Honam Mathematical J. **31** (2009), No. 2, pp. 137–145

COMMON LOCAL SPECTRAL PROPERTIES OF INTERTWINING LINEAR OPERATORS

Jong-Kwang Yoo and Hyuk Han

Abstract. Let $T \in \mathcal{L}(X)$, $S \in \mathcal{L}(Y)$, $A \in \mathcal{L}(X,Y)$ and $B \in \mathcal{L}(Y,X)$ such that $SA = AT$, $TB = BS$, $AB = S$ and $BA = T$. Then S and T shares that same local spectral properties SVEP, property (β), property $(\beta)_\epsilon$, property (δ) and decomposability. From these common local spectral properties, we give some results related with Aluthge transforms and subscalar operators.

1. Introduction

Let X and Y be Banach spaces over the complex plane $\mathbb{C}$, let $\mathcal{L}(X,Y)$ be the space of all bounded linear operators from X to Y. And let $\mathcal{L}(X)$ denote the Banach algebra of all bounded linear operators on a Banach space X.

Given an operator $T \in \mathcal{L}(X)$, $\sigma(T)$ denotes the spectrum of T and $\mathrm{Lat}(T)$ denotes the collection of all closed T-invariant linear subspaces of X, and for an $Y \in \mathrm{Lat}(T)$, $T|Y$ denotes the restriction of T on Y. An operator $T \in \mathcal{L}(X)$ is called *decomposable* if for every open covering $\{U, V\}$ of the complex plane $\mathbb{C}$, there exist $Y, Z \in \mathrm{Lat}(T)$ such that

$$\sigma(T|Y) \subset U, \ \ \sigma(T|Z) \subset V \ \text{ and } \ Y + Z = X.$$

It has been shown by Albrecht[1] that this simple definition of operator decomposability is equivalent to the original definition due to Foias[5]. Decomposable operators are rich. For example, normal operators, spectral operators in the sense of Dunford, operators with totally disconnected spectrum and hence compact operators are decomposable[7].

Received March 30, 2009. Accepted May 19, 2009.
2000 Mathematics Subject Classification: 47A11, 47A53.
Key words and phrases: Local Spectral Theory, Bishop's Property β, Decomposable Operators, Subscalar Operators.
Corresponding author : Hyuk Han(hyukhan@kongju.ac.kr).

Let $D(\lambda, r)$ be the open disc centered at $\lambda \in \mathbb{C}$ with radius $r > 0$. We say that T has the *single valued extension property*, abbreviate it SVEP, at $\lambda \in \mathbb{C}$ if there exists $r > 0$ such that for every open subset $U \subset D(\lambda, r)$, the only analytic solution of the equation $(T - \mu)f(\mu) = 0$ is the constant function $f \equiv 0$. We define the *analytic residuum*, denoted by $\mathcal{S}(T)$, the open set where T fails to have the constant function $f \equiv 0$. An operator $T \in \mathcal{L}(X)$ said to have the single-valued extension property, when T satisfies this property at every complex number. Hence T has the SVEP if and only if $\mathcal{S}(T) = \emptyset$.

Given an arbitrary operator $T \in \mathcal{L}(X)$ and $x \in X$, the *local resolvent set* $\rho_T(x)$ of T at $x \in X$ is defined as the set of all $\lambda \in \mathbb{C}$ for which there exist an analytic X-valued function f on some open neighborhood U of λ such that $(T - \mu)f(\mu) = x$ for all $\mu \in U$. The complement of the local resolvent set is said to be the *local spectrum* and denoted by $\sigma_T(x)$. That is,

$$\sigma_T(x) = \mathbb{C} \setminus \rho_T(x).$$

It may happen that the local spectrum $\sigma_T(x)$ is the empty set.

Let U be an open subset of the complex plane and $\mathcal{O}(U, X)$ be the Fréchet algebra of all analytic X-valued functions on U endowed with uniform convergence on compact sets of U. The operator T is said to satisfy *Bishop's property* (β) at $\lambda \in \mathbb{C}$ if there exists $r > 0$ such that for every open subset $U \subset D(\lambda, r)$ and for any sequence $\{f_n\} \subset \mathcal{O}(U, X)$, if $\lim_{n\to\infty}(T - \mu)f_n(\mu) = 0$ in $\mathcal{O}(U, X)$, then $\lim_{n\to\infty} f_n(\mu) = 0$ in $\mathcal{O}(U, X)$. We denote by $\sigma_\beta(T)$ by the set where T fails to satisfy (β) and we say that T satisfies Bishop's property (β) precisely when $\sigma_\beta(T) = \emptyset$.

An operator $T \in L(X)$ is said to have the *decompositions property* (δ) if given an arbitrary open covering $\{U, V\}$ of $\mathbb{C}$ and for every $x \in X$ there exist a pair of elements $u, v \in X$ and a pair of analytic functions $f : \mathbb{C} \setminus U^- \longrightarrow X$ and $g : \mathbb{C} \setminus V^- \longrightarrow X$ such that $x = u + v$,

$$\begin{aligned} u &= (T - \lambda)f(\lambda) \quad \text{for all} \quad \lambda \in \mathbb{C} \setminus U^-, \\ v &= (T - \lambda)g(\lambda) \quad \text{for all} \quad \lambda \in \mathbb{C} \setminus V^-. \end{aligned}$$

It is well known that the properties (β) and (δ) are dual to each other[7]. That is, the operator T satisfies has Bishop's property (β) if and only if its adjoint T^* satisfies the decomposition property (δ) on the dual space, and if two properties are interchanged the corresponding statement true. Also it is well known that T is decomposable in the sense of Foias if and only if T satisfies both (β) and (δ), and hence T is decomposable if and only if T^* is decomposable. It has also been shown that an operator $T \in L(X)$ has property (β) if and only if T is similar to the restriction of a decomposable operator to one of its closed invariant subspaces and an operator $T \in L(X)$ has property (δ) if and only if T is similar to a quotient of a decomposable operator[7].

The property $(\beta)_\epsilon$ is defined in a similar way as for property (β). To be precise; let $\mathcal{E}(U, X)$ be the Fréchet algebra of all infinitely differentiable X-valued functions on $U \subset \mathbb{C}$ endowed with the topology of uniform convergence on compact subsets of U of all derivatives. The operator $T \in \mathcal{L}(X)$ is said to have *property* $(\beta)_\epsilon$ at $\lambda \in \mathbb{C}$ if there exists U a neighborhood of λ such that for each open set $O \subset U$ and for any sequence $\{f_n\}$ of X-valued functions in $\mathcal{E}(O, X)$ the convergence of $(T-\mu)f_n(\mu)$ to zero in $\mathcal{E}(O, X)$ yields to the convergence of f_n to zero in $\mathcal{E}(O, X)$. Denote by $\sigma_{(\beta)_\epsilon}(T)$ the set where T fails to satisfy $(\beta)_\epsilon$. We will say that T satisfies property $(\beta)_\epsilon$ if $\sigma_{(\beta)_\epsilon}(T) = \emptyset$.

An important generalization of normal operators to the setting of Banach spaces is the class of generalized scalar operators. We denote by $C^\infty(\mathbb{C})$ the Fréchet algebra of all infinitely differentiable complex valued functions $\varphi(z)$, $z = x_1 + ix_2$, x_1, $x_2 \in \mathbb{R}$, defined on the complex plane $\mathbb{C}$ with the topology of uniform convergence of every derivative on each compact subset of $\mathbb{C}$. That is, with the topology generated by the family of pseudo-norm

$$|\varphi|_{K,m} = \max_{|p| \le m} \sup_{z \in K} |D^p \varphi(z)|,$$

where K is an arbitrary compact subset of $\mathbb{C}$, m a non-negative integer, $p = (p_1, p_2)$, p_1, $p_2 \in \mathbb{N}$, $|p| = p_1 + p_2$ and

$$D^p \varphi = \frac{\partial^{|p|}\varphi}{\partial {x_1}^{p_1} \partial {x_2}^{p_2}}\,, \quad (z = x_1 + ix_2).$$

An operator $T \in L(X)$ on a complex Banach space X is called a *generalized scalar operator* if there exists a continuous algebra homomorphism $\Phi : C^\infty(\mathbb{C}) \to L(X)$ satisfying $\Phi(1) = I$ and $\Phi(z) = T$, where I is the identity operator on X and z denotes the identity function on $\mathbb{C}$. Such a continuous function Φ is in fact an operator valued distribution and it is called a *spectral distribution* for T. The class of generalized scalar operators were introduced by Colojoară and Foiás[5]. An important subclass of the decomposable operators is formed by the generalized scalar operators. An operator $T \in \mathcal{L}(X)$ on a complex Banach space X is said to be *subscalar* if T is similar to the restriction of a generalized scalar operator to one of its closed invariant subspaces. It is clear that every subscalar operator has property (β), since the restriction of an operator with property (β) to a closed invariant subspace certainly inherits this property. Moreover it is well known that $T \in \mathcal{L}(X)$ is subscalar if and only if T has property $(\beta)_\epsilon$ [6].

Let H be a Hilbert space over the complex plane $\mathbb{C}$ with the inner product $\langle \cdot, \cdot \rangle$. An operator $T \in \mathcal{L}(H)$ is said to be *hyponormal* if its self commutator $[T^*, T] = T^*T - TT^*$ is positive, that is,

$$\langle (T^*T - TT^*)\xi, \xi \rangle \geq 0,$$

or equivalently, $\|T^*\xi\| \leq \|T\xi\|$ for every $\xi \in H$. It is well known that hyponormal operators on a Hilbert space H is subscalar[9].

2. Common local spectral properties of intertwining Linear Operators

Theorem 1. *Let $T \in \mathcal{L}(X)$, $S \in \mathcal{L}(Y)$, $A \in \mathcal{L}(X, Y)$ and $B \in \mathcal{L}(Y, X)$ such that $SA = AT$ and $TB = BS$. Suppose that $AB = S$ and $BA = T$. Then T has the single valued extension property (resp. property (β)) at $\lambda \in \mathbb{C}$ if and only if S has the single valued extension property (resp. property (β)) at $\lambda \in \mathbb{C}$. Moreover, T has the single valued extension property (resp. property (β)) if and only if S has the single valued extension property (resp. property (β)).*

Proof. We only give the proof for property (β), the case of the single valued extension property is similar. Let $\lambda \in \mathbb{C} \setminus \sigma_\beta(S)$ and let $\{f_n\}$ be a sequence of X-valued analytic functions in a open neighborhood U of λ such that $\lim_{n\to\infty}(T - \mu)f_n(\mu) = 0$ in $\mathcal{O}(U, X)$. Then we have,

$$\begin{aligned} 0 &= \lim_{n\to\infty} A(T-\mu)f_n(\mu) \\ &= \lim_{n\to\infty} (S-\mu)Af_n(\mu) \end{aligned}$$

in $\mathcal{O}(U, Y)$. Since $\lambda \in \mathbb{C} \setminus \sigma_\beta(S)$, it follows that $\lim_{n\to\infty} Af_n(\mu) = 0$ in $\mathcal{O}(U, Y)$. Then we have,

$$\begin{aligned} 0 &= \lim_{n\to\infty} BAf_n(\mu) \\ &= \lim_{n\to\infty} Tf_n(\mu) \end{aligned}$$

in $\mathcal{O}(U, X)$. Since $\mu f_n(\mu) = Tf_n(\mu) - (T - \mu)f_n(\mu)$, we deduce that $\{\mu f_n(\mu)\}$ converges to 0 on compact sets of U. Since f_n is analytic, the maximum modulus principle implies $\{f_n\}$ converges to 0 on compact sets of U. Thus $\lambda \in \mathbb{C} \setminus \sigma_\beta(T)$. The reverse implication is obtained by the symmetry. $\square$

By passing to adjoint in Theorem 1, and by using the duality of property (β) and (δ), we obtain

Corollary 2. *Let $T \in \mathcal{L}(X)$, $S \in \mathcal{L}(Y)$, $A \in \mathcal{L}(X,Y)$ and $B \in \mathcal{L}(Y,X)$ such that $SA = AT$ and $TB = BS$. Suppose that $AB = S$ and $BA = T$. Then T has the decomposition property (δ) if and only if S has the decomposition property (δ). Moreover, T is decomposable if and only if S is decomposable.*

The following lemma is found in [8].

Lemma 3. *Let O be an open subset of $\mathbb{C}$ and $\{f_n\}$ be a sequence in $\mathcal{E}(O,X)$ such that $\{\mu f_n(\mu)\}$ converges to zero in $\mathcal{E}(O,X)$. Then $\{f_n\}$ converges to zero in $\mathcal{E}(O,X)$.*

Theorem 4. *Let $T \in \mathcal{L}(X)$, $S \in \mathcal{L}(Y)$, $A \in \mathcal{L}(X,Y)$ and $B \in \mathcal{L}(Y,X)$ such that $SA = AT$ and $TB = BS$. Suppose that $AB = S$ and $BA = T$. Then $\sigma_{(\beta)_\epsilon}(T) = \sigma_{(\beta)_\epsilon}(S)$. In particular, T is subscalar if and only if S is subscalar.*

Proof. Suppose that $\lambda \in \mathbb{C} \setminus \sigma_{(\beta)_\epsilon}(S)$. Then there exists a neighborhood O of λ such that $O \cap \sigma_{(\beta)_\epsilon}(S) = \phi$. If $\{f_n\}$ is any sequence in $\mathcal{E}(O,X)$ such that $(T-\mu)f_n(\mu)$ converges to zero in $\mathcal{E}(O,X)$, then

$$\begin{aligned} 0 &= \lim_{n\to\infty} A(T-\mu)f_n(\mu) \\ &= \lim_{n\to\infty} (S-\mu)Af_n(\mu). \end{aligned}$$

in $\mathcal{E}(O,Y)$. Since $\lambda \in \mathbb{C} \setminus \sigma_{(\beta)_\epsilon}(S)$, it follows that $\lim_{n\to\infty} Af_n(\mu) = 0$ in $\mathcal{E}(O,Y)$. Then we have,

$$\begin{aligned} 0 &= \lim_{n\to\infty} BAf_n(\mu) \\ &= \lim_{n\to\infty} Tf_n(\mu). \end{aligned}$$

Since $\mu f_n(\mu) = Tf_n(\mu) - (T-\mu)f_n(\mu)$, we deduce that $\{\mu f_n(\mu)\}$ converges to 0 in $\mathcal{E}(O,X)$. By Lemma 3, $\{f_n\}$ converges to zero in $\mathcal{E}(O,X)$. Hence $\lambda \in \mathbb{C} \setminus \sigma_{(\beta)_\epsilon}(T)$. The reverse implication is obtained by the the symmetry. $\square$

Corollary 5. *Let $A : X \longrightarrow Y$ and $B : Y \longrightarrow X$ be bounded linear operators. Then AB has property (β)(resp. (δ) or decomposable or* *subscalar) if and only if BA has property (β)(resp. (δ) or decomposable or subscalar).*

Proof. Let $S = AB$ and $T = BA$. Then we have,

$$SA = ABA = AT \quad \text{and} \quad TB = BAB = BS.$$

Hence by Theorem 1 and Theorem 4, we have this corollary. $\square$

Let $T \in \mathcal{L}(H)$ be a bounded operator on a Hilbert space H and $U|T|$ be the polar decomposition of T, where $|T| = (TT^*)^{\frac{1}{2}}$ and U is the appropriate partial isometry. The *generalized Aluthge transform* associated with T and $s, t \geq 0$ is defined by

$$T(s,t) = |T|^s U |T|^t.$$

In the case $s = t = \frac{1}{2}$, the operator

$$\widetilde{T} = |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}}$$

is called the *Aluthge transform* of T and was first considered by Aluthge[2] to extend some inequalities related to hyponormality. Let $s \leq t$, $A = |T|^r$ and $B = |T|^s U |T|^{t-r}$. Then we have,

$$AB = T(s+r, t-r) \quad \text{and} \quad BA = T(s,t).$$

Therefore, $T(s,t)$ and $T(s+r, t-r)$ almost have the same local spectral properties. In particular, T and $\widetilde{T}$ almost have the same local spectral properties.

Corollary 6. *Let $T \in \mathcal{L}(H)$, $s \geq 0$ and $0 \leq r \leq t$. Then $T(s,t)$ has the property (β) (resp. (δ) or decomposable or subscalar) if and only if $T(s+r, t-r)$ has the property (β) (resp. (δ) or decomposable or subscalar).*

Theorem 7. *Let $T \in \mathcal{L}(X)$, $S \in \mathcal{L}(Y)$, $A \in \mathcal{L}(X,Y)$ and $B \in \mathcal{L}(Y,X)$ such that $SA = AT$, $TB = BS$. Suppose that $AB = S$ and $BA = T$. Then*

(1) $\sigma_S(Ax) \subset \sigma_T(x) \subset \sigma_S(Ax) \cup \{0\}$ for every $x \in X$.
(resp. $\sigma_T(By) \subset \sigma_S(y) \subset \sigma_T(By) \cup \{0\}$ for every $y \in Y$.)

(2) In particular, if A is injective then $\sigma_T(x) = \sigma_S(Ax)$ for every $x \in X$.
(resp. if B is injective then $\sigma_S(y) = \sigma_T(By)$ for every $y \in Y$.

Proof. (1) Let $\lambda \notin \sigma_T(x)$ and $x(\mu)$ be an X-valued analytic function on a neighborhood O of λ such that $(T - \mu)x(\mu) = x$ for every $\mu \in O$.

Since $SA = AT$, we have

$$\begin{aligned} Ax &= A(T-\mu)x(\mu) \\ &= (S-\mu)Ax(\mu) \end{aligned}$$

for all $x \in O$. Since $Ax(\cdot) : O \longrightarrow Y$ is an Y-valued analytic function on a neighborhood O of λ, we have

$$\lambda \notin \sigma_S(Ax).$$

To show the second inclusion, let $\lambda \notin \sigma_S(Ax) \cup \{0\}$ and $y(\mu)$ be an Y-valued analytic function on an open neighborhood O of λ with $0 \notin O$ such that $(S-\mu)y(\mu) = Ax$ for all $\mu \in O$. Since $BA = T$ and $TB = BS$, we have

$$\begin{aligned} Tx &= BAx \\ &= B(S-\mu)y(\mu) \\ &= (T-\mu)By(\mu). \end{aligned}$$

Therefore, we have

$$T(By(\mu) - x) = \mu By(\mu).$$

Define the X-valued analytic function $z(\cdot) : O \longrightarrow X$ by

$$z(\mu) = \frac{1}{\mu}(By(\mu) - x).$$

Then it is easy to see that

$$x = (T-\mu)z(\mu) \quad \text{for every} \quad \mu \in O,$$

and hence $\lambda \notin \sigma_T(x)$.

(2) Since $\sigma_{T+\lambda I}(x) = \sigma_T(x) + \lambda$ for every $\lambda \in \mathbb{C}$ and every $x \in X$, it suffices to consider the case $\lambda = 0$. Suppose that $0 \in \sigma_S(Ax)$. Then, by (1) of this Theorem we have,

$$\sigma_T(x) = \sigma_S(Ax) \quad \text{for every} \quad x \in X.$$

Suppose that $0 \notin \sigma_S(Ax)$ and let $y(\mu)$ be an Y-valued analytic function on a neighborhood U of 0 such that $(S-\mu)y(\mu) = Ax$ for every $\mu \in U$. Since $0 \in U$, $ABy(0) = Sy(0) = Ax$. From the injectivity of A, it follows that $By(0) = x$. Moreover, we have

$$\begin{aligned} \mu y(\mu) &= Sy(\mu) - Ax \\ &= A(By(\mu) - x). \end{aligned}$$

Therefore, we have

$$y(\mu) = A[\frac{1}{\mu}(By(\mu) - x)]$$

for every $\mu \in U \setminus \{0\}$. Define the function $z(\cdot) : U \longrightarrow X$ by

$$z(\mu) = \begin{cases} \frac{1}{\mu}(By(\mu) - x) & \text{if} \quad \mu \neq 0 \\ By'(0) & \text{if} \quad \mu = 0. \end{cases}$$

Then clearly $z(\mu)$ is analytic on U and it is easily see that $A[x - (T - \mu)z(\mu)] = 0$ for every $\mu \in U$. Since A is injective, we have

$$(T - \mu)z(\mu) = x$$

for every $\mu \in U$. Hence $0 \notin \sigma_T(x)$. □

Corollary 8. *Let $T \in \mathcal{L}(X)$, $S \in \mathcal{L}(Y)$, $A \in \mathcal{L}(X, Y)$ and $B \in \mathcal{L}(Y, X)$ such that $SA = AT$, $TB = BS$. Suppose that $AB = S$ and $BA = T$. Suppose that A and B are injective. Then $\sigma(T) = \sigma(S)$.*

Proof. For an arbitrary operator $T \in \mathcal{L}(X)$, the following equality is well known

$$\sigma(T) = \bigcup_{x \in X} \sigma_T(x) \bigcup \mathcal{S}(T).$$

Suppose that A and B are injective. Then by Theorem 1 and Theorem 7 we have,

$$\begin{aligned} \sigma(T) &= \bigcup_{x \in X} \sigma_T(x) \bigcup \mathcal{S}(T) \\ &= \bigcup_{x \in X} \sigma_S(Ax) \bigcup \mathcal{S}(S) \\ &\subset \bigcup_{y \in Y} \sigma_S(y) \bigcup \mathcal{S}(S) \\ &= \sigma(S). \end{aligned}$$

The reverse inclusion is obtained by the symmetry. □

References

[1] E. Albrecht, *On decomposable operators*, Integral Equations Operator Theory, **2**(1979), 1–10.

[2] A. Aluthge, *On p-hyponormal operators for* $0 < p < 1$, Integral Equations Operator Theory, **13**(1990), 307–315.

[3] C. Benhida and E. H. Zerouali, *Local spectral theory of linear operators RS and SR*, Integral Equations and Operator Theory, **54(1)**(2006), 1–8.

[4] Lin Chen, Yan Zikun and Ruan Yingbin, *Common operator properties of operators RS and SR and p-Hyponormal operators*, Integral Equations Operator Theory, **43**(2002), 313–325.

[5] I. Colojoarvă and C. Foiás, *Theory of Generalized Spectral Operators*, Gorden and Breach, New York, 1968.

[6] J. Eschmeier and M. Putinar, *Bishop's condition (β) and rich extensions of linear operators*, Indiana Univ. Math. J., **37**(1988), 325–348.

[7] K. B. Laursen and M. M. Neumann, *An Introduction to Local Spectral Theory*, London Mathematical Society Monographs New Series 20, Oxford Science Publications, Oxford, 2000.

[8] B. Malgrange, *Ideals of Differentiable Functions*, Oxford University Press, London, 1967.

[9] M. Putinar, *Hyponormal operators are subscalar*, J. Operator Theory, **12**(1984), 385–395.

Jong-Kwang Yoo
Department of Liberal Arts and Science
Chodang University
Muan 534-701, Korea
E-mail: jkyoo@chodang.ac.kr

Hyuk Han
Department of Liberal Arts
Kongju National University
Yesan 340-702, Korea
E-mail: hyukhan@kongju.ac.kr