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SOFT IDEALS IN SOFT BCC-ALGEBRAS

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Abstract. Soft set theory by Molodtsov is applied to ideals in BCC-algebras. The notion of soft BCC-ideals of soft BCC-algebras and idealistic soft BCC-algebras are introduced, and several examples are provided. Relations between a fuzzy BCC-ideal and an idealistic soft BCC-algebra are given, and the characterization of idealistic soft BCC-algebras is established.

1. Introduction

Various problems in system identification involve characteristics which are essentially non-probabilistic in nature [12]. In response to this situation Zadeh [13] introduced *fuzzy set theory* as an alternative to probability theory. Uncertainty is an attribute of information. In order to suggest a more general framework, the approach to uncertainty is outlined by Zadeh [14]. To solve complicated problems in economics, engineering, and environment, we can't successfully use classical methods because of various uncertainties typical for those problems. There are three theories: theory of probability, theory of fuzzy sets, and the interval mathematics which we can consider as mathematical tools for dealing with uncertainties. But all these theories have their own difficulties. Uncertainties can't be handled using traditional mathematical tools but may be dealt with using a wide range of existing theories such as the probability theory, the theory of (intuitionistic) fuzzy sets, the theory of vague sets, the theory of interval mathematics, and the theory of rough sets. However, all of these theories have their own difficulties which are pointed out in [11]. Maji et al. [10] and Molodtsov [11] suggested that one reason for these difficulties may be due to the inadequacy

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of the parametrization tool of the theory. To overcome these difficulties, Molodtsov [11] introduced the concept of soft set as a new mathematical tool for dealing with uncertainties that is free from the difficulties that have troubled the usual theoretical approaches. Molodtsov pointed out several directions for the applications of soft sets. At present, works on the soft set theory are progressing rapidly. Maji et al. [10] described the application of soft set theory to a decision making problem. Maji et al. [9] also studied several operations on the theory of soft sets. Chen et al. [2] presented a new definition of soft set parametrization reduction, and compared this definition to the related concept of attributes reduction in rough set theory. The algebraic structure of set theories dealing with uncertainties has been studied by some authors. The most appropriate theory for dealing with uncertainties is the theory of fuzzy sets developed by Zadeh [13]. Aktaş and Çağman [1] studied the basic concepts of soft set theory, and compared soft sets to fuzzy and rough sets, providing examples to clarify their differences. They also discussed the notion of soft groups. Jun [5] introduced the notion of soft BCK/BCI-algebras and soft subalgebras, and then derived their basic properties. Jun et al. [8] dealt with the algebraic structure of BCK/BCI-algebras by applying soft set theory. They discussed the algebraic properties of soft sets in BCK/BCI-algebras, and introduced the notion of soft ideals and idealistic soft BCK/BCI-algebras. They investigated relations between soft BCK/BCI-algebras and idealistic soft BCK/BCI-algebras. Also, Jun et al. [7] applied the notion of soft sets by Molodtsov to the ideal theory of d-algerbas, and provided their various properties. In this paper, we deal with the ideal structure of BCC-algebras by applying soft set theory. We introduce the notion of soft BCC-ideals in BCC-algebras and idealistic soft BCC-algebras, and give several examples. We give relations between a fuzzy BCC-ideal and an idealistic soft BCC-algebra. We establish the characterization of idealistic soft BCC-algebras. We also discuss the intersection, union, "AND" operation, and "OR" operation of soft BCC-ideals and idealistic soft BCC-algebras.

2. Basic results on BCC-algebras

Let $K(\tau)$ be the class of all algebras of type $\tau = (2, 0)$. By a *BCC-algebra* we mean a system $(X; \to, 0) \in K(\tau)$ in which the following axioms hold:

(C1) $(\forall x, y, z \in X)\ ((x \to y) \to (z \to y)) \to (x \to z) = 0)$.
(C2) $(\forall x \in X)\ (0 \to x = 0 \ \&\ x \to 0 = x)$.

(C3) $(\forall x, y \in X)$ $(x \to y = 0$ & $y \to x = 0 \Rightarrow x = y)$.

For any BCC-algebra X, the relation $\leq$ defined by

$$(\forall x, y \in X)\,(x \leq y \iff x \to y = 0)$$

is a partial order on X. In a BCC-algebras X, the following hold (see [4]).

(a1) $(\forall x \in X)$ $(x \leq x)$.
(a2) $(\forall x, y \in X)$ $(x \to y \leq x)$.
(a3) $(\forall x, y, z \in X)$ $(x \leq y \implies x \to z \leq y \to z$ & $z \to y \leq z \to x)$.

A nonempty subset S of a BCC-algebra X is said to be a *subalgebra* of X if $x \to y \in S$ whenever $x, y \in S$. A nonempty subset I of a BCC-algebra X is called a *BCC-ideal* of X if it satisfies:

(I1) $0 \in I$.
(I2) $(\forall x, y, z \in X)$ $((x \to y) \to z \in I$ & $y \in I \implies x \to z \in I)$.

Note that a BCC-ideal of a BCC-algebra X is a subalgebra of X. A mapping $f : X \to Y$ of BCC-algebras is called a *homomorphism* if $f(x \to y) = f(x) \to f(y)$ for all $x, y \in X$. For a homomorphism $f : X \to Y$ of BCC-algebras, the *kernel* of f, denoted by $\ker(f)$, is defined to be the set

$$\ker(f) = \{x \in X \mid f(x) = 0\}.$$

Let X be a BCC-algebra. A fuzzy set $\mu : X \to [0, 1]$ is called a *fuzzy BCC-ideal* of X (see [3]) if it satisfies:

(F1) $(\forall x \in X)$ $(\mu(0) \geq \mu(x))$.
(F2) $(\forall x, y, z \in X)$ $(\mu(x \to z) \geq \min\{\mu((x \to y) \to z), \mu(y)\})$.

3. Basic results on soft sets

Molodtsov [11] defined the soft set in the following way: Let U be an initial universe set and E be a set of parameters. Let $\mathscr{P}(U)$ denotes the power set of U and $A \subset E$.

Definition 3.1. [11] A pair (δ, A) is called a *soft set* over U, where δ is a mapping given by

$$\delta : A \to \mathscr{P}(U).$$

In other words, a soft set over U is a parameterized family of subsets of the universe U. For $\varepsilon \in A$, $\delta(\varepsilon)$ may be considered as the set of ε-approximate elements of the soft set (δ, A). Clearly, a soft set is not a set. For illustration, Molodtsov considered several examples in [11].

Definition 3.2. [9] Let (δ, A) and (γ, B) be two soft sets over a common universe U. The *intersection* of (δ, A) and (γ, B) is defined to be the soft set (ρ, C) satisfying the following conditions:

(i) $C = A \cap B$,

(ii) $(\forall e \in C)$ $(\rho(e) = \delta(e)$ or $\gamma(e)$, (as both are same sets)).

In this case, we write $(\delta, A)\widetilde{\cap}(\gamma, B) = (\rho, C)$.

Definition 3.3. [9] Let (δ, A) and (γ, B) be two soft sets over a common universe U. The *union* of (δ, A) and (γ, B) is defined to be the soft set (ρ, C) satisfying the following conditions:

(i) $C = A \cup B$,

(ii) for all $e \in C$,

$$\rho(e) = \begin{cases} \delta(e) & \text{if } e \in A \setminus B, \\ \gamma(e) & \text{if } e \in B \setminus A, \\ \delta(e) \cup \gamma(e) & \text{if } e \in A \cap B. \end{cases}$$

In this case, we write $(\delta, A)\widetilde{\cup}(\gamma, B) = (\rho, C)$.

Definition 3.4. [9] If (δ, A) and (γ, B) are two soft sets over a common universe U, then "(δ, A) *AND* (γ, B)" denoted by $(\delta, A)\widetilde{\wedge}(\gamma, B)$ is defined by $(\delta, A)\widetilde{\wedge}(\gamma, B) = (\rho, A \times B)$, where $\rho(x, y) = \delta(x) \cap \gamma(y)$ for all $(x, y) \in A \times B$.

Definition 3.5. [9] If (δ, A) and (γ, B) are two soft sets over a common universe U, then "(δ, A) *OR* (γ, B)" denoted by $(\delta, A)\widetilde{\vee}(\gamma, B)$ is defined by $(\delta, A)\widetilde{\vee}(\gamma, B) = (\rho, A \times B)$, where $\rho(x, y) = \delta(x) \cup \gamma(y)$ for all $(x, y) \in A \times B$.

Definition 3.6. [9] For two soft sets (δ, A) and (γ, B) over a common universe U, we say that (δ, A) is a *soft subset* of (γ, B), denoted by $(\delta, A)\widetilde{\subset}$ (γ, B), if it satisfies:

(i) $A \subset B$,

(ii) For every $\varepsilon \in A$, $\delta(\varepsilon)$ and $\gamma(\varepsilon)$ are identical approximations.

4. Soft BCC-ideals in BCC-algebras

In what follows let X and A be a BCC-algebra and a nonempty set, respectively, and R will refer to an arbitrary binary relation between an element of A and an element of X, that is, R is a subset of $A \times X$ without otherwise specified. A set-valued function $\delta : A \to \mathscr{P}(X)$ can be defined as $\delta(x) = \{y \in X \mid (x, y) \in R\}$ for all $x \in A$. The pair (δ, A) is then a soft set over X.

Definition 4.1. [6] Let (δ, A) be a soft set over X. Then (δ, A) is called a *soft BCC-algebra* over X if $\delta(x)$ is a subalgebra of X for all $x \in A$.

Definition 4.2. [6] Let (δ, A) and (γ, B) be two soft BCC-algebras over X. Then (δ, A) is called a *soft BCC-subalgebra* of (γ, B), denoted by $(\delta, A)\widetilde{<}(\gamma, B)$, if it satisfies:

(i) $A \subset B$,
(ii) $\delta(x)$ is a subalgebra of $\gamma(x)$ for all $x \in A$.

Definition 4.3. Let S be a subalgebra of X. A subset I of X is called a *BCC-ideal* of X related to S (briefly, *S-BCC-ideal* of X), denoted by $I \lhd S$, if it satisfies:

(i) $0 \in I$,
(ii) $(\forall x, z \in S)$ $(\forall y \in I)$ $((x \to y) \to z \in I \;\Rightarrow\; x \to z \in I)$.

Note that if S is a subalgebra of X and I is a subset of X that contains S, then I is an S-BCC-ideal of X. Obviously, every BCC-ideal of X is an S-BCC-ideal of X for every subalgebra S of X, and hence every BCC-ideal of X is an S-BCC-ideal of X for some subalgebra S of X. But the converse is not true in general as seen in the following example.

Example 4.4. Let $X = \{0, a, b, c, d\}$ be a BCC-algebra with the following Cayley table:

$\to$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	0
b	b	b	0	0	b
c	c	b	a	0	b
d	d	a	d	a	0

Then $S = \{0, b\}$ is a subalgebra of X and $I = \{0, b, d\} \lhd S$, but I is not a BCC-ideal of X since $(d \to d) \to c = 0 \in I$ and $d \to c = a \notin I$.

If S_1 and S_2 are subalgebras of X such that $S_1 \subset S_2$, then every S_2-BCC-ideal of X is an S_1-BCC-ideal of X. But the converse is not true in general.

Example 4.5. Let $X = \{0, 1, 2, 3, 4\}$ be a set with the following Cayley table:

$\rightarrow$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	1	0	0
4	4	3	4	3	0

Then $(X, \rightarrow, 0)$ is a BCC-algebra (see [4]). Note that $S_1 := \{0, 1, 2\}$ and $S_2 := \{0, 1, 2, 3\}$ are subalgebras of X. Let $I := \{0, 1, 2\}$. Then I is an S_1-BCC-ideal of X, but I is not an S_2-BCC-ideal of X because $(3 \rightarrow 2) \rightarrow 1 = 0 \in I$ and $3 \rightarrow 1 = 3 \notin I$.

Definition 4.6. Let (δ, A) be a soft BCC-algebra over X. A soft set (γ, I) over X is called a *soft BCC-ideal* of (δ, A), denoted by $(\gamma, I)\widetilde{\lhd}(\delta, A)$, if it satisfies:

(i) $I \subset A$,
(ii) $(\forall x \in I)\ (\gamma(x) \lhd \delta(x))$.

Let us illustrate this definition using the following example.

Example 4.7. Let $X = \{0, 1, 2, 3, 4\}$ be a BCC-algebra with the following Cayley table:

$\rightarrow$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	1	0	0
4	4	3	3	3	0

For $A = X$, let $\delta : A \rightarrow \mathscr{P}(X)$ be a set-valued function defined by

$$\delta(x) = \{y \in X \mid y \rightarrow (y \rightarrow x) \in \{0, 1\}\}$$

for all $x \in A$. Then $\delta(0) = X$, $\delta(1) = \{0, 1, 2, 3\}$ and $\delta(2) = \delta(3) = \delta(4) = \{0, 1\}$ are subalgebras of X, and so (δ, A) is a soft BCC-algebra over X. Now consider $I := \{0, 1\} \subset A$ and define a set-valued function $\gamma : I \rightarrow \mathscr{P}(X)$ by

$$\gamma(x) = \{y \in X \mid y \rightarrow x \in \{0\}\}$$

for all $x \in I$. We can verify that $\gamma(0) = \{0\} \lhd \delta(0) = X$ and $\gamma(1) = \{0, 1\} \lhd \delta(1) = \{0, 1, 2, 3\}$. Hence $(\gamma, I)\widetilde{\lhd}(\delta, A)$.

Note that every soft BCC-ideal is a soft BCC-subalgebra, but the converse is not true in general as seen in the following example.

Example 4.8. Let $X = \{0, 1, 2, 3, 4, 5\}$ be a BCC-algebra with the following Cayley table:

$\rightarrow$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	1	0	0	0	0	1
2	2	2	0	0	1	1
3	3	2	1	0	1	1
4	4	4	4	4	0	1
5	5	5	5	5	5	0

Consider $A = X$ and define a set-valued function $\delta : A \rightarrow \mathscr{P}(X)$ by

$$\delta(x) = \{y \in X \mid (y \rightarrow x) \rightarrow x \in \{0\}\}$$

for all $x \in A$. Then (δ, A) is a soft BCC-algebra over X since $\delta(0) = \{0\}$, $\delta(1) = \{0, 1\}$, $\delta(2) = \delta(3) = \{0, 1, 2, 3\}$, $\delta(4) = \{0, 1, 2, 3, 4\}$ and $\delta(5) = \{0, 5\}$ are subalgebras of X. Now take $I := \{2, 4\} \subset A$ and let $\gamma : I \rightarrow \mathscr{P}(X)$ be a set-valued function defined by

$$\gamma(x) = \{y \in X \mid y \rightarrow x \in \{0\}\}$$

for all $x \in I$. Then $\gamma(2) = \{0, 1, 2\}$ and $\gamma(4) = \{0, 1, 4\}$ are subalgebras of X, and hence (γ, I) is also a soft BCC-algebra over X. Obviously $\gamma(2)$ and $\gamma(4)$ are subalgebras of $\delta(2)$ and $\delta(4)$, respectively. Thus $(\gamma, I)\widetilde{<}(\delta, A)$. But $\gamma(4)$ is not a $\delta(4)$-BCC-ideal of X since $(2 \rightarrow 4) \rightarrow 1 = 0 \in \gamma(4)$ and $2 \rightarrow 1 = 2 \notin \gamma(4)$. This means that (γ, I) is not a soft BCC-ideal of (δ, A).

Theorem 4.9. *Let (δ, A) be a soft BCC-algebra over X. For any soft sets (γ_1, I_1) and (γ_2, I_2) over X where $I_1 \cap I_2 \neq \emptyset$, we have*

$$(\gamma_1, I_1)\widetilde{\lhd}(\delta, A),\ (\gamma_2, I_2)\widetilde{\lhd}(\delta, A) \ \Rightarrow\ (\gamma_1, I_1)\widetilde{\cap}(\gamma_2, I_2)\widetilde{\lhd}(\delta, A).$$

Proof. Using Definition 3.2, we can write

$$(\gamma_1, I_1)\widetilde{\cap}(\gamma_2, I_2) = (\gamma, I),$$

where $I = I_1 \cap I_2$ and $\gamma(x) = \gamma_1(x)$ or $\gamma_2(x)$ for all $x \in I$. Obviously, $I \subset A$ and $\gamma : I \rightarrow \mathscr{P}(X)$ is a mapping. Hence (γ, I) is a soft set over X. Since $(\gamma_1, I_1)\widetilde{\lhd}(\delta, A)$ and $(\gamma_2, I_2)\widetilde{\lhd}(\delta, A)$, we know that $\gamma(x) = \gamma_1(x) \lhd \delta(x)$ or $\gamma(x) = \gamma_2(x) \lhd \delta(x)$ for all $x \in I$. Hence

$$(\gamma_1, I_1)\widetilde{\cap}(\gamma_2, I_2) = (\gamma, I)\widetilde{\lhd}(\delta, A).$$

This completes the proof. □

Corollary 4.10. *Let (δ, A) be a soft BCC-algebra over X. For any soft sets (γ, I) and (ρ, I) over X, we have*

$$(\gamma, I)\widetilde{\lhd}(\delta, A),\ (\rho, I)\widetilde{\lhd}(\delta, A) \Rightarrow (\gamma, I)\widetilde{\cap}(\rho, I)\widetilde{\lhd}(\delta, A).$$

Proof. Straightforward. □

Theorem 4.11. *Let (δ, A) be a soft BCC-algebra over X. For any soft sets (γ, I) and (ρ, J) over X in which I and J are disjoint, we have*

$$(\gamma, I)\widetilde{\lhd}(\delta, A),\ (\rho, J)\widetilde{\lhd}(\delta, A) \Rightarrow (\gamma, I)\widetilde{\cup}(\rho, J)\widetilde{\lhd}(\delta, A).$$

Proof. Assume that $(\gamma, I)\widetilde{\lhd}(\delta, A)$ and $(\rho, J)\widetilde{\lhd}(\delta, A)$. By means of Definition 3.3, we can write $(\gamma, I)\widetilde{\cup}(\rho, J) = (\kappa, U)$ where $U = I \cup J$ and for every $x \in U$,

$$\kappa(x) = \begin{cases} \gamma(x) & \text{if } x \in I \setminus J, \\ \rho(x) & \text{if } x \in J \setminus I, \\ \gamma(x) \cup \rho(x) & \text{if } x \in I \cap J. \end{cases}$$

Since $I \cap J = \emptyset$, either $x \in I \setminus J$ or $x \in J \setminus I$ for all $x \in U$. If $x \in I \setminus J$, then $\kappa(x) = \gamma(x) \lhd \delta(x)$ since $(\gamma, I)\widetilde{\lhd}(\delta, A)$. If $x \in J \setminus I$, then $\kappa(x) = \rho(x) \lhd \delta(x)$ since $(\rho, J)\widetilde{\lhd}(\delta, A)$. Thus $\kappa(x) \lhd \delta(x)$ for all $x \in U$, and so $(\gamma, I)\widetilde{\cup}(\rho, J) = (\kappa, U)\widetilde{\lhd}(\delta, A)$. □

If I and J are not disjoint in Theorem 4.11, then Theorem 4.11 is not true in general as seen in the following example.

Example 4.12. Consider the BCC-algebra X in Example 4.7. Let $A = X$ and $\delta : A \to \mathscr{P}(X)$ be a set-valued function defined by

$$\delta(x) = \{y \in X \mid y \to (x \to y) \in \{0, 3\}\}$$

for all $x \in A$. Then (δ, A) is a soft BCC-algebra over X since $\delta(0) = \delta(1) = \delta(2) = \{0, 3\}$, $\delta(3) = \{0, 1, 3\}$ and $\delta(4) = X$ are subalgebras of X. If we take $I := \{2, 3\} \subset A$ and define a set-valued function $\gamma : I \to \mathscr{P}(X)$ by

$$\gamma(x) = \{y \in X \mid y \to (y \to x) \in \{0\}\}$$

for all $x \in I$, then we can verify that $\gamma(2) = \{0, 1\} \lhd \delta(2)$ and $\gamma(3) = \{0\} \lhd \delta(3)$, and hence $(\gamma, I)\widetilde{\lhd}(\delta, A)$. Now let (ρ, J) be a soft set over X, where $J = \{2\} \subset A$ and $\rho : J \to \mathscr{P}(X)$ is a set-valued function defined by

$$\rho(x) = \{y \in X \mid (y \to x) \to x \in \{0\}\}$$

for all $x \in J$. Then $\rho(2) = \{0, 2\} \lhd \delta(2)$, which means that $(\rho, J)\widetilde{\lhd}(\delta, A)$. But $\gamma(2) \cup \rho(2) = \{0, 1, 2\}$ is not a $\delta(2)$-BCC-ideal of X since $(3 \to 2) \to$

$0 = 1 \in \{0, 1, 2\}$ and $3 \to 0 = 3 \notin \{0, 1, 2\}$. Hence $(\gamma, I)\widetilde{\cup}(\rho, J)$ is not a soft BCC-ideal of (δ, A).

Definition 4.13. Let (δ, A) be a soft set over X. Then (δ, A) is called an *idealistic soft BCC-algebra* over X if $\delta(x)$ is a BCC-ideal of X for all $x \in A$.

Let us illustrate this definition using the following example.

Example 4.14. Consider the BCC-algebra X in Example 4.4. Let $A = X$ and define a set-valued function $\delta : A \to \mathscr{P}(X)$ by

$$\delta(x) = \{y \in X \mid y \to (y \to x) \in \{0\}\}$$

for all $x \in A$. Then $\delta(0) = X$, $\delta(a) = \delta(d) = \{0, b\}$, $\delta(b) = \{0, a, d\}$ and $\delta(c) = \{0\}$. We can verify that $\delta(x) \lhd X$ for all $x \in A$, and hence (δ, A) is an idealistic soft BCC-algebra over X.

Proposition 4.15. *Let (δ, A) and (δ, B) be soft sets over X where $B \subseteq A \subseteq X$. If (δ, A) is an idealistic soft BCC-algebra over X, then so is (δ, B).*

Proof. Straightforward. □

The converse of Proposition 4.15 is not true in general as seen in the following example.

Example 4.16. Let (δ, A) be a soft set over X which is defined in Example 4.8. If we take $B := \{0, 1, 2, 3, 4\} \subseteq A$, we can verify that $\delta(0) = \{0\}$, $\delta(1) = \{0, 1\}$, $\delta(2) = \delta(3) = \{0, 1, 2, 3\}$ and $\delta(4) = \{0, 1, 2, 3, 4\}$ are BCC-ideals of X. Thus (δ, B) is an idealistic soft BCC-algebra over X. But $\delta(5)(= \{0, 5\})$ is not a BCC-ideal of X since $(2 \to 5) \to 4 = 0 \in \delta(5)$ and $2 \to 4 = 1 \notin \delta(5)$. This means that (δ, A) is not an idealistic soft BCC-algebra over X.

Since every BCC-ideal of a BCC-algebra is a subalgebra, we know that every idealistic soft BCC-algebra over a BCC-algebra X is a soft BCC-algebra over X, but the converse is not true as seen in the following example.

Example 4.17. (1) Consider a soft set (δ, A) over X which is given in Example 4.7. We know that (δ, A) is a soft BCC-algebra over X (see Example 4.7). But $\delta(2)(= \{0, 1\})$ is not a BCC-ideal of X since $(4 \to 1) \to 2 = 1 \in \delta(2)$ and $4 \to 2 = 3 \notin \delta(2)$. Hence (δ, A) is not an idealistic soft BCC-algebra over X.

(2) Consider a soft BCC-algebra (δ, A) over X which is described in Example 4.8. We know that (δ, A) is not an idealistic soft BCC-algebra over X (see Example 4.16).

Theorem 4.18. *Let (δ, A) and (γ, B) be two idealistic soft BCC-algebras over X. If $A \cap B \neq \emptyset$, then the intersection $(\delta, A)\widetilde{\cap}(\gamma, B)$ is an idealistic soft BCC-algebra over X.*

Proof. Using Definition 3.2, we can write $(\delta, A)\widetilde{\cap}(\gamma, B) = (\rho, C)$, where $C = A \cap B$ and $\rho(x) = \delta(x)$ or $\gamma(x)$ for all $x \in C$. Note that $\rho : C \to \mathscr{P}(X)$ is a mapping, and therefore (ρ, C) is a soft set over X. Since (δ, A) and (γ, B) are idealistic soft BCC-algebras over X, it follows that $\rho(x) = \delta(x)$ is a BCC-ideal of X, or $\rho(x) = \gamma(x)$ is a BCC-ideal of X for all $x \in C$. Hence $(\rho, C) = (\delta, A)\widetilde{\cap}(\gamma, B)$ is an idealistic soft BCC-algebra over X. □

Corollary 4.19. *Let (δ, A) and (γ, A) be two idealistic soft BCC-algebras over X. Then their intersection $(\delta, A)\widetilde{\cap}(\gamma, A)$ is an idealistic soft BCC-algebra over X.*

Proof. Straightforward. □

Theorem 4.20. *Let (δ, A) and (γ, B) be two idealistic soft BCC-algebras over X. If A and B are disjoint, then the union $(\delta, A)\widetilde{\cup}(\gamma, B)$ is an idealistic soft BCC-algebra over X.*

Proof. Using Definition 3.3, we can write $(\delta, A)\widetilde{\cup}(\gamma, B) = (\rho, C)$, where $C = A \cup B$ and for every $x \in C$,

$$\rho(x) = \begin{cases} \delta(x) & \text{if } x \in A \setminus B, \\ \gamma(x) & \text{if } x \in B \setminus A, \\ \delta(x) \cup \gamma(x) & \text{if } x \in A \cap B. \end{cases}$$

Since $A \cap B = \emptyset$, either $x \in A \setminus B$ or $x \in B \setminus A$ for all $x \in C$. If $x \in A \setminus B$, then $\rho(x) = \delta(x)$ is a BCC-ideal of X since (δ, A) is an idealistic soft BCC-algebra over X. If $x \in B \setminus A$, then $\rho(x) = \gamma(x)$ is a BCC-ideal of X since (γ, B) is an idealistic soft BCC-algebra over X. Hence $(\rho, C) = (\delta, A)\widetilde{\cup}(\gamma, B)$ is an idealistic soft BCC-algebra over X. □

In Theorem 4.20, if A and B are not disjoint, then the result is not valid as seen in the following example.

Example 4.21. Let $X = \{0, a, b, c, d\}$ be a BCC-algebra defined in Example 4.4. Consider an idealistic soft BCC-algebra (δ, A) over X

which is described in Example 4.14. If we take $B = \{b, d\}$, then B is not disjoint with $A(= X)$. Define a set-valued function $\gamma : B \to \mathscr{P}(X)$ by

$$\gamma(x) = \{y \in X \mid y \to x \in \{0\}\}$$

for all $x \in B$. We obtain that $\gamma(b) = \{0, b\} \lhd X$ and $\gamma(d) = \{0, a, d\} \lhd X$ (see Example 4.14). This means that (γ, B) is an idealistic soft BCC-algebra over X. Now, let $(\delta, A)\widetilde{\cup}(\gamma, B) = (\rho, C)$. Then

$$\rho(b) = \delta(b) \cup \gamma(b) = \{0, a, d\} \cup \{0, b\} = \{0, a, b, d\}$$

and

$$\rho(d) = \delta(d) \cup \gamma(d) = \{0, b\} \cup \{0, a, d\} = \{0, a, b, d\}.$$

But $\rho(b)$ and $\rho(d)$ are not BCC-ideals of X since $(c \to a) \to 0 = b \in \{0, a, b, d\}$ and $c \to 0 = c \notin \{0, a, b, d\}$. Hence $(\delta, A)\widetilde{\cup}(\gamma, B)$ is not an idealistic soft BCC-algebra over X.

Theorem 4.22. *If (δ, A) and (γ, B) are idealistic soft BCC-algebras over X, then $(\delta, A)\widetilde{\wedge}(\gamma, B)$ is an idealistic soft BCC-algebra over X.*

Proof. By means of Definition 3.4, we know that

$$(\delta, A)\widetilde{\wedge}(\gamma, B) = (\rho, A \times B),$$

where $\rho(x, y) = \delta(x) \cap \gamma(y)$ for all $(x, y) \in A \times B$. Since $\delta(x)$ and $\gamma(y)$ are BCC-ideals of X, the intersection $\delta(x) \cap \gamma(y)$ is also a BCC-ideal of X. Hence $\rho(x, y)$ is a BCC-ideal of X for all $(x, y) \in A \times B$, and therefore $(\delta, A)\widetilde{\wedge}(\gamma, B) = (\rho, A \times B)$ is an idealistic soft BCC-algebra over X. □

Definition 4.23. An idealistic soft BCC-algebra (δ, A) over X is said to be *trivial* (resp., *whole*) if $\delta(x) = \{0\}$ (resp., $\delta(x) = X$) for all $x \in A$.

Example 4.24. Let $X = \{0, 1, 2, 3, 4\}$ be a BCC-algebra which is given in Example 4.5. Consider $A = \{0, 1, 2\} \subset X$ and a set-valued function $\delta : A \to \mathscr{P}(X)$ defined by $\delta(x) = \{y \in X \mid y \to (x \to y) \in \{0\}\}$ for all $x \in A$. Then $\delta(0) = \delta(1) = \delta(2) = \{0\}$, and so (δ, A) is a trivial idealistic soft BCC-algebra over X. Now, let $\gamma : A \to \mathscr{P}(X)$ be a set-valued function defined by $\gamma(x) = \{y \in X \mid x \to y \in \{0, x\}\}$ for all $x \in A$. Then $\gamma(0) = \gamma(1) = \gamma(2) = X$. Hence (γ, A) is a whole idealistic soft BCC-algebra over X.

Let $f : X \to Y$ be a mapping of BCC-algebras. For a soft set (δ, A) over X, $(f(\delta), A)$ is a soft set over Y where $f(\delta) : A \to \mathscr{P}(Y)$ is defined by $f(\delta)(x) = f(\delta(x))$ for all $x \in A$.

Theorem 4.25. *Let $f : X \to Y$ be an onto homomorphism of BCC-algebras and let (δ, A) be an idealistic soft BCC-algebra over X.*

(i) *If* $\delta(x) \subseteq \ker(f)$ *for all* $x \in A$, *then* $(f(\delta), A)$ *is the trivial idealistic soft BCC-algebra over* Y.
(ii) *If* (δ, A) *is whole, then* $(f(\delta), A)$ *is the whole idealistic soft BCC-algebra over* Y.

Proof. (i) Assume that $\delta(x) \subseteq \ker(f)$ for all $x \in A$. Then $f(\delta)(x) = f(\delta(x)) = \{0_Y\}$ for all $x \in A$. Hence $(f(\delta), A)$ is the trivial idealistic soft BCC-algebra over Y by Definition 4.23.

(ii) Suppose that (δ, A) is whole. Then $\delta(x) = X$ for all $x \in A$, and so $f(\delta)(x) = f(\delta(x)) = f(X) = Y$ for all $x \in A$. It follows from Definition 4.23 that $(f(\delta), A)$ is the whole idealistic soft BCC-algebra over Y. □

Theorem 4.26. *For every fuzzy BCC-ideal* μ *of* X, *there exists an idealistic soft BCC-algebra* (δ, A) *over* X.

Proof. Let μ be a fuzzy BCC-ideal of X. Then $U(\mu; t) := \{x \in X \mid \mu(x) \geq t\}$ is a BCC-ideal of X for all $t \in \mathrm{Im}(\mu)$. If we take $A = \mathrm{Im}(\mu)$ and consider a set-valued function $\delta : A \to \mathscr{P}(X)$ given by $\delta(t) = U(\mu; t)$ for all $t \in A$, then (δ, A) is an idealistic soft BCC-algebra over X. □

Conversely, the following theorem is straightforward.

Theorem 4.27. *For any fuzzy set* μ *in* X, *if an idealistic soft BCC-algebra* (δ, A) *over* X *is given by* $A = \mathrm{Im}(\mu)$ *and* $\delta(t) = U(\mu; t)$ *for all* $t \in A$, *then* μ *is a fuzzy BCC-ideal of* X.

Let μ be a fuzzy set in X and let (δ, A) be a soft set over X in which $A = \mathrm{Im}(\mu)$ and $\delta : A \to \mathscr{P}(X)$ is a set valued function defined by

$$(\forall t \in A)\,(\delta(t) = \{x \in X \mid \mu(x) + t > 1\}). \tag{4.1}$$

Then there exists $t \in A$ such that $\delta(t)$ is not a BCC-ideal of X as seen in the following example.

Example 4.28. Consider the fuzzy set μ and the soft set (δ, A) in Example 4.4. Then $\delta(0.6) = \{0, c, d\}$ is not a BCC-ideal of X since $(a \to c) \to b = 0 \to b = 0 \in \delta(0.6)$ and $a \to b = a \notin \delta(0.6)$.

Theorem 4.29. *Let* μ *be a fuzzy set in* X *and let* (δ, A) *be a soft set over* X *in which* $A = [0, 1]$ *and* $\delta : A \to \mathscr{P}(X)$ *is given by* (4.1). *Then the following assertions are equivalent.*

(i) μ *is a fuzzy BCC-ideal of* X.
(ii) $(\forall t \in A)$ $(\delta(t) \neq \emptyset \Rightarrow (\delta, A)$ is an idealistic soft BCC-algebra over X.

Proof. Assume that μ is a fuzzy BCC-ideal of X. Let $t \in A$ be such that $\delta(t) \neq \emptyset$. If we select $x \in \delta(t)$, then $\mu(0) + t \geq \mu(x) + t > 1$, and so $0 \in \delta(t)$. Let $t \in A$ and $x, y, z \in X$ be such that $y \in \delta(t)$ and $(x \to y) \to z \in \delta(t)$. Then $\mu(y) + t > 1$ and $\mu((x \to y) \to z) + t > 1$. It follows from (F2) that

$$\begin{aligned} \mu(x \to z) + t &\geq \min\{\mu((x \to y) \to z), \mu(y)\} + t \\ &= \min\{\mu((x \to y) \to z) + t, \mu(y) + t\} \\ &> 1. \end{aligned}$$

Hence $x \to z \in \delta(t)$, and therefore $\delta(t)$ is a BCC-ideal of X for all $t \in A$. Consequently, (δ, A) is an idealistic soft BCC-algebra over X. Conversely, suppose that (ii) holds. If there exists $x_0 \in X$ such that $\mu(0) < \mu(x_0)$, then we can select $t_0 \in A$ such that $\mu(0) + t_0 < 1 < \mu(x_0) + t_0$. It follows that $0 \notin \delta(t_0)$, a contradiction. Hence $\mu(0) \geq \mu(x)$ for all $x \in X$. Now, assume that $\mu(a \to c) < \min\{\mu((a \to b) \to c), \mu(b)\}$ for some $a, b, c \in X$. Take $s_0 \in A$ such that

$$\mu(a \to c) + s_0 < 1 < \min\{\mu((a \to b) \to c), \mu(b)\} + s_0.$$

Then $(a \to b) \to c \in \delta(s_0)$ and $b \in \delta(s_0)$ but $a \to c \notin \delta(s_0)$. This is a contradiction. Therefore $\mu(x \to z) \geq \min\{\mu((x \to y) \to z), \mu(y)\}$ for all $x, y, z \in X$. □

Corollary 4.30. *Let μ be a fuzzy set in X such that $\mu(x) > 0.5$ for some $x \in X$, and let (δ, A) be a soft set over X in which*

$$A := \{t \in \mathrm{Im}(\mu) \mid t > 0.5\}$$

and $\delta : A \to \mathscr{P}(X)$ is given by (4.1). If μ is a fuzzy BCC-ideal of X, then (δ, A) is an idealistic soft BCC-algebra over X.

Proof. Straightforward. □

Theorem 4.31. *Let μ be a fuzzy set in X and let (δ, A) be a soft set over X in which $A = (0.5, 1]$ and $\delta : A \to \mathscr{P}(X)$ is defined by*

$$(\forall t \in A)\,(\delta(t) = U(\mu; t)).$$

Then (δ, A) is an idealistic soft BCC-algebra over X if and only if the following assertions are valid.

(i) $(\forall x \in X)$ $(\max\{\mu(0), 0.5\} \geq \mu(x))$.
(ii) $(\forall x, y, z \in X)$ $(\max\{\mu(x \to z), 0.5\} \geq \min\{\mu((x \to y) \to z), \mu(y)\})$.

Proof. Assume that (δ, A) is an idealistic soft BCC-algebra over X. If there exists $x_0 \in X$ such that $\max\{\mu(0), 0.5\} < \mu(x_0)$, then we can select $t_0 \in A$ such that $\max\{\mu(0), 0.5\} < t_0 < \mu(x_0)$. It follows that

$\mu(0) < t_0$ so that $0 \notin \delta(t_0)$. This is a contradiction, and so (i) is valid. Suppose that (ii) is not valid. Then there exist $a, b, c \in X$ such that

$$\max\{\mu(a \to c), 0.5\} < \min\{\mu((a \to b) \to c), \mu(b)\}.$$

Take $u_0 \in A$ such that

$$\max\{\mu(a \to c), 0.5\} < u_0 < \min\{\mu((a \to b) \to c), \mu(b)\}.$$

Then $(a \to b) \to c \in \delta(u_0)$ and $b \in \delta(u_0)$, but $a \to c \notin \delta(u_0)$. This is a contradiction. Therefore (ii) holds.

Conversely, suppose that (i) and (ii) are valid. Let $t \in A$. For any $x \in \delta(t)$, we have

$$\max\{\mu(0), 0.5\} \geq \mu(x) \geq t > 0.5$$

and so $\mu(0) \geq t$, i.e., $0 \in \delta(t)$. Let $x, y, z \in X$ be such that $y \in \delta(t)$ and $(x \to y) \to z \in \delta(t)$. Then $\mu((x \to y) \to z) \geq t$ and $\mu(y) \geq t$. It follows from the second condition that

$$\max\{\mu(x \to z), 0.5\} \geq \min\{\mu((x \to y) \to z), \mu(y)\} \geq t > 0.5$$

so that $\mu(x \to z) \geq t$, i.e., $x \to z \in \delta(t)$. Therefore (δ, A) is an idealistic soft BCC-algebra over X. $\square$

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