one of the following holds:

(i) There exists an $y_0\in Y$ *such that* $f(x,y_0)\le a$ *for all* $x\in X$.

(ii) There exists an $x_0\in X$ *such that* $g(x_0,y)\ge a$ *for all* $y\in Y$.

Proof. Let

$$S_1=\{(x,y)\in X\times Y : f(x,y)>a\},\ S_2=\{(x,y)\in X\times Y : g(x,y)<a\},$$
$$T_1=\{(x,y)\in X\times Y : s(x,y)>a\},\ T_2=\{(x,y)\in X\times Y : t(x,y)<a\}.$$

Suppose that the conclusion does not hold. Then for each $y\in Y$, we have $S_1(y)\neq\emptyset$; and for each $x\in X$, we have $S_2(x)\neq\emptyset$. Moreover, each $S_1(x)$ and each $S_2(y)$ are compactly open by (4.2) and (4.3). Therefore, (4.2) and (4.3) imply (3.1) and (3.2) by exchanging the roles of indices 1 and 2. Now we show that (3.3) also holds. From (4.4),

$$\{y\in Y : f(x,y)\le a \text{ for all } x\in L_N\}=\{y\in Y : y\notin S_1(x) \text{ for all } x\in L_N\}.$$

Since each $S_1(x)$ is compactly open, by the remark for Theorem 3, condition (3.3) is satisfied. Therefore, by Theorem 3, we have $T_1\cap T_2\neq\emptyset$. This contradicts (4.1). This completes our proof.

□

PARTICULAR FORMS. 1. Ben-El-Mechaiekh et al [4, Théorème 5.4]: X and Y are compact convex spaces. Note that compactness of X is superfluous.

2. Granas [9, 3.1 Théorème et 13.6 Théorème]: X and Y are convex spaces with stronger compactness condition than ours. The arguments are based on the KKM theorem.

3. Shih and Tan [30, Theorem 4]: X and Y are convex spaces.

4. Ding [7, Theorem 5.2 and Corollary 5.1]: A C-space version of Theorem 4 and its simple consequence.

3. Von Neumann Type Minimax Theorems

From Theorem 4, we deduce the following von Neumann type minimax theorem for G-convex spaces:

THEOREM 5. *Let (X,Γ) and (Y,Γ') be Hausdorff G-convex spaces and $f,s,t,g : X \times Y \to \mathbb{R}$ functions satisfying (4.1)-(4.3) and (4.4) for any $a \in \mathbb{R}$. Then we have*

$$\inf_{y\in Y} \sup_{x\in X} f(x,y) \leq \sup_{x\in X} \inf_{y\in Y} g(x,y).$$

Further if $f = g$, then we have

$$\inf_{y\in Y} \sup_{x\in X} f(x,y) = \sup_{x\in X} \inf_{y\in Y} f(x,y).$$

Proof. We may assume that

$$-\infty < u = \inf_{y\in Y} \sup_{x\in X} f(x,y)$$

and let $a < u$. Since $\inf_{y\in Y} \sup_{x\in X} f(x,y) > a$, for each $y \in Y$, there exists an $x \in X$ such that $f(x,y) > a$. Hence the conclusion (i) of Theorem 4 does not hold. Therefore, by Theorem 4 (ii), there exists an $x_0 \in X$ such that $g(x_0, y) \geq a$ for all $y \in Y$. Since a is arbitrary and $a < u$, we have

$$u \leq \sup_{x \in X} \inf_{y \in Y} g(x, y).$$

This completes the proof of the first part. Further if $f = g$, then we always have

$$\sup_{x \in X} \inf_{y \in Y} f(x, y) \leq \inf_{y \in Y} \sup_{x \in X} f(x, y).$$

Therefore, by the first part, we have the minimax equality. □

Remark. In Theorem 5, if $f = g$, we may assume that (4.4) holds for any $a < u$ whenever $-\infty < u = \inf_{y \in Y} \sup_{x \in X} f(x, y)$.

Particular forms. 1. Von Neumann [18] : X and Y are compact convex subsets of Euclidean spaces and $f = g$ is continuous.

2. Nikaidô [19]: Euclidean spaces in the above are replaced by Hausdorff topological spaces, and f is continuous in each variable.

3. Sion [32]: X and Y are compact convex spaces and $f = g = s = t$.

4. Brézis, Nirenberg and Stampachia [6, Proposition 1]: X and Y are convex subsets of Hausdorff topological vector spaces, $f = g$ and assumed that

(4.4)′ for some $\tilde{x}$ and some $\lambda > \sup_{x \in X} \inf_{y \in Y} f(x, y)$, the set $\{y \in Y : f(\tilde{x}, y) \leq \lambda\}$ is compact,

instead of (4.4).

We show that (4.4)′ implies (4.4) for any $a < u$ as in Remark. In fact, for any $N \in \langle X \rangle$, let $L_N = \mathrm{co}(\{\tilde{x}\} \cup N)$. Note that $\{y \in Y : f(x, y) \leq a\}$ is closed since $f(x, \cdot)$ is l.s.c. for each $x \in X$ by (4.2). therefore

$$\{y \in Y : f(x, y) \leq a \text{ for all } x \in L_N\} = \bigcap_{x \in L_N} \{y \in Y : f(x, y) \leq a\}$$

is a closed subset of the compact set $\{y \in Y : f(\tilde{x}, y) \leq \lambda\}$ since $\tilde{x} \in L_N$ and $a < u < \lambda$. Therefore, $(4.4)' \Rightarrow (4.4)$.

5. Liu [17, Theorem 1]: X and Y are convex spaces, Y is compact, and $f = s$, $t = g$.

6. Komiya [14, Theorem 3]: X and Y are compact convex spaces in the sense of Komiya, and $f = g$.

7. Ben-El-Mechaiekh et al [4, Corollaire 5.5]: X and Y are compact convex spaces. Note that compactness of X is superfluous.

8. Lassonde [15, Theorem 1.11]: X and Y are convex spaces and $f = s$, $t = g$.

9. Simons [31, Theorem 1.4 and Corollary 1.5]: X and Y are convex spaces, Y is compact, and $f = s$, $t = g$.

10. Shih and Tan [29, Theorems 4 and 5]: X and Y are convex spaces.

11. Bielawski [5, (4.13) Theorem]: X and Y are compact spaces having certain simplicial convexities, and $f = g$.

12. Granas [9, 3.1 et 3.2 Théorèmes] : X and Y are convex spaces and Y is compact.

13. Horvath [13, Proposition 5.2]: X and Y are C-spaces, Y is compact, and $f = g$.

14. Ben-El-Mechaiekh [3, Corollary 7]: A particular form of Theorem 5 slightly extending the result of Brézis et al. [6].

15. Guillerme [11, Théorèmes IV.2 et V.2]: Particular forms of Theorem 5 for convex spaces X and Y.

16. Ding [7, Theorem 5.3 and Corollary 5.2]: A C-space version of Theorem 5 and its simple consequence.

Let $(X, D; \Gamma)$ be a G-convex space, Y a set and $f : D\times Y \to \overline{\mathbb{R}}, g : X\times Y \to \overline{\mathbb{R}}$ be two functions. g is said to be f-*qausiconcave* on X if for any $A \in \langle D\rangle$ and for each $y \in Y$,

$$g(z, y) \geq \min_{x\in A} f(x, y) \text{ for all } z \in \Gamma_A.$$

It is clear that if $D = X$, $f \leq g$ on $X \times Y$ and for each $y \in Y$, the function $x \mapsto f(x, y)$ or $x \mapsto g(x, y)$ is quasiconcave, then g is f-quasiconcave on X. See Balaj [2].

$T \in \mathbb{V}(X, Y) \Longleftrightarrow$ for topological spaces X and Y, a multimap $T : X \multimap Y$ is u.s.c. with compact acyclic values.

Note that $\mathbb{V}(X, Y) \subset \mathfrak{A}^\kappa_c(X, Y)$.

The following lemma is a particular case of Corollary for $\mathbb{V}$ of $\mathfrak{A}^\kappa_c$ in [26].

LEMMA. *Let $(X, D; \Gamma)$ be a G-convex space, Y a Hausdorff space, $F \in \mathbb{V}(X, Y)$ and $H : X \multimap Y$ such that, for any $N \in \langle D\rangle$, $F(\Gamma_N) \subset H(N)$. Then the family $\{\overline{H(x)} : x \in D\}$ has the finite intersection property.*

Let X and Y be two topological spaces. A function $f : X \times Y \to \overline{\mathbb{R}}$ is said to be *transfer lower semicontinuous* or *transfer l.s.c.* in the second variable [16, 34] if for each $a \in \mathbb{R}$ and all $x \in X$, $y \in Y$ with $f(x, y) > a$, there exists a $x' \in X$ and a neighborhood $V(y)$ of y such that $f(x', z) > a$ for all $z \in V(y)$.

Note that if F is transfer l.s.c. in the second variable, then $\bigcap_{x\in X}\{y \in Y : f(x, y) \leq a\} = \bigcap_{x\in X} \overline{\{y \in Y : f(x, y) \leq a\}}$, see Tian [33].

THEOREM 6. *Let $(X, D; \Gamma)$ be a G-convex space, Y a Hausdorff compact space, $f : D \times Y \to \overline{\mathbb{R}}, g : X \times Y \to \overline{\mathbb{R}}$ two functions and $a \in \mathbb{R}$ satisfying*

(6.1) *g is l.s.c. on $X \times Y$;*

(6.2) *g is f-quasiconcave on X;*

(6.3) *for each $x \in X$, the set $\{y \in Y : g(x, y) \leq a\}$ is acyclic or empty; and*

(6.4) *for each $x \in D$, $f(x, \cdot)$ is transfer l.s.c. in the second variable.*

Then

$$\inf_{y\in Y}\sup_{x\in X} f(x,y) \le \sup_{x\in X}\min_{y\in Y} g(x,y).$$

Proof. Since $g(x,y)$ is l.s.c. on Y, $\min_{y\in Y} g(x,y)$ exists for each $x\in X$. We may assume that $v=\sup_{x\in X}\min_{y\in Y} g(x,y)<\infty$. and let $a>v$. Define the multimaps $F : X \multimap Y$, $H : D \multimap Y$ by

$$F(x)=\{y\in Y : g(x,y)\le a\},\quad H(x)=\{y\in Y : f(x,y)\le a\}.$$

Note that $\bigcap_{x\in D} H(x)=\bigcap_{x\in D}\overline{H(x)}$ and for each $x\in X$, $F(x)$ is nonempty acyclic.

Since g is l.s.c. on $X\times Y$, GrF is closed in $X\times Y$, hence F has closed values. Since Y is compact, F is u.s.c. and thus $F\in\mathbb{V}(X,Y)$. By (6.2), for each $A\in\langle D\rangle$, $F(\Gamma_A)\subset H(A)$. By Lemma, $\{\overline{H(x)} : x\in X\}$ satisfies the finite intersection property, therefore $\bigcap_{x\in D} H(x)=\bigcap_{x\in D}\overline{H(x)}\neq\emptyset$. i.e. $\inf_{y\in Y}\sup_{x\in X} f(x,y)\le a$ for all $a>\sup_{x\in X}\min_{y\in Y} g(x,y)$. Therefore

$$\inf_{y\in Y}\sup_{x\in X} f(x,y) \le \sup_{x\in X}\min_{y\in Y} g(x,y).$$

□

Particular form. Balaj [1, Theorem 3]: X is a convex space and $f(x,\cdot)$ is l.s.c. for each $x\in X$.

Corollary 7. *Let $(X;\Gamma)$ be a G-convex space, Y a Hausdorff compact space, $f : X\times Y\to\overline{\mathbb{R}}$ a l.s.c. function such that for each $a\in\mathbb{R}$, the following hold:*

(7.1) *for each $y\in Y$, $\{x\in X : f(x,y)>a\}$ is nonempty and Γ-convex; and*

(7.2) *for each $x\in X$, the set $\{y\in Y : f(x,y)\le a\}$ is acyclic.*

Then

$$\min_{y\in Y}\sup_{x\in X} f(x,y) = \sup_{x\in X}\min_{y\in Y} f(x,y).$$

Proof. Since $f(x,\cdot)$ is l.s.c. on Y, $\min_{y\in Y} f(x,y)$ exists for each $x \in X$. Since $\sup_{x\in X} f(x,y)$ is l.s.c. on $y \in Y$, being the supremum of l.s.c. function $\min_{y\in Y}$ $\sup_{x\in X} f(x,y)$ exists. Note that $\min_{y\in Y} f(x,y) \le f(x,y) \le \sup_{x\in X} f(x,y)$ for all $x \in X$ and $y \in Y$. Therefore, we have

$$\sup_{x\in X}\min_{y\in Y} f(x,y) \le \min_{y\in Y}\sup_{x\in X} f(x,y).$$

The equality holds by Theorem 6. □

Particular forms. 1. von Neumann [18]: X and Y are compact convex subsets of Euclidean spaces, f is continuous and Γ-convexity and acyclicity are replaced by convexity.

2. Some related results are given in Park et al. [24].

4. Systems of Γ-convex Inequalities

Let X be a set and $\mathcal{F} = \{f\}$, $\mathcal{G} = \{g\}$ two families of real functions on X. we denote $\mathcal{F} \le \mathcal{G}$ if and only if for each $f \in \mathcal{F}$, there exists a $g \in \mathcal{G}$ such that $f(x) \le g(x)$ for each $x \in X$.

A family $\mathcal{H} = \{h\}$ of real functions on X is said to be *concave* whenever for each $h_1, h_2, \cdots, h_n \in \mathcal{H}$ and each $\alpha_1, \alpha_2, \cdots, \alpha_n \in [0,1]$ with $\Sigma_{i=1}^n \alpha_i = 1$, there exists an $h \in \mathcal{H}$ satisfying

$$h(x) \ge \Sigma_{i=1}^n \alpha_i h_i(x) \text{ for each } x \in X;$$

see Pietsch [27, p.40].

Theorem 8. *Let $(X;\Gamma)$ be a Hausdorff G-convex space, and $\mathcal{F},\mathcal{G},\mathcal{H}$ three collections of real valued functions on X. Suppose that*

(8.1) $\mathcal{F} \le \mathcal{G} \le \mathcal{H}$;

(8.2) *each $f \in \mathcal{F}$ is compactly l.s.c. on X;*

(8.3) *each convex combination of functions in $\mathcal{G}$ is quasiconvex;*

(8.4) *$\mathcal{H}$ is concave; and*

(8.5) *there exists a nonempty compact subset K of X and for each $\lambda \in \mathbb{R}$ such that for each $x \in X\backslash K$ and each convex combination f of functions in $\mathcal{F}$, we have $f(x) > \lambda$.*

Then the following minimax inequality holds:

$$\min_{x\in K}\sup_{f\in\mathcal{F}} f(x) \le \sup_{h\in\mathcal{H}}\inf_{x\in X} h(x).$$

Proof. Let $a = \sup_{h\in\mathcal{H}}\inf_{x\in X} h(x)$ and assume $a < \infty$. For each $f \in \mathcal{F}$, let $S(f) = \{x \in K : f(x) \le a\}$, which is closed in K, since f is compactly l.s.c. If the family $\{S(f) : f \in \mathcal{F}\}$ has the finite intersection property, the compactness of K implies the conclusion. Let $\{f_1, f_2, \cdots, f_n\} \subset \mathcal{F}$. Choose $\{g_i\}_{i=1}^n \subset \mathcal{G}$ and $\{h_i\}_{i=1}^n \subset \mathcal{H}$ so that $f_i \le g_i \le h_i$ for each i. Define $\bar{f}, \bar{g} : \Delta_{n-1} \times X \to \mathbb{R}$ by

$$\bar{f}(\alpha, x) = \Sigma_{i=1}^n \alpha_i f_i(x) \text{ and } \bar{g}(\alpha, x) = \Sigma_{i=1}^n \alpha_i g_i(x)$$

for $(\alpha, x) \in \Delta_{n-1} \times X$. Then

(1) $\bar{f} \le \bar{g}$ by definition;

(2) for each $\alpha \in \Delta_{n-1}$, $\bar{f}(\alpha,\cdot)$ is compactly l.s.c. on X (since each $f_i(\cdot)$ is compactly l.s.c. by (8.2)), and $A \in \langle\{x \in X : \bar{g}(\alpha, x) < a\}\rangle$ implies $\Gamma_A \subset \{x \in X : \bar{g}(\alpha, x) < a\}$ by (8.3);

(3) for each $x \in X$, $\bar{g}(\cdot, x)$ is u.s.c. (in fact, $\alpha \mapsto \Sigma_{i=1}^n \alpha_i g_i(x)$ is continuous on Δ_{n-1}), and $B \in \langle\{\alpha \in \Delta_{n-1} : \bar{f}(\alpha, x) > a\}\rangle$ implies co$B \subset \{\alpha \in \Delta_{n-1} : \bar{f}(\alpha, x) > a\}$; and

(4) for each $N \in \langle \Delta_{n-1} \rangle$, by putting $L_N = \Delta_{n-1}$, we have

$$\bigcap_{\alpha \in \Delta_{n-1}} \{x \in X : \bar{f}(\alpha, x) \leq a\} \subset K$$

by (8.5). Therefore the requirements of Theorem 5 replacing (X, Y) by (Δ_{n-1}, X) are all satisfied for $\bar{f} = s$ and $t = \bar{g}$. Hence we have $\inf_{x \in X} \sup_{\alpha \in \Delta_{n-1}} \bar{f}(\alpha, x) \leq \sup_{\alpha \in \Delta_{n-1}} \inf_{x \in X} \bar{g}(\alpha, x)$. On the other hand, there exists an $h \in \mathcal{H}$ such that

$$\bar{g}(\alpha, x) = \Sigma_{i=1}^{n} \alpha_i g_i(x) \leq \Sigma_{i=1}^{n} \alpha_i h_i(x) \leq h(x)$$

for all $x \in X$ by (8.4).

Therefore $\sup_{\alpha \in \Delta_{n-1}} \inf_{x \in X} \bar{g}(\alpha, x) \leq \sup_{h \in \mathcal{H}} \inf_{x \in X} h(x) = a$. Consequently, there exists an $x_0 \in X$ such that $\bar{f}(\alpha, x_0) = \Sigma_{i=1}^{n} \alpha_i f_i(x_0) \leq a$ for all $\alpha \in \Delta_{n-1}$.

This clearly implies $f_i(x_0) \leq a$ for each i, and $x_0 \in K$ by (8.5). Hence we have

$$x_0 \in \bigcap_{i=1}^{n} S(f_i).$$

$\square$

Particular forms. 1. Granas and Liu [10, Theorem 9.2]: X is a compact convex space.

2. Shih and Tan [30, Theorem 7]: X is a normal closed convex space.

3. Balaj [1, Theorem 5] is compact acyclic case of Theorem 8 with a superfluous condition that $g \in \mathcal{G}$ is l.s.c.

Theorem 9. *Under the hypothesis of Theorem 8, given any $a \in \mathbb{R}$, one of the following holds:*

(i) *There is an $h \in \mathcal{H}$ such that $\inf_{x \in X} h(x) > a$.*

(ii) *There is an $x_0 \in K$ such that $f(x_0) \leq a$ for all $f \in \mathcal{F}$.*

Note that (ii) is equivalent to the following:

(ii)′ For each $h \in \mathcal{H}$, there exists an $x \in X$ such that $h(x) \leq \lambda$.

THEOREM 10. *Let $(X;\Gamma)$ be a Hausdorff G-convex space, Y a set, $\lambda \in \mathbb{R}$ and $f, g, h : X \times Y \to \mathbb{R}$ functions. Suppose that*

(10.1) $f(x,y) \le g(x,y) \le h(x,y)$ *for* $(x,y) \in X \times Y$;

(10.2) $x \mapsto f(x,y)$ *is compactly l.s.c. on* X *for each* $y \in Y$;

(10.3) *for each* $\{y_1, y_2, \cdots y_k\} \in \langle Y \rangle$ *and each* $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_k) \in \Delta_{k-1}$, *the function* $x \mapsto \Sigma_{i=1}^k \alpha_i g(x, y_i)$ *is quasiconvex;*

(10.4) *the family* $\{h(\cdot,y)\}_{y\in Y}$ *is concave; and*

(10.5) *there exists a nonempty compact subset* K *of* X *and for any* $\lambda \in \mathbb{R}$ *such that for each* $x \in X \backslash K$, $\{y_1, y_2, \cdots y_k\} \in \langle Y \rangle$ *and* $\alpha \in \Delta_{k-1}$, *we have* $\Sigma_{i=1}^k \alpha_i f(x, y_i) > \lambda$.

Then we have the following:

(a) *either*

(i) *there exists an* $y_0 \in Y$ *such that* $h(x,y_0) > \lambda$ *for all* $x \in X$; *or*

(ii) *There exists an* $x_0 \in K$ *such that* $f(x_0, y) \le \lambda$ *for all* $y \in Y$.

(b) *The following minimax inequality holds:*

$$\min_{x\in K} \sup_{y\in Y} f(x,y) \le \sup_{y\in Y} \inf_{x\in X} h(x,y).$$

Proof. Apply Theorems 8 and 9 when $\mathcal{F} = \{f(\cdot,y)\}_{y\in Y}, \mathcal{G} = \{g(\cdot,y)\}_{y\in Y}$ and $\mathcal{H} = \{h(\cdot,y)\}_{y\in Y}$. □

REMARK. In Theorem 10, by putting $f = g$ or $g = h$, we obtain two other minimax theorems for two functions.

In case $f = g = h$ in Theorem 10, we obtain following generalizations of the well known minimax theorems due to Ky Fan, Nikaido, and Kneser:

Theorem 11. *Let $(X;\Gamma)$ be a Hausdorff G-convex space, Y a set, and $f : X \times Y \to \mathbb{R}$ a function such that $x \mapsto f(x,y)$ quasiconvex, and compactly l.s.c. on X for each $y \in Y$, and (10.5) holds. Suppose that one of the following conditions holds:*

(i) *(Ky Fan) the family $\{f(\cdot,y)\}_{y\in Y}$ is concave;*

(ii) *(Nikaido) Y is a convex subset of a vector space and $y \mapsto f(x,y)$ is concave on Y for each $x \in X$;*

(iii) *(Kneser) Y is a vector space and $y \mapsto f(x,y)$ is affine on Y for each $x \in X$;*

Then we have:

(a) *For each $\lambda \in \mathbb{R}$, either*

(i) *there exists an $y_0 \in Y$ such that $f(x,y_0) > \lambda$ for all $x \in X$; or*

(ii) *there exists an $x_0 \in K$ such that $f(x_0,y) \le \lambda$ for all $y \in Y$.*

(b) *The following minimax equality holds:*

$$\min_{x\in K} \sup_{y\in Y} f(x,y) = \sup_{y\in Y} \inf_{x\in X} f(x,y).$$

References

1. M. Balaj, *Acyclic versions for some minimax inequalities*, Nonlinear Anal. Forum **10** (2005), 169–174.
2. M. Balaj, *Two minimax inequalities in G-convex space*, Applied Math. Letters **19** (2006), 235–239.
3. H. Ben-El-Mechaiekh, *Note on a class of set-valued maps having continuous selections*, Fixed point Theory and Applications (M.A. Th/'era and J.-B. Baillon, Eds.), Longman Sci. & Tech., Essex (1991), 33-43.
4. H. Ben-El-Mechaiekh, P. Deguire, A. Granas, *Points fixes et coïncidences pour les fonctions multivoques II (Application de type ϕ et ϕ^*)*, C.R. Acad. Sci. Paris **295** (1982), 381–384.

5. R. Bielawski, *Simplicial convexity and its applications*, J. Math. Anal. Appl. **127** (1987), 155-171.
6. H. Brézis, L. Nirenberg and G. Stampachia, *A remark on Ky Fan's minimax principle*, Boll. Un. Mat. Ital. **6** (1972), 293–300.
7. X.-P. Ding, *Continuous selection theorem, coincidence theorem and intersection theorems concerning sets with H-convex sections*, J. Austral. Math. Soc. (Series A) **52** (1992), 11–25.
8. Ky Fan, *Applications of a theorem concerning sets with convex sections*, Math. Ann. **163** (1966), 189–203.
9. A. Granas, *Sur Quelques méthodes topologiques en analyse convexe*, Méthodes Topolo- giques en Analyse Convexe, Sém. Math. sup. **110** (1990), Press. Univ. Montréal, 11–77.
10. A. Granas and F.C. Liu, *Coincidences for set-valued maps and minimax inequlities*, J. Math. Pures et Appl. **65** (1986), 119–148.
11. J.Guillerme,, *Les inégalités "Inf-Sup $\leq$ Sup-Inf"*, Fixed Point Theory and Applications (M.A. Théra and J.-B. Baillon, Eds.), Longman Sci. & Tech., Essex (1991), 193–214.
12. C.-W. Ha, *A non-compact minimax theorem*, Pacific J. Math. **97** (1981), 115–117.
13. C.D. Horvath, *Contractibility and generalized convexity*, J. Math. Anal. Appl. **156** (1991), 341–357.
14. H. Komiya, *Convexity on a topological space*, Fund. Math. **111** (1981), 107–113.
15. M. Lassonde, *On the use of KKM multifunctions in fixed point theory and related topics*, J. Math. Anal. Appl. **97** (1983), 151–201.
16. L.J. Lin, *Applications of a fixed point theorem in G-convex spaces*, Nonlinear Anal. **46** (2001), 601–608.
17. F.C. Liu, *A note on the von Neumann-Sion minimax principle*, Bull. Inst. Math. Acad. Sinica **6** (1978), 517–522.
18. J. von Neumann, *Zur Theorie der Gesellschaftsspiele*, Math. Ann. **100** (1928), 295–320.
19. H. Nikaidô, *On von Neumann's minimax theorem*, Pacific J. Math. **4** (1954), 65–72.
20. S. Park, *Continuous selection theorems in generalized convex spaces*, Numer. Funct. Anal. Optimiz. **20** (1999), 567–583.
21. S. Park, *Elements of the KKM theory for generalized convex spaces*, Korean J. Comp. Appl. Math. **7** (2000), 1–28.
22. S. Park, *Remarks on topologies of generalized convex spaces*, Nonlinear Funct. Anal. Appl. **5** (2000), 67–79.
23. S. Park, *Fixed point theorems in locally G-convex spaces*, Nonlinear Anal. **48** (2002), 869-879.
24. S. Park, J. S. Bae and H. K. Kang, *Geometric properties, minimax inequalities, and fixed point theorems on convex spaces*, Proc. Amer. Math. Soc. **121** (1994),

429–440.
25. S. Park and H. Kim, *Coincidence theorems of admissible maps on generalized convex spaces*, J. Math. Anal. Appl. **197** (1996), 173–187.
26. S. Park and H. Kim, *Foundations of the KKM theory on generalized convex spaces*, J. Math. Anal. Appl. **209** (1997), 551–571.
27. A. Pietsch, Operator Ideals, North-Holland, Amsterdam (1980).
28. M.-H.Shih and K.-K.Tan, *The Ky Fan minimax principle, sets with convex sections, and variational inequalities*, Differential Geometry, Calculus of Variations, and Their Applications (G.M. Rassias and T.M. Rassias, Eds.), Marcel Dekker Inc, New York (1984).
29. M.-H.Shih and K.-K.Tan, *Non-compact sets with convex sections, II*, J. Math. Anal. Appl. **120** (1986), 264–270.
30. M.-H.Shih and K.-K.Tan, *A geometric property of convex sets with applications to minimax type inequalities and fixed point theorems*, J. Austral. Math. Soc. (Ser. A) **45** (1988), 169–183.
31. S. Simons, *Two-function minimax theorems and variational inequalities for functions on compact and noncompact sets, with some comments on fixed-point theorems*, Proc. Symp. Pure Math. **45** (1986 Pt.2), 377-392.
32. M. Sion, *On general minimax theorems*, Pacific J. Math. **8** (1958), 171–176.
33. G.Q. Tian, *Generalizations of the FKKM Theorem and the Ky Fan minimax inequality, with applications to maximal elements, price equilibrium, and complementarity*, J. Math. Anal. Appl. **170** (1992), 457–471.
34. H. Zhang and X. Wu, *A new existence theorem of maximal elements in noncompact H-spaces with applications to minimax inequalities and variational inequalities*, Acta Math. Hungar. **80** (1998), 115–127.

Hoonjoo Kim
Department of mathematics education,
Daebul University,
Young-Am 526-890, Korea
E-mail: hoonjoo@mail.daebul.ac.kr