Honam Mathematical J. **31** (2009), No. 2, pp. 247–258

# MIXED VECTOR $FQ$-IMPLICIT VARIATIONAL INEQUALITIES WITH $FQ$-COMPLEMENTARITY PROBLEMS

Byung-Soo Lee

**Abstract.** This paper introduces new mixed vector $FQ$-implicit variational inequality problems and corresponding mixed vector $FQ$-implicit complementarity problems for set-valued mappings, and studies the equivalence between them under certain assumptions in Banach spaces. It also derives some new existence theorems of solutions for them with examples under suitable assumptions without monotonicity.

This paper generalizes and extends many results in [8, 10, 19-22].

## 1. Introduction

Variational inequality problems and complementarity problems have many applications in nonlinear analysis including optimization, economics, finance, engineering, mechanics and game theory [4, 6, 11, 12]. In particular, there have been many discussions on the relations between complementarity problems and corresponding variational inequality problems [1-3, 5, 8, 11-19]. In [2], the authors considered the relation between the multivalued implicit variational inequality problems and the multivalued implicit complementarity problems. Cottle and Yao [5] considered some existences of solutions for a nonlinear complementarity problem involving a pseudo-monotone mapping on a closed convex cone in Hilbert spaces, and showed some necessary and sufficient conditions for the existence of solutions to some variational inequality problems. In

Received January 21, 2009. Accepted March 18, 2009.

**2000 Mathematics Subject Classification:** 90C33,49J40.

**Key words and phrases:** Mixed vector $FQ$-implicit complementarity problem; Mixed vector $FQ$-implicit variational inequality problem; KKM-mapping; Fan-KKM Theorem; Positively homogeneous mapping; Convex cone; Upper semicontinuity; Lower semicontinuity.

This research was supported by Kyungsung University Research Grants in 2009.

2001, Yin, Xu and Zhang [22] introduced a class of $F$-complementarity problem (F-CP) for finding $x \in K$ such that

$$\langle Tx, x\rangle + F(x) \;=\; 0 \text{ and } \langle Tx, y\rangle + F(y) \;\geq\; 0 \text{ for all } y \in K,$$

where $K$ is a nonempty closed and convex cone of a real Banach space $X$, $T : K \to X^\star$, the dual space, is a mapping and $F : K \to (-\infty, +\infty)$ is a function, and proved that it is equivalent to the following variational inequality problems:

find $x \in K$ such that

$$\langle Tx, y - x\rangle + F(y) - F(x) \;\geq\; 0 \text{ for all } y \in K,$$

where $F$ is a positively homogeneous and convex function. They also proved the existence of solutions for ($F$-CP) under some assumptions with $F$-pseudo-monotonicity.

In 2003, Fang and Huang [8] studied a class of vector $F$-complementarity problems with demi-pseudomonotone mappings in Banach spaces by considering the solvability of the problems. Huang and Li [10] studied a class of scalar $F$-implicit variational inequality problems and another class of $F$-implicit complementarity problems in Banach spaces in 2004. Recently, the result of the scalar case in [10] was extended and generalized to the vector case by Li and Huang [20]. The equivalence between the $F$-implicit variational inequality problem and $F$-implicit complementarity problem was presented and some new existence theorems of solutions for $F$-implicit variational inequality problems were also proved.

Recently, Lee, Khan and Salahuddin [19] generalized some results of [10, 20] to more generalized vector case. They introduced a new class of generalized vector $F$-implicit complementarity problems and corresponding new class of generalized vector $F$-implicit variational inequality problems in Banach spaces and proved the equivalence between them under certain assumptions. Furthermore, they derived some new existence theorems of solutions for the generalized vector $F$-implicit complementarity problems and the generalized vector $F$-implicit variational inequality problems by using Fan-KKM Theorem [7] under some suitable assumptions without any monotonicity.

In this paper, we consider the following mixed vector $FQ$-implicit variational inequality problems for set-valued mappings (FQ-VI);
find $x \in K$ such that

$$p - s + w - z \in P(x)$$

$$\text{for any } p \in Q(x, g(y)),$$

any $s \in Q(x, h(x))$,
any $w \in F(g(y))$, and
any $z \in F(h(x))$, where $y \in K$.

We also discuss the following mixed vector $FQ$-implicit complementarity problems for set-valued mappings (FQ-CP) corresponding to (FQ-VI);
find $x \in K$ such that
(a) $p + w \in P(x)$
for any $p \in Q(x, g(y))$ and any $w \in F(g(y))$, where $y \in K$.
and
(b) $s + z = 0$
for any $s \in Q(x, h(x))$ and any $z \in F(h(x))$,
where $K$ is a cone of a real Banach space $X$ and $\{P(x) : x \in K\}$ is a family of cones with the apex at the origin in a real Banach space $Y$. Mappings $g$, $h : K \to K$ are single-valued, $F : K \to 2^Y$ and $Q : K \times K \to 2^Y$ are set-valued. We study the equivalence between (FQ-VI) and (FQ-CP) under certain assumptions and derive some existence theorems of solutions for them by using Fan-KKM Theorem under some suitable assumptions without monotonicity.

## 2. Preliminaries

**Remark 2.1.** $P(x)$, $x \in K$ is a closed set such that
(i) $\lambda P(x) \subset P(x)$, $\lambda > 0$, $x \in K$,
(ii) $P(x) + P(x) \subset P(x)$, $x \in K$,
(iii) $P(x) \cap (-P(x)) = \{0\}$, $x \in K$.

An ordered Banach space $(Y, P(x))$ is a real Banach space with an ordering defined by a closed cone $P(x) \subset Y$ as for any $y$, $z \in Y$,

$$y \geq z \quad \text{if and only if} \quad y - z \in P(x),$$
$$y \not\geq z \quad \text{if and only if} \quad y - z \notin P(x).$$

**Remark 2.2.**

$$z \leq 0 \quad \text{if and only if} \quad z \in -P(x),$$
$$z \not\leq 0 \quad \text{if and only if} \quad z \notin -P(x),$$
$$z \geq 0 \quad \text{if and only if} \quad z \in P(x),$$
$$z \not\geq 0 \quad \text{if and only if} \quad z \notin P(x).$$

**Definition 2.1.** Let $X, Y$ be two vector spaces and $K$ be a cone of $X$. A set-valued mapping $F : K \to 2^Y$ is said to be positively homogeneous if $F(\alpha x) = \alpha F(x)$ for all $x \in K$ and $\alpha \geq 0$. $F$ is said to be linear if $F(\alpha x + \beta y) = \alpha F(x) + \beta F(y)$ for $x$, $y \in K$, $\alpha + \beta = 1$, $\alpha$, $\beta \geq 0$.

**Example 2.1.** Put $X = Y = \mathbb{R}$ , $K = \mathbb{R}^+ = [0, \infty)$ and set a set-valued mapping $F : \mathbb{R}^+ \to 2^{\mathbb{R}}$ by $F(x) = [-x, x]$ for $x \in \mathbb{R}^+$, then $F$ is positively homogeneous and linear.

**Definition 2.2.** A set-valued mapping $F : M(\subset X) \to 2^X$ is called a KKM-mapping, if for any finite subset $A$ of $X$, $coA \subset \bigcup\limits_{x \in A} F(x)$, where $coA$ denotes the convex hull of $A$.

**Lemma 2.1.** [8] *Let $(Y, P)$ be an ordered Banach space induced by a pointed closed cone $P$. If $x$, $y \in P$, then $x + y \in P$.*

**Definition 2.3.** A set-valued mapping $W : K \subset X \to 2^Y$ is upper semicontinuous at $x_0 \in K$ if for every open set $V$ containing $W(x_0)$ there exists an open set $U$ containing $x_0$ such that $W(U) \subset V$. $W$ is lower semicontinuous at $x_0 \in K$ if for every open set $V$ intersecting $W(x_0)$ there exists an open set $U$ containing $x_0$ such that $W(x) \cap V \neq \emptyset$ for every $x \in U$. $W$ is upper semicontinuous (resp. lower semicontinuous) on $K$ if it is upper semicontinuous (resp. lower semicontinuous) at every point of $K$. $W$ is continuous on $K$ if it is both upper semicontinuous and lower semicontinuous on $K$.

**Lemma 2.2.** *Let $W : X \to 2^Y$ be a set-valued mapping and $x_0 \in X$.*

(i) *$W$ is upper semicontinuous at $x_0$ if and only if for any net $\{x_\alpha\} \subset X$ with $x_\alpha \to x_0$ and for any net $\{y_\alpha\}$ in $Y$ with $y_\alpha \in W(x_\alpha)$ such that $y_\alpha \to y_0$ in $Y$, we have $y_0 \in W(x_0)$.*

(ii) *$W$ is lower semicontinuous at $x_0$ if and only if for any net $\{x_\alpha\} \subset X$ with $x_\alpha \to x_0$, and for any $y_0 \in W(x_0)$, there exists a net $\{y_\alpha\}$ such that $y_\alpha \in W(x_\alpha)$ and $y_\alpha \to y_0$.*

**Lemma 2.3.** [9] *Let $W : X \to 2^Y$ be a compact set-valued mapping and $x_0 \in X$. Then $W$ is upper semicontinuous at $x_0$ if and only if for any net $\{x_\alpha\} \subset X$ such that $x_\alpha \to x_0$ and for every $y_\alpha \in W(x_\alpha)$, there exists $y_0 \in W(x_0)$ and a subnet $\{y_{\alpha_\beta}\}$ of $\{y_\alpha\}$ such that $y_{\alpha_\beta} \to y_0$.*

**Fan-KKM Theorem.** [7] Let $M$ be a nonempty subset of a Hausdorff topological vector space $X$ and $G : M \to 2^X$ be a KKM-mapping. If $G(x)$ is closed in $X$ for every $x \in M$ and compact for some $x \in M$, then $\bigcap\limits_{x \in M} G(x)$ is nonempty.

## 3. Main Results

In this section, unless otherwise specified, we assume that $K$ is a nonempty closed convex cone of a real Banach space $X$ and $\{P(x) : x \in K\}$ is a family of nonempty pointed closed convex cones with the apex at the origin in a real Banach space $Y$.

(FQ-CP) implies (FQ-VI) easily, now we consider the converse.

**Theorem 3.1.** *Assume that $F : K \to 2^Y$ is positively homogeneous, $Q : K \times K \to 2^Y$ is positively homogeneous in the second argument and $g$, $h : K \to K$ is surjective. If $x$ solves (FQ-VI), then it solves (FQ-CP).*

*Proof.* Let $x(\in K)$ solve (FQ-VI), then

$$p - s + w - z \in P(x) \tag{3.1}$$

for any $p \in Q(x, g(y))$, $s \in Q(x, h(x))$, $w \in F(g(y))$ and $z \in F(h(x))$, where $y \in K$.

Since $g$ is surjective and $K$ is a cone, we can put $g(y) = 2h(x)$, for some $y \in K$. On the other hand, by the positive homogeneity of $F$ and $Q$ in the second argument, we have

$$\begin{aligned} &Q(x, g(y)) = Q(x, 2h(x)) = 2Q(x, h(x)) \quad \text{and} \\ &F(g(y)) = F(2h(x)) = 2F(h(x)). \end{aligned}$$

Hence for any $s \in Q(x, h(x))$ and any $z \in F(h(x))$. From (3.1)

$$s + z = 2s - s + 2z - z \in P(x).$$

Putting $g(y) = \frac{1}{2}h(x)$ for some $y \in K$, by the similar method, we have

$$s + z \in -P(x)$$

for any $s \in Q(x, h(x))$ and any $z \in F(h(x))$.

Since $P(x)$ is a pointed cone,

$$s + z = 0.$$

Thus we obtain

$$\begin{aligned} p + w &= p - s + w - z + s + z \\ &= p - s + w - z \\ &\in P(x) \end{aligned}$$

for any $p \in Q(x, g(y))$, $s \in Q(x, h(x))$, $w \in F(g(y))$ and $z \in F(h(x))$, where $y \in K$ □

Now we consider the existence of solutions to (FQ-VI) and the properties of the solution set.

**Theorem 3.2.** *Let $K$ be a nonempty closed convex subset of $X$ and $P : K \to 2^Y$ be upper semicontinuous on $K$. Assume that*

(a) *$g, h : K \to K$ are continuous, $F : K \to 2^Y$ is lower semicontinuous and $Q : K \times K \to 2^Y$ is lower semicontinuous in two arguments.*

(b) *there exists a single-valued mapping $T : K \times K \to Y$ satisfying*

  (b1) *for $x \in K$, $T(x, x) \in P(x)$,*

  (b2) *for $x, y \in K$,*

$$p - s + w - z - T(x, y) \in P(x)$$
$$\text{for any } p \in Q(x, g(y)),\ s \in Q(x, h(x)),\ w \in F(g(y)) \text{ and } z \in F(h(x)),$$

  (b3) *for $x \in K$ the set $\{y \in K : T(x, y) \notin P(x)\}$ is convex,*

(c) *there exists a nonempty compact convex subset $D$ of $K$ such that for all $x \in K \setminus D$ there exists a $y \in D$ satisfying*

$$p - s + w - z \notin P(x)$$

*for some $p \in Q(x, g(y))$, $s \in Q(x, h(x))$, $w \in F(g(y))$ and $z \in F(h(x))$.*

*Then (FQ-VI) has a nonempty closed solution set.*

*Proof.* Define a set-valued mapping $G : K \to 2^D$ by

$$G(y) = \{x \in D : p - s + w - z \in P(x) \text{ for any } p \in Q(x, g(y)) \\ , s \in Q(x, h(x)) w \in F(g(y)) \text{ and } z \in F(h(x))\} \text{ for } y \in K,$$

then $G(y)$ is closed in $D$ by the condition (a). In fact, let $\{x_\alpha\}$ be a net in $G(y)$ converging to $x_0$, then

$$p_\alpha - s_\alpha + w - z_\alpha \in P(x_\alpha)$$
$$\text{for any } p_\alpha \in Q(x_\alpha, g(y)),\ s_\alpha \in Q(x_\alpha, h(x_\alpha)),\ w \in F(g(y)) \text{ and } z_\alpha \in F(h(x_\alpha)).$$

Since $F$ and $Q$ are lower semicontiuous on $K$ and $K \times K$, respectively, and $g$ and $h$ are continuous, for any $p_0 \in Q(x_0, g(y))$, $s_0 \in Q(x_0, h(x_0))$ and $z_0 \in F(h(x_0))$, there exist nets $\{p_\alpha\}$ in $Q(x_\alpha, g(y))$, $\{s_\alpha\}$ in $Q(x_\alpha, h(x_\alpha))$ and $\{z_\alpha\}$ in $F(h(x_\alpha))$ such that $p_\alpha \to p_0$, $s_\alpha \to s_0$ and $z_\alpha \to z_0$. Hence $p_\alpha - s_\alpha + w - z_\alpha \to p_0 - s_0 + w - z_0$. Since $p_\alpha - s_\alpha + w - z_\alpha \in P(x_\alpha)$ and $P$ is upper semicontinuous, we have

$$p_0 - s_0 + w - z_0 \in P(x_0)$$
$$\text{for any } p_0 \in Q(x_0, g(y)),\ s_0 \in Q(x_0, h(x_0)),\ w \in F(g(y))$$

and $z_0 \in F(h(x_0))$.

Hence $x_0 \in G(y)$, which means that $G(y)$ is closed in $D$. Since every element $x \in \bigcap_{y \in K} G(y)$ is a solution of (FQ-VI), we have to show that $\bigcap_{y \in K} G(y)$ is nonempty. So it is sufficient to prove that $\{G(y) : y \in K\}$ has the finite intersection property since $K$ is compact.

Let $\{y_i : i = 1, 2, \cdots, n\}$ be a finite subset of $K$ and set $B = \overline{co}(D \cup \{y_i : i = 1, 2, \cdots, n\})$. Then $B$ is a compact and convex subset of $K$. Define a set-valued mapping $F_1 : B \to 2^B$ by, for any $y \in B$

$$\begin{aligned} F_1(y) = \{x \in B : p - s + w - z \in P(x) \text{ for any } p \in Q(x, g(y)), \\ s \in Q(x, h(x)), w \in F(g(y)) \text{ and } z \in F(h(x))\}, \end{aligned}$$

then $F_1(y)$ is nonempty. In fact, by the condition (b1), for $y \in B$, $T(y, y) \in P(y)$ and by the condition (b2), for $y \in B$

$$p - s + w - z - T(y, y) \in P(y)$$

for any $p \in Q(y, g(y))$, $s \in Q(y, h(y))$, $w \in F(g(y))$ and $z \in F(h(y))$.

Hence by Lemma 2.1, for $y \in B$

$$p - s + w - z \in P(y)$$

for any $p \in Q(y, g(y))$, $s \in Q(y, h(y))$, $w \in F(g(y))$ and $z \in F(h(y))$,

which shows that $F_1(y)$ is nonempty.

By the similar method to the case of $G(y)$, for any $y \in B$, $F_1(y)$ is closed. Since $F_1(y)$ is a closed subset of a compact set $B$, $F_1(y)$ is compact. Now we define another set-valued mapping $F_2 : B \to 2^B$ by, for any $y \in B$

$$F_2(y) = \{x \in B : T(x, y) \in P(x)\},$$

then $F_2$ is a KKM-mapping. In fact, suppose that there exists a finite subset $\{u_i : i = 1, 2, \cdots, n\}$ of $B$ and $\lambda_i \geq 0$ $(i = 1, 2, \cdots, n)$ with $\sum_{i=1}^{n} \lambda_i = 1$ such that $u = \sum_{i=1}^{n} \lambda_i u_i \notin \bigcup_{j=1}^{n} F_2(u_j)$. Then $T(u, u_j) \notin P(u)$ $(j = 1, 2, \cdots, n)$. So from the condition (b3), $T(u, u) \notin P(u)$, which contradicts the condition (b1). Hence $F_2$ is a KKM-mapping. Now we show that $F_1$ is also a KKM-mapping. If, for $y \in B$, $x \in F_2(y)$, then $T(x, y) \in P(x)$. Hence by the condition (b2), from Lemma 2.1 we have

$$p - s + w - z \in P(x)$$

for any $p \in Q(x, g(y))$, $s \in Q(x, h(x))$, $w \in F(g(y))$ and $z \in F(h(x))$.

Thus $x \in F_1(y)$. Consequently, by the F-KKM Theorem, there exists $x^* \in B$ such that $x^* \in F_1(y)$ for all $y \in B$, that is,

$$\begin{aligned} &p^* - s^* + w - z^* \in P(x^*) \\ \text{for any } &p^* \in Q(x^*, g(y)),\ s^* \in Q(x^*, h(x^*)),\ w \in F(g(y)) \\ &\text{and } z^* \in F(h(x^*)). \end{aligned}$$

By the condtion (c) we have $x^* \in D$. Moreover, $x^* \in G(y_i)$ $(i = 1, 2, \cdots, n)$, i.e., $\bigcap_{i=1}^{n} G(y_i)$ is nonempty, so $\{G(y) : y \in K\}$ has the finite intersection property. Since $g$, $h$ are continuous $F$, $Q$ are lower semicontinuous and $P$ is upper semicontinuous, the solution set of (FQ-VI) is closed. □

**Example 3.1.** Let $X = Y = \mathbb{R}^2$ and $K = \mathbb{R}^2_+ = [0, \infty) \times [0, \infty)$. Put, for $x \in K$ $P(x) = K$ and $D = [0, 1] \times [0, 1]$. Define $g$, $h : K \to K$, $F : K \to 2^Y$ and $Q : K \times K \to 2^Y$ by, for $x = (x_1, x_2)$ and $y = (y_1, y_2) \in K$,

$$\begin{aligned} g(y) &= \left(y_2, \frac{y_1}{2}\right), \\ h(x) &= \left(\frac{x_2}{3}, \frac{x_1}{4}\right), \\ F(x) &= \left[\frac{x_1}{2}, x_1\right] \times \{0\}, \\ Q(x, y) &= \left[\frac{y_1 + y_2}{2}, y_1 + y_2\right] \times \{0\}. \end{aligned}$$

Then

$$\begin{aligned} Q(x, g(y)) &= Q\left((x_1, x_2), \left(y_2, \frac{y_1}{2}\right)\right) = \left[\frac{y_2}{2} + \frac{y_1}{4}, y_2 + \frac{y_1}{2}\right] \times \{0\}, \\ Q(x, h(x)) &= Q\left((x_1, x_2), \left(\frac{x_2}{3}, \frac{x_1}{4}\right)\right) = \left[\frac{x_2}{6} + \frac{x_1}{8}, \frac{x_2}{3} + \frac{x_1}{4}\right] \times \{0\}, \\ F(g(y)) &= F\left(\left(y_2, \frac{y_1}{2}\right)\right) = \left[\frac{y_2}{2}, y_2\right] \times \{0\}, \\ F(h(x)) &= F\left(\left(\frac{x_2}{3}, \frac{x_1}{4}\right)\right) = \left[\frac{x_2}{6}, \frac{x_2}{3}\right] \times \{0\}. \end{aligned}$$

If we take a mapping $T : K \times K \to Y$ defined by

$$T(x, y) = \left(\left(\frac{y_1}{4} + \frac{2y_2}{3}\right) - \left(\frac{x_1}{4} + \frac{2x_2}{3}\right), 0\right),$$

then for any $p \in Q(x, g(y))$, $s \in Q(x, h(x))$, $w \in F(g(y))$ and $z \in F(h(x))$,

$$p - s + w - z - T(x, y) \in P(x).$$

In fact, for

$$\begin{aligned} p &= \Big(\frac{y_1}{4} + \frac{y_2}{2}, 0\Big) \in Q(x, g(y)) \\ s &= \Big(\frac{x_2}{3} + \frac{x_1}{4}, 0\Big) \in Q(x, h(x)) \\ w &= \Big(\frac{y_2}{2}, 0\Big) \in F(g(y)) \\ z &= \Big(\frac{x_2}{3}, 0\Big) \in F(h(x)), \end{aligned}$$

$$p - s + w - z - T(x, y) = \Big(\frac{y_2}{3}, 0\Big) \in K.$$

For $x \in K$, let $A = \{z \in K : T(x, z) \notin K\}$ then $A$ is convex. In fact, for $z^1 = (z_1^1, z_2^1)$, $z^2 = (z_1^2, z_2^2) \in A$, and $0 \le \alpha \le 1$,

$$\begin{aligned} \frac{z_1^1}{4} + \frac{2z_2^1}{3} &< \frac{x_1}{4} + \frac{2x_2}{3}, \\ \frac{z_1^2}{4} + \frac{2z_2^2}{3} &< \frac{x_1}{4} + \frac{2x_2}{3}, \end{aligned}$$

hence

$$\frac{\alpha z_1^1}{4} + \frac{2\alpha z_2^1}{3} + \frac{(1-\alpha) z_1^2}{4} + \frac{2(1-\alpha) z_2^2}{3} < \frac{x_1}{4} + \frac{2x_2}{3},$$

which shows that $\alpha z^1 + (1-\alpha) z^2 \in A$. If $x \in K \backslash D$, then $x_1 > 1$ and $x_2 > 1$. Hence there exists $y = (y_1, y_2) = \big(\frac{1}{3}, \frac{2}{3}\big) \in D$ such that

$$p - s + w - z = \left(\frac{3y_1 + 12y_2 - 3x_1 - 8x_2}{12}, 0\right) \notin K$$

for

$$\begin{aligned} p &= \Big(\frac{y_1}{4} + \frac{y_2}{2}, 0\Big) \in Q(x, g(y)) \\ s &= \Big(\frac{x_2}{3} + \frac{x_1}{4}, 0\Big) \in Q(x, h(x)) \\ w &= \Big(\frac{y_2}{2}, 0\Big) \in F(g(y)) \\ z &= \Big(\frac{x_2}{3}, 0\Big) \in F(h(x)). \end{aligned}$$

The upper semicontinuity of $P$ and the lower semicontinuities of $F$ and $Q$ are easily shown. Thus all the conditions in Theorem 3.2 hold. And

$\{(0,0)\}(\subset K)$ is the solution set of (FQ-VI), which is closed. In fact, if there exists $x = (x_1, x_2) \neq (0,0)$ in $K$ such that

$$p - s + w - z \in P(x) = K$$

for any

$$\begin{aligned} &p \in Q(x, g(y)) \\ &s \in Q(x, h(x)) \\ &w \in F(g(y)) \\ &z \in F(h(x)), \quad \text{for} \quad y \in K, \end{aligned}$$

then for $y = (y_1, y_2) \in K$,

$$\left(\frac{y_1}{4} + \frac{y_2}{2} - \frac{x_2}{3} - \frac{x_1}{4} + \frac{y_2}{2} - \frac{x_2}{3}, 0\right) \in K.$$

Hence $3y_1 + 12y_2 \geq 3x_1 + 8x_2 > 0$, which is a contradiction for $y_1 = \frac{1}{4}x_1$ and $y_2 = \frac{1}{6}x_2$, for example.

Putting $D = K$ in the condition (c) of Theorem 3.2, we have the following corollary.

**Theorem 3.3.** *Let $K$ be a nonempty compact and convex subset of $X$. If we assume the conditions (a) and (b) of Theorem 3.2, then the solution set of (FQ-VI) is nonempty closed.*

**Theorem 3.4.** *Assume that $g$ and $h$ are continuous, $F$ and $Q$ are lower semicontinuous, $g$ is surjective, $F$ and $Q$ are positively homogeneous. If we also assume the conditions (b) and (c) in Theorem 3.2, then (FQ-CP) has a solution. Furthemore, the solution set of (FQ-CP) is also nonempty closed.*

**Remark 3.1.** Our results reduce to the costant cone and the single-valued mappings $F$ and $Q$ shown in the previous results in [8, 10, 19, 20, 22]. Our results can be considered as the vector version of the recent result by Wu and Huang [21].

## References


[1] A. Carbone, *A note on complementarity problem*, Internat. J. Math. & Math. Sci. **Vol. 21**(3) (1998), 621–623.

[2] S. S. Chang and N. J. Huang, *Generalized multivalued implicit complementarity problems in Hilbert spaces*, Math. Japonica **36**(6) (1991), 1093–1100.

[3] G. Y. Chen and X. Q. Yang, *The vector complementarity problems and its equivalences with the weak minimal element in ordered spaces*, J. Math. Anal. Appl. **153**(1990), 136–158.

[4] R. W. Cottle, J. S. Pang and R. E. Stone, *The Linear Complementarity Problem*, Academic Press, New York, 1992.

[5] R. W. Cottle and J. C. Yao, *Pseudo-monotone complementarity problems in Hilbert space*, J. Opti, Th. & Appl. **Vol. 75**(2) (1992), 281–295.

[6] F. Facchinei and J. S. Pang, *Finite-dimensional Variational Inequalities and Complementarity Problems*, Springer-Verlag, New York, 2003.

[7] K. Fan, *A generalization of Tychonoff's fixed point theorem*, Math. Ann. **142** (1961), 305–310.

[8] Y. P. Fang and N. J. Huang, *The vector $F$-complementarity problem with demipseudomonotone mappings in Banach spaces*, Appl. Math. Lett. **16** (2003), 1019–1024.

[9] F. Ferro, *A minimax theorem for vector-valued functions*, J. Optim, Theory Appl. **60** (1989), 19–31.

[10] N. J. Huang and J. Li, *$F$-implicit complementarity problems in Banach spaces*, Z. Anal. Anwendungen **23** (2004), 293–302.

[11] G. Isac, *Complementarity Problems*, Lecture Notes in Math. No. 1528, Springer-Verlag, New York, 1992.

[12] G. Isac, *Topological Methods in Complementarity Theory*, Kluwer Academic Publishers, Dordrecht, Boston, London, 2000.

[13] G. Isac, *Condition $(S)^1_+$, Altman's condition and the scalar asymptotic derivative: applications to complementarity theory*, Nonlinear Analysis Forum **Vol. 5** (2000), 1–13.

[14] G. Isac, *A special variational inequality and the implicit complementarity problems*, J. Fac. Sci. Univ. Tokyo **37** (1990), 109–127.

[15] G. Isac, *A generalization of Karamardian's condition in complementarity theory*, Nonlinear Analysis Forum **Vol. 4** (1999), 49–63.

[16] G. Isac, *On the implicit complementarity problem in Hilbert spaces*, Bull. Australian Math. Soc. **32** (1985), 251–260.

[17] G. Isac and Jinlu Li, *Complementarity problems, Karamardian's condition and a generalization of Harker-Pang condition*, Nonlinear Analysis Forum **Vol. 6**(2) (2001), 383–390.

[18] S. Karamardian, *Generalized complementarity problem*, J. Optim. Theory Appl. **8** (1971), 161–168.

[19] B. S. Lee, M. F. Khan and Salahuddin, *Vector $F$-implicit complementarity problems with corresponding variational inequality problems*, Appl. Math. Lett. **20** (2007), 433–438.

[20] J. Li and N. J. Huang, *Vector $F$-implicit complementarity problems in Banach spaces*, Appl. Math. Lett. **19** (2006), 464–471.

[21] K. Q. Wu and N. J. Huang, *General mixed vector $F$-implicit complementarity problems in Banach spaces*, Journal of Applied Analysis **Vol. 14**(1) (2008), 73–88.

[22] H. Y. Yin, C. X. Xu and Z. X. Zhang, *The $F$-complementarity problems and its equivalence with the least element problem*, Acta Math. Sinica **44** (2001), 679-686.

Department of Mathematics,
Kyungsung University
Busan 608-736, Korea
*E-mail*: bslee@ks.ac.kr