Honam Mathematical J. **31** (2009), No. 1, pp. 75–85

# SOLVABILITY FOR A CLASS OF THE SYSTEM OF THE NONLINEAR SUSPENSION BRIDGE EQUATIONS

Tacksun Jung and Q-Heung Choi

**Abstract.** We show the existence of the nontrivial periodic solution for a class of the system of the nonlinear suspension bridge equations with Dirichlet boundary condition and periodic condition by critical point theory and linking arguments. We investigate the geometry of the sublevel sets of the corresponding functional of the system, the topology of the sublevel sets and linking construction between two sublevel sets. Since the functional is strongly indefinite, we use the linking theorem for the strongly indefinite functional and the notion of the suitable version of the Palais-Smale condition.

## 1. Introduction

In this paper we investigate the existence of the nontrivial periodic solution for a class of the system of the nonlinear suspension bridge equations with Dirichlet boundary condition and periodic condition

$$
\begin{aligned}
(u_1)_{tt} + (u_1)_{xxxx} - F_{r_1}(x,t,u_1,\dots,u_n) &= 0 \qquad \text{in } (-\frac{\pi}{2},\frac{\pi}{2}) \times R, \quad (1.1)\\
(u_2)_{tt} + (u_2)_{xxxx} - F_{r_2}(x,t,u_1,\dots,u_n) &= 0 \qquad \text{in } (-\frac{\pi}{2},\frac{\pi}{2}) \times R,\\
\vdots \qquad \vdots \qquad \vdots \qquad & \vdots\\
(u_n)_{tt} + (u_n)_{xxxx} - F_{r_n}(x,t,u_1,\dots,u_n) &= 0 \qquad \text{in } (-\frac{\pi}{2},\frac{\pi}{2}) \times R,
\end{aligned}
$$

$$
u_i(\pm\frac{\pi}{2},t) = (u_i)_{xx}(\pm\frac{\pi}{2},t) = 0, \qquad i = 1,\dots,n,
$$

$$
u_i(x,t) = u_i(-x,t) = u_i(x,-t) = u_i(x,t+\pi), \qquad i = 1,\dots,n,
$$

Received September 27, 2008. Accepted March 7, 2009.

**2000 Mathematics Subject Classification:** 35Q72.

**Key words and phrases:** System of the nonlinear suspension bridge equations, linking theorem for strongly indefinite functional, variational linking inequality, $(P.S)^*$ condition.

where $F : (-\frac{\pi}{2}, \frac{\pi}{2}) \times R \times R^n \to R$ is a differentiable function with $F(x,t,0,\dots,0) = 0$, $F_x(x,t,0,\dots,0) = 0$ and $F_t(x,t,0,\dots,0) = 0$, and $F_{r_i}(x,t,r_1,\dots,r_n) = \frac{\partial F}{\partial r_i}(x,t,r_1,\dots,r_n)$. Let $u = (u_1,\dots,u_n)$. We assume that $F$ satisfies the following conditions:

$(F1)$ $\lim_{(u_1,\dots,u_n)\to(0,\dots,0)} \frac{F_{r_i}(x,t,u)}{|u_1|+\dots+|u_n|} = 0.$

$(F2)$ $\lim_{|u_1|+\dots+|u_n|\to\infty} \frac{F_{r_i}(x,t,u)}{|u_1|+\dots+|u_n|} = \infty$, $i = 1,\dots,n$.

$(F3)$ $u \cdot F_u(x,t,u) \ge \mu F(x,t,u)$ $\forall x,\ t,\ \mu > 2$,

$(F4)$ $|F_{r_1}(x,t,r_1,\dots,r_n)| + \dots + |F_{r_n}(x,t,r_1,\dots,r_n)| \le \gamma(|r_1|^\nu + \dots + |r_n|^\nu)$, $\forall x,\ t,\ r_1,\dots,r_n,\ \gamma > 0,\ \nu > 1,\ i = 1,\dots,n.$

As the physical model for these systems we can find crossing $n$ beams with travelling waves supported by cables with a load $f$ as follows:

$$u_{tt} + u_{xxxx} = bu^2 + f(x,t) \quad \text{in} \quad (-\frac{\pi}{2}, \frac{\pi}{2}) \times R,$$

$$u(\pm\frac{\pi}{2}, t) = u_{xx}(\pm\frac{\pi}{2}, t) = 0,$$

$$u(x,t) = u(-x,t) = u(x,-t) = u(x,t+\pi).$$

Choi and Jung ([3],[4],[5]) investigate the existence and multiplicity of solutions for the single nonlinear suspension bridge equation with Dirichlet boundary condition.

Let $u = (u_1,\dots,u_n)$ and

$$F_u(x,t,u) = (F_{u_1}(x,t,u_1,\dots,u_n),\dots,F_{u_n}(x,t,u_1,\dots,u_n))$$

and $|\cdot|$ denote the Euclidean norm in $R^n$. The system (1.1) can be rewritten by

$$\begin{cases} u_{tt} + u_{xxxx} &= F_u(x,t,u), \\ u(\pm\frac{\pi}{2}, t) &= u_{xx}(\pm\frac{\pi}{2}, t) = (0,\dots,0), \\ u(x,t+\pi) &= u(x,t) = u(-x,t) = u(x,-t), \end{cases} \tag{1.2}$$

where $u_{tt} + u_{xxxx} = ((u_1)_{tt} + (u_1)_{xxxx},\dots,(u_n)_{tt} + (u_n)_{xxxx})$.

The main result of this paper is the following:

THEOREM 1.1. *Assume that the nonlinear term $F$ satisfies the conditions $(F1) - (F4)$. Then system (1.1) has at least one nontrivial periodic solution.*

As well known the solutions of system (1.1) coincide with the critical points of the functional $I : H \to R \in C^{1,1}$ defined by

$$I(u) = \frac{1}{2}\int_\Omega [-|u_t|^2 + |u_{xx}|^2] dxdt - \int_\Omega F(x,t,u) dxdt, \tag{1.3}$$

where $u = (u_1, \ldots, u_n)$, $-|u_t|^2 + |u_{xx}|^2 = \sum_{i=1}^{n}(-|(u_i)_t|^2 + |(u_i)_{xx}|^2)$, $n \geq 1$, and the space $H$ is introduced in section 2. For the proof of Theorem 1.1 we use the variational method and critical point theory for strongly indefinite functional. In the proof we study the geometry and topology of the sublevel sets of $I$. Since the functional is strongly indefinite, we use the linking theorem for strongly indefinite functional and the notion of the suitable version of the Palais-Smale condition.

The proof of Theorem 1.1 is organized as follows: In section 2, we approach the variational method, obtain some results on the nonlinear term $F$ and recall the linking theorem for strongly indefinite functional. In section 3, we prove Theorem 1.1.

## 2. Critical Point Theory for the Strongly Indefinite Functional

The eigenvalue problem

$$v_{tt} + v_{xxxx} = \lambda v \qquad \text{in } (-\frac{\pi}{2}, \frac{\pi}{2}) \times R. \tag{2.1}$$

$$v(\pm\frac{\pi}{2}, t) = v(\pm\frac{\pi}{2}, t) = 0,$$

$$v(x,t) = v(-x,t) = v(x,-t) = v(x, t+\pi)$$

has infinitely many eigenvalues

$$\lambda_{mn} = (2n+1)^4 - 4m^2 \qquad (m, n = 0, 1, 2, \ldots)$$

and corresponding normalized eigenfunctions $\phi_{mn}$, $m$, $n > 0$, given by

$$\phi_{0n} = \frac{\sqrt{2}}{\pi}\cos(2n+1)x \qquad \text{for } \ n \geq 0,$$

$$\phi_{mn} = \frac{2}{\pi}\cos 2mt\cos(2n+1)x \qquad \text{for } \ m > 0, n \geq 0.$$

Let $\Omega$ be the square $[-\frac{\pi}{2}, \frac{\pi}{2}] \times [-\frac{\pi}{2}, \frac{\pi}{2}]$ and $E'$ the Hilbert space defined by

$$E' = \{v \in L^2(\Omega) | \ v \text{ is even in } x \text{ and } t, \ \int_Q v = 0 \ \}.$$

The set of functions $\{\phi_{mn}\}$ is an orthonormal basis in $E'$. Let us denote an element $v$, in $E'$, as

$$v = \sum h_{mn}\phi_{mn}$$

and we define a subspace $E$ of $E'$ as

$$E = \{v \in E' | \ \sum |\lambda_{mn}| h_{mn}^2 < \infty\}.$$

This is a complete normed space with a norm $\|v\| = [\sum |\lambda_{mn}|h_{mn}^2]^{\frac{1}{2}}$.

Let $H$ be the $n$ cartesian product space of $E$, i.e.,

$$H = E \times E \times \ldots \times E.$$

The norm in $H$ is given by

$$\|u\|^2 = \|P^+u\|^2 + \|P^-u\|^2, \qquad u = (u_1, \ldots, u_n)$$

where $\|P^+u\|^2 = \sum_{i=1}^n \|P^+u_i\|^2$, $\|P^-u\|^2 = \sum_{i=1}^n \|P^-u_i\|^2$.

Let $H^+$ and $H^-$ be the subspaces of $H$ on which the functional

$$u \mapsto Q(u) = \int_\Omega [-|u_t|^2 + |u_{xx}|^2]dxdt, \qquad u = (u_1, \ldots, u_n)$$

is positive definite and negative definite, respectively. Then

$$H = H^+ \oplus H^-.$$

Let $P^+$ be the projection from $H$ onto $H^+$ and $P^-$ the projection from $H$ onto $H^-$. The functional $I(u)$ can be rewritten by

$$I(u) = \frac{1}{2}\|P^+u\|^2 - \frac{1}{2}\|P^-u\|^2 - \int_\Omega F(x,t,u)dxdt = \frac{1}{2}Q(u) - \int_\Omega F(x,t,u)dxdt. \tag{2.2}$$

Let $(H_n)_n$ be a sequence of closed finite dimensional subspace of $H$ with the following assumptions: $H_n = H_n^- \oplus H_n^+$ where $H_n^+ \subset H^+$, $H_n^- \subset H^-$ for all $n$ ($H_n^+$ and $H_n^-$ are subspaces of $H$), $\dim H_n < +\infty$, $H_n \subset H_{n+1}$, $\cup_{n\in N} H_n$ is dense in $H$.

Since each eigenvalue has a finite multiplicity and $|\lambda_{mn}| \geq 1$ for all $m$, $n$, we have some properties for a single equation:

Lemma 2.1. *(i)* $\|u\| \geq \|u\|_{L^2(\Omega)}$, *where* $\|u\|_{L^2(\Omega)}$ *denotes the* $L^2$ *norm of* $u$.
*(ii)* $\|u\| = 0$ *if and only if* $\|u\|_{L^2(\Omega)} = 0$.
*(iii)* $u_{tt} + u_{xxxx} \in E$ *implies* $u \in E$.

Lemma 2.2. *Suppose that* $c$ *is not an eigenvalue of* $L$, $Lu = u_{tt} + u_{xxxx}$, *and let* $f \in E'$. *Then we have* $(L-c)^{-1}f \in E$.

*Proof.* When $n$ is fixed, we define

$$\lambda_n^+ = \inf_m\{\lambda_{mn} : \ \lambda_{mn} > 0\} = 8n^2 + 8n + 1,$$

$$\lambda_n^- = \sup_m\{\lambda_{mn} : \lambda_{mn} < 0\} = -8n^2 - 8n - 3.$$

We see that $\lambda_n^+ \to +\infty$ and $\lambda_n^- \to -\infty$ as $n \to \infty$. Hence the number of elements in the set $\{\lambda_{mn} : \ |\lambda_{mn}| < |c|\}$ is finite, where $\lambda_{mn}$ is an eigenvalue of $L$. Let

$$f = \sum h_{mn}\phi_{mn}.$$

Then

$$(L-c)^{-1}f = \sum \frac{1}{\lambda_{mn}-c} h_{mn}\phi_{mn}.$$

Hence we have the inequality

$$\|(L-c)^{-1}f\| = \sum |\lambda_{mn}| \frac{1}{(\lambda_{mn}-c)^2} h_{mn}^2 \le C \sum h_{mn}^2$$

for some $C$, which means that

$$\|(L-c)^{-1}f\| \le C_1 \|f\|_{L^2(\Omega)}, \qquad C_1 = \sqrt{C}. \quad \square$$

Now we return to the case of system. By the following Proposition 2.1, the weak solutions of system (1.1) coincide with the critical points of the associated functional $I$.

Proposition 2.1. *Assume that $F$ satisfies the conditions $(F1)$-$(F4)$. Then the functional $I(u)$ is continuous, Frèchet differentiable in $H$ with Frèchet derivative*

$$\nabla I(u)v = \int_Q [(u_{tt} + u_{xxxx}) \cdot v - F_u(u) \cdot v] dxdt. \tag{2.3}$$

*Moreover $DI \in C$. That is $I \in C^1$.*

*Proof.* For $u, v \in H$,

$$\begin{aligned}
& |I(u+v) - I(u) - \nabla I(u)v| \\
= \ & |\frac{1}{2}\int_\Omega (u_{tt} + u_{xxxx} + v_{tt} + v_{xxxx}) \cdot (u+v) dxdt - \int_\Omega F(u+v) dxdt \\
& -\frac{1}{2}\int_\Omega (u_{tt} + u_{xxxx}) \cdot u dxdt + \int_\Omega F(u) dxdt - \int_\Omega (u_{tt} + u_{xxxx} - F_u(u)) \cdot v dxdt| \\
= \ & |\frac{1}{2}\int_\Omega [(u_{tt} + u_{xxxx}) \cdot v + (v_{tt} + v_{xxxx}) \cdot u + (v_{tt} + v_{xxxx}) \cdot v] dxdt \\
& - \int_\Omega [F(u+v) - F(u)] dxdt - \int_\Omega [(u_{tt} + u_{xxxx} - F_u(u)) \cdot v] dxdt|.
\end{aligned}$$

We have

$$|\int_\Omega [F(u+v) - F(u)] dxdt| \le |\int_\Omega [F_u(u) \cdot v + o(|v|)] dxdt| = O(|v|). \tag{2.4}$$

Thus we have

$$|I(u+v) - I(u) - \nabla I(u)v| = O(|v|^2). \tag{2.5}$$

Next we prove that $I(u)$ is continuous. For $u, v \in H$,

$$\begin{aligned}
& |I(u+v) - I(u)| \\
= & \ |\frac{1}{2}\int_\Omega (u_{tt} + u_{xxxx} + v_{tt} + v_{xxxx}) \cdot (u+v) dxdt - \int_\Omega F(u+v) dxdt \\
& -\frac{1}{2}\int_\Omega (u_{tt} + u_{xxxx}) \cdot u dxdt + \int_\Omega F(u) dxdt| \\
= & \ |\frac{1}{2}\int_\Omega [(u_{tt} + u_{xxxx}) \cdot v + (v_{tt} + v_{xxxx}) \cdot u + (v_{tt} + v_{xxxx}) \cdot v] dxdt \\
& - \int_\Omega (F(u+v) - F(u)) dxdt| = O(|v|).
\end{aligned}$$

Similarly, it is easily checked that $I$ is $C^1$. □

PROPOSITION 2.2. *Assume that $F$ satisfies the conditions (F1)-(F4). Then there exist $a_0 > 0$, $b_0 \in R$ and $\mu > 2$ such that*

$$F(x,t,u) \geq a_0|u|^\mu - b_0, \qquad \forall x, t, u. \tag{2.6}$$

*Proof.* Let $u \in H$ be such that $|u|^2 \geq R^2$. Let us set $\varphi(\xi) = F(x,t,\xi u)$ for $\xi \geq 1$. Then

$$\varphi(\xi)' = u \cdot F_u(x,t,\xi u) \geq \frac{\mu}{\xi}\varphi(\xi).$$

Multiplying by $\xi^{-\mu}$, we get

$$(\xi^{-\mu}\varphi(\xi))' \geq 0,$$

hence $\varphi(\xi) \geq \varphi(1)\xi^\mu$ for $\xi \geq 1$. Thus we have

$$\begin{aligned}
F(x,t,u) &\geq F\big(x,t,\frac{R|u|}{\sqrt{|u|^2}}\big)\big(\frac{\sqrt{|u|^2}}{R}\big)^\mu \\
&\geq c_0\big(\frac{\sqrt{|u|^2}}{R}\big)^\mu \geq a_0|u|^\mu - b_0,
\end{aligned}$$

for some $a_0, b_0$, where $c_0 = \inf\{F(x,t,u) |\ (x,t) \in \Omega,\ |u|^2 = R^2\}$. □

PROPOSITION 2.3. *Assume that $F$ satisfies the conditions (F1)-(F4). Then* *if* $\|u_n\| \to +\infty$ *and*

$$\frac{\int_\Omega u_n \cdot F_u(x,t,u_n)dxdt - 2\int_\Omega F(x,t,u_n)dxdt}{\|u_n\|} \to 0,$$

*then there exist* $(u_{h_n})_n$ *and* $w \in H$ *such that*

$$\frac{\mathrm{grad}(\int_\Omega F(x,t,u_{h_n})dxdt)}{\|u_{h_n}\|} \to w \text{ and } \frac{u_{h_n}}{\|u_{h_n}\|} \rightharpoonup (0,\dots,0).$$

*Proof.* By (F3) and Proposition 2.2, for $u \in H$,

$$\int_\Omega [u \cdot F_u(x,t,u)]dxdt - 2\int_\Omega F(x,t,u)dxdt \geq$$

$$(\mu-2)\int_\Omega F(x,t,u)dxdt \geq (\mu-2)(a_0\|u\|^\mu_{L^\mu} - b_1).$$

By (F4),

$$\|\mathrm{grad}(\int_\Omega F(x,t,u)dxdt)\| \leq C'\||u|^\nu\|_{L^r}$$

for $r > 1$ and suitable constants $C'$. To get the conclusion it suffices to estimate $\|\frac{|u|^\nu}{\|u\|}\|_{L^r}$ in terms of $\frac{\|u\|^\mu_{L^\mu}}{\|u\|}$. If $\mu \geq r\nu$, then this is an consequence of *Hölder* inequality. If $\mu < r\nu$, by the standard interpolation arguments, it follows that $\|\frac{|u|^\nu}{\|u\|}\|_{L^r} \leq C\big(\frac{\|u\|^\mu_{L^\mu}}{\|u\|}\big)^{\frac{\nu}{\mu}}\|u\|^l$, where $l$ is such that $l = -1 + \frac{\nu}{\mu}$. Thus we prove the proposition. □

For finding at least one nontrivial solution we shall use the following linking theorem for strongly indefinite functional (cf. [8]).

LEMMA 2.3. *(Linking Theorem) Let* $H$ *be a real Hilbert space with* $H = H_1 \oplus H_2$ *and* $H_2 = H_1^\perp$. *We suppose that*
*(I1)* $I \in C^1(H,R)$ *satisfies* $(P.S.)^*$ *condition;*
*(I2)* $I(u) = \frac{1}{2}(Lu,u) + bu$, *where* $Lu = L_1P_1u + L_2P_2u$ *and* $L_i : H_i \to H_i$ *is bounded and selfadjoint,* $i = 1,2$;
*(I3)* $b'$ *is compact;*
*(I4) there exists a subspace* $\tilde{H} \subset H$ *and sets* $S \subset H$, $T \subset \tilde{H}$ *and constants* $\gamma > w$ *such that*
*(i)* $S \subset H_1$ *and* $I|_S \geq \gamma$,
*(ii)* $T$ *is bounded and* $I|_{\partial T} \leq w$,
*(iii)* $S$ *and* $\partial T$ *link.*
*Then* $I$ *possesses a critical value* $c \geq \gamma$.

## 3. Proof of Theorem 1.1

From now on we shall show that $I$ satisfies the linking conditions (I1)-(I4) under the assumptions (F1)-(F4). Assume that the (F1)-(F4) hold. Let us set

$$\begin{aligned}
B_r &= \{u \in H^+ | \ \|u\| \le r\} \subset H^+, \\
S_r &= \{u \in H^+ | \ \|u\| = r\} \subset H^+, \\
S(\bar\rho) &= \{u \in H^+ | \ \|u\| = \bar\rho\} \subset H^+, \\
\Delta_R(S(\bar\rho), H^-) &= \{u_1 + u_2 | \ u_1 \in H^-, u_2 \in S(\bar\rho) \subset H^+, \bar\rho > 0, \|u_1 + u_2\| \le R\}, \\
\Sigma_R(S(\bar\rho), H^-) &= \{u_1 + u_2 | \ u_1 \in H^-, u_2 \in S(\bar\rho) \subset H^+, \bar\rho > 0, \|u_1 + u_2\| = R\} \\
&\quad \cup \{u_1 | \ \|u_1\| \le R, u_2 \in S(\bar\rho)\}.
\end{aligned}$$

We have the following variation linking inequality:

Lemma 3.1. *Assume that $F$ satisfies the conditions $(F1) - (F4)$ and let $Y = H_n^+$ be any closed subspace of $H^+$. Then there exist $\bar\rho$, $R > 0$ and $r$ with $R > r$ such that*

$$\sup_{u \in \Sigma_R(S(\bar\rho), H^-)} I(u) < 0 < \inf_{\substack{u \in H^+ \\ u \in S_r}} I(u) \qquad \text{and}$$

$$\inf_{\substack{u \in H^+ \\ u \in B_r}} I(u) > -\infty, \qquad \sup_{u \in \Delta_R(S(\bar\rho), H^-)} I(u) < \infty,$$

*where $S(\bar\rho) = \{u | \ \|u\| = \bar\rho\} \subset Y$ and $S_r = \{u | \ \|u\| = r\} \subset H^+$.*

*Proof.* First we will prove that there exists $B_r$ with radius $r > 0$ and $B_r \cap S(\bar\rho) \ne \emptyset$ such that $\inf_{\substack{u \in H^+ \\ u \in S_r = \partial B_r}} I(u) > 0$. Let $u \in H^+$. Then we have that $\|P^- u\| = 0$ and

$$I(u) = \frac{1}{2}\|P^+ u\|^2 - \int_\Omega F(x, t, u) dx dt.$$

By $(F1)$ and $(F4)$, $F(x, t, u_1, \dots, u_n) \le a|u|^\beta$, $a > 0$ and $\beta > 2$. So we have

$$I(u) \ge \frac{1}{2}\|P^+ u\|^2 - a\|u\|^\beta_{L^2(\Omega)}.$$

Since $\beta > 2$, there exists a small sphere $S_r = \partial B_r$ with radius $r$ contained in $H^+$ such that for $u \in S_r$, $\inf_{\substack{u \in H^+ \\ u \in S_r}} I(u) > 0$ and $\inf_{\substack{u \in H^+ \\ u \in B_r}} I(u) > -a\|u\|^\beta_{L^2(\Omega)} > -\infty$. Next, we will prove that there exist $\bar\rho$, $R > 0$ and $r > 0$, $R > r$ such that $B_r \cap S(\bar\rho) \ne \emptyset$ and $\sup_{u \in \Sigma_R(S(\bar\rho), H^-)} I(u) < 0$.

Let $u \in H^- \oplus H^+$, $P^+u \in S(\bar{\rho}) \subset Y \subset H^+$ and $\bar{\rho}$ is a small number. Then we have

$$I(u) = \frac{1}{2}\bar{\rho}^2 - \frac{1}{2}\|P^-u\|^2 - \int_\Omega F(x,t,u)dxdt.$$

By Proposition 2.2, there exist $a_0 > 0$, $b_0 \in R$ and $\mu > 2$ such that $F(x,t,u) \geq a_0|u|^\mu - b_0$, $\forall x,t,u$. Thus we have

$$I(u) \leq \frac{1}{2}\bar{\rho}^2 - \frac{1}{2}\|P^-u\|^2 - a_0\|u\|^\mu_{L^2(\Omega)} + b_0\pi^2.$$

Since $\mu > 2$, there exist $R > \bar{\rho}$ such that if $u \in \Sigma_R(S(\bar{\rho}), H^-)$, $I(u) < 0$. Thus we have $\sup_{u \in \Sigma_R(S(\bar{\rho}),H^-)} I(u) < 0$. Moreover if $u \in \Delta_R(S(\bar{\rho}), H^-)$, then $I(u) < \frac{1}{2}\bar{\rho}^2 + b_0\pi^2 < \infty$. Thus we have $\sup_{u \in \Delta_R(S(\bar{\rho}),H^-)} I(u) < \infty$. □

Let $Y = H_n^+$ for some $n$, and denote by $P_Y$ the orthogonal projection from $H$ onto $Y$.

LEMMA 3.2. *Assume that $F$ satisfies the condition $(F1) - (F4)$. Then $I$ satisfies the $(P.S.)^*_c$ condition with respect to $(H_n)_n$ for every real number $c \in R$.*

*Proof.* Let $c \in R$ and $(h_n)$ be a sequence in $N$ such that $h_n \to +\infty$, $(u_n)_n$ be a sequence such that

$$u_n = (u_1, \ldots, u_n) \in H_{h_n}, \forall n, I(u_n) \to c, P_{H_{h_n}} \nabla I(u_n) \to 0.$$

We claim that $(u_n)_n$ is bounded. By contradiction we suppose that $\|u_n\| \to +\infty$ and set $\hat{u_n} = \frac{u_n}{\|u_n\|}$. Then

$$\langle P_{H_{h_n}} \nabla I(u_n), \hat{u_n}\rangle = \langle \nabla I(u_n), \hat{u_n}\rangle = 2\frac{I(u_n)}{\|u_n\|} -$$

$$\frac{\int_\Omega F_u(x,t,u_n)\cdot u_n dxdt - 2\int_\Omega F(x,t,u_n)dxdt}{\|u_n\|} \longrightarrow 0.$$

Hence

$$\frac{\int_\Omega F_u(x,t,u_n)\cdot u_n dxdt - 2\int_\Omega F(x,t,u_n)dxdt}{\|u_n\|} \longrightarrow 0.$$

By Proposition 2.3,

$$\frac{\operatorname{grad}\int_\Omega F(x,t,u_n)dxdt}{\|u_n\|}$$

converges and $\hat{u_n} \rightharpoonup 0$. We get

$$\frac{P_{H_{h_n}}\nabla I(u_n)}{\|u_n\|} = P_{H_{h_n}}((\hat{u_n})_{tt} + (\hat{u_n})_{xxxx}) - \frac{P_{H_{h_n}}\operatorname{grad}(\int_\Omega F(x,t,u_n)dxdt)}{\|u_n\|} \longrightarrow 0,$$

so $(P_{H_{h_n}}((\hat{u_n})_{tt}+(\hat{u_n})_{xxxx}))$ converges. Since the inverse of the operator $(\cdot)_{tt}+(\cdot)_{xxxx}$ is a compact mapping, up to subsequence, $(\hat{u_n})_n$ has a limit. Since $\hat{u_n} \rightharpoonup (0,\ldots,0)$, we get $\hat{u_n} \to (0,\ldots,0)$, which is a contradiction to the fact that $\|\hat{u_n}\| = 1$. Thus $(u_n)_n$ is bounded. We can now suppose that $u_n \rightharpoonup u$ for some $u \in H$.

We claim that $u_n$ converges to $u$ strongly. We have

$$\begin{aligned}&\langle P_{H_{h_n}} \nabla I(u_n), u_n\rangle \\ &= P_{H_{h_n}} \int_\Omega [((u_n)_{tt} + (u_n)_{xxxx}) \cdot u_n \\ &-F_u(x,t,u_n)\cdot u_n]dxdt \longrightarrow 0.\end{aligned}$$

Since $u_n$ converges to $u$ weakly, $\int_\Omega F_u(x,t,u_n)\cdot u_n dxdt \to \int_\Omega F_u(x,t,u)\cdot udxdt$ and

$$\begin{aligned}&P_{H_{h_n}} \int_\Omega [((u_n)_{tt} + (u_n)_{xxxx}) \cdot u_n]dxdt \\ &= \|P_{H^+}P_{H_{h_n}}u_n\|^2 - \|P_{H^-}P_{H_{h_n}}u_n\|^2 \\ &\longrightarrow \int_\Omega (u_{tt}+u_{xxxx})\cdot udxdt = \|P_{H^+}u\|^2 - \|P_{H^-}u\|^2.\end{aligned}$$

Thus we have $\|P_{H^+}P_{H_{h_n}}u_n\|^2 \to \|P_{H^+}u\|^2$ and $\|P_{H^-}P_{H_{h_n}}u_n\|^2 \to \|P_{H^-}u\|^2$, so we have $\|P_{H^+}P_{H_{h_n}}u_n\|^2 + \|P_{H^-}P_{H_{h_n}}u_n\|^2 \to \|P_{H^+}u\|^2 + \|P_{H^-}u\|^2$. Thus $\|P_{H_{h_n}}u_n\|^2 \to \|u\|^2$. Thus we have that $u_n$ converges to $u$ strongly with $\nabla I(u) = \lim \nabla I(u_n) = 0$. Thus we prove the lemma. □

PROOF OF THEOREM 1.1

The space $H$ can be composed of $H = H^+ \oplus H^-$. By Proposition 2.1, $I$ is $C^1(H, R^1)$, and by Lemma 3.2, $I(u)$ satisfies the $(P.S.)^*_c$ condition with respect to $(H_n)_n$, for any $c \in R$, so the condition (I1) of Lemma 2.3 is satisfied. By Proposition 2.3, the mapping $u \mapsto \text{grad}(\int_\Omega F(x,t,u)dxdt)$ is a compact mapping. If we set $Lu = u_{tt}+u_{xxxx}$ and $bu = -\int_\Omega F(x,t,u)dxdt$, $H_1 = H^-$ and $H_2 = H^+$, then $I(u)$ is of the form $I(u) = \frac{1}{2}(Lu,u) + bu$, where $Lu = L_1P_1u + L_2P_2u$ and $L_i : H_i \to H_i$ is bounded and self-adjoint, $i = 1, 2$, and $b'$ is compact, so the condition (I2) and condition (I3) of Lemma 2.3 is satisfied.

Let $S = S_r \subset H^+$ and $T = \Delta_R(S(\bar{\rho}), H^-) \subset H^- \oplus H^+$. Then $S_r$ and $\Sigma_R(S(\bar{\rho}), H^-)$ link and by Lemma 3.1, the condition (I4) of Lemma 2.3 is satisfied. Thus by Lemma 2.3, $I$ has at least one nontrivial critical value $c > 0$. Thus we prove the theorem. □

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