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A Class of Lorentzian $\alpha-$Sasakian Manifolds

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Abstract. In this study we consider $\varphi-$conformally flat, $\varphi-$conharmonically flat, $\varphi-$projectively flat and $\varphi-$concircularly flat Lorentzian $\alpha-$Sasakian manifolds. In all cases, we get the manifold will be an $\eta-$Einstein manifold.

1. Introduction

Let (M^n, g), $n = \dim M > 3$, be connected semi Riemannian manifold of class C^∞ and ∇ be its Levi-Civita connection. The Riemannian-Christoffel curvature tensor R, the Weyl conformal curvature tensor C (see [19]), the conharmonic curvature tensor K (see [9]), the projective curvature tensor P (see [19]) and the concircular curvature tensor $\tilde{C}$ (see [19]) of (M^n, g) are defined by

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, \tag{1.1}$$

$$\begin{aligned} C(X,Y)Z = R(X,Y)Z & \\ &+ \frac{1}{n-2}\Big[S(X,Z)Y - S(Y,Z)X + g(X,Z)QY - g(Y,Z)QX\Big] \\ &- \frac{\tau}{(n-1)(n-2)}\left[g(X,Z)Y - g(Y,Z)X\right], \end{aligned} \tag{1.2}$$

$$\begin{aligned} K(X,Y)Z = R(X,Y)Z & \\ &- \frac{1}{n-2}\Big[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY\Big], \end{aligned} \tag{1.3}$$

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(1.4) $$P(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}[g(Y,Z)QX - g(X,Z)QY],$$

(1.5) $$\tilde{C}(X,Y)Z = R(X,Y)Z - \frac{\tau}{n(n-1)}[g(Y,Z)X - g(X,Z)Y].$$

respectively, where Q is the Ricci operator defined by $S(X,Y) = g(QX,Y)$, S is the Ricci tensor, $\tau = \mathrm{tr(S)}$ is the scalar curvature and $X,Y,Z \in \chi(M)$, $\chi(M)$ is being Lie algebra of vector fields of M.

In [17], S. Tanno classified connected almost contact metric manifolds whose automorphism groups possess the maximum dimension. For such a manifold, the sectional curvature of plane sections containing ξ is a constant, say c. He showed that they can be divided into three classes:

(1) homogeneous normal contact Riemannian manifolds with $c > 0$,

(2) global Riemannian products of a line or a circle with a Kaehler manifold of constant holomorphic sectional curvature if $c = 0$,

(3) a warped product space $\mathbb{R} \times_f \mathbb{C}$ if $c < 0$. It is known that the manifolds of class (1) are characterized by admitting a Sasakian structure. Kenmotsu [11] characterized the differential geometric properties of the manifolds of class (3); the structure so obtained is now known as Kenmotsu structure. In general, these structures are not Sasakian [14].

In the Gray-Hervella classification of almost Hermitian manifolds [8], there appears a class, W_4, of Hermitian manifolds which are closely related to locally conformal Kaehler manifolds [10]. An almost contact metric structure on a manifold M is called a trans-Sasakian structure [13] if the product manifold $M \times \mathbb{R}$ belongs to the class W_4. The class $C_6 \oplus C_5$ ([13],[14]) coincides with the class of the trans-Sasakian structures of type (α,β). In fact, in [13], local nature of the two subclasses, namely, C_5 and C_6 structures, of trans-Sasakian structures are characterized completely.

Also, in [15], Özgür and De studied quasi-conformally flat and quasi-conformally semisymmetric Kenmotsu manifolds. Then, in [20], Yıldız and Murathan studied Lorentzian $\alpha-$Sasakian manifolds.

We note that trans-Sasakian structures of type $(0,0)$, $(0,\beta)$ and $(\alpha,0)$ are *cosymplectic* [2], $\beta-$*Kenmotsu* [11] and $\alpha-$*Sasakian* [11] respectively. In [18] it is proved that trans-Sasakian structures are generalized quasi-Sasakian. Thus, trans-Sasakian structures also provide a large class of generalized quasi-Sasakian structures.

An almost contact metric structure (φ,ξ,η,g) on M is called a *trans-Sasakian structure* [13] if $(M \times \mathbb{R}, J, G)$ belongs to the class W_4[8], where J is the almost complex structure on $M \times \mathbb{R}$ defined by

$$J(X, fd/dt) = (\varphi X - f\xi, \eta(X)d/dt),$$

for all vector fields X on M and smooth functions f on $M \times \mathbb{R}$, and G is the product metric on $M \times \mathbb{R}$. This may be expressed by the condition [3]

(1.6) $$(\nabla_X \varphi)Y = \alpha(g(X,Y)\xi - \eta(Y)X) + \beta(g(\varphi X,Y)\xi - \eta(Y)\varphi X),$$

for some smooth functions α and β on M, and we say that the trans-Sasakian structure is of type (α, β).

From (1.6) it follows that

$$\nabla_X \xi = -\alpha\varphi X + \beta(X - \eta(X)\xi), \tag{1.7}$$

$$(\nabla_X \eta)Y = -\alpha g(\varphi X, Y) + \beta g(\varphi X, \varphi Y). \tag{1.8}$$

Trans-Sasakian manifolds have been studied by De and Tripathi [7] and they obtained the following results:

$$\begin{aligned} R(X,Y)\xi = {} & (\alpha^2 - \beta^2)(\eta(Y)X - \eta(X)Y) + 2\alpha\beta(\eta(Y)\varphi X - \eta(X)\varphi Y) \\ & + (Y\alpha)\varphi X - (X\alpha)\varphi Y + (Y\beta)\varphi^2 X - (X\beta)\varphi^2 Y, \end{aligned} \tag{1.9}$$

$$\begin{aligned} R(\xi,Y)X = {} & (\alpha^2 - \beta^2)(g(X,Y)\xi - \eta(X)Y) + 2\alpha\beta(g(\varphi X,Y)\xi - \eta(X)\varphi Y) \\ & + (X\alpha)\varphi Y + g(\varphi X, Y)(\mathrm{grad}\alpha) \\ & + (X\beta)(Y - \eta(Y)\xi) - g(\varphi X, \varphi Y)(\mathrm{grad}\beta), \end{aligned} \tag{1.10}$$

$$R(\xi, X)\xi = (\alpha^2 - \beta^2 - \xi\beta)(\eta(X)\xi - X), \tag{1.11}$$

$$2\alpha\beta + \xi\alpha = 0, \tag{1.12}$$

$$S(X,\xi) = ((n-1)(\alpha^2 - \beta^2) - \xi\beta)\eta(X) - (n-2)X\beta - (\varphi X)\alpha, \tag{1.13}$$

$$Q\xi = ((n-1)(\alpha^2 - \beta^2) - \xi\beta)\xi - (n-2)\mathrm{grad}\beta + \varphi(\mathrm{grad}\alpha). \tag{1.14}$$

Definition 1.1. A trans-Sasakian structure of type (α, β) is α-Sasakian if $\beta = 0$ and α nonzero constant [10].

If $\alpha = 1$, then α-Sasakian manifold is a Sasakian manifold.

2. Lorentzian $\alpha-$Sasakian manifolds

A differentiable manifold of dimension n is called Lorentzian $\alpha-$Sasakian manifold if it admits a $(1,1)-$tensor field φ, a contravariant vector field ξ, a covariant vector field η and Lorentzian metric g which satisfy ([2], [5], [6], [7], [8], [12])

$$\eta(\xi) = -1, \tag{2.1}$$

$$\varphi^2 = I + \eta \otimes \xi, \tag{2.2}$$

$$g(\varphi X,\varphi Y)=g(X,Y)+\eta(X)\eta(Y), \tag{2.3}$$

$$g(X,\xi)=\eta(X), \tag{2.4}$$

$$\varphi\xi=0,\quad \eta(\varphi X)=0,$$

for all $X,Y\in TM$.

From (1.7) and (1.8), a Lorentzian $\alpha-$Sasakian manifold M is satisfying

$$\nabla_X\xi=-\alpha\varphi X, \tag{2.5}$$

$$(\nabla_X\eta)Y=-\alpha g(\varphi X,Y), \tag{2.6}$$

where ∇ denotes the operator of covariant differentation with respect to the Lorentzian metric g.

A Lorentzian $\alpha-$Sasakian manifold M is said to be $\eta-$Einstein if its Ricci tensor S is of the form

$$S(X,Y)=ag(X,Y)+b\eta(X)\eta(Y), \tag{2.7}$$

for any vector fields X,Y, where a,b are functions on M.

Further, from equations (1.9)-(1.14) on a Lorentzian $\alpha-$Sasakian manifold M the following relations hold:

$$R(\xi,X)Y=\alpha^2(g(X,Y)\xi+\eta(Y)X), \tag{2.8}$$

$$R(X,Y)\xi=\alpha^2(\eta(Y)X+\eta(X)Y), \tag{2.9}$$

$$R(\xi,X)\xi=\alpha^2(\eta(X)\xi+X), \tag{2.10}$$

$$S(X,\xi)=(n-1)\alpha^2\eta(X), \tag{2.11}$$

$$Q\xi=(n-1)\alpha^2\xi, \tag{2.12}$$

$$S(\xi,\xi)=-(n-1)\alpha^2, \tag{2.13}$$

$$S(\varphi X,\varphi Y)=S(X,Y)+(n-1)\alpha^2\eta(X)\eta(Y). \tag{2.14}$$

3. Main results

In this section we consider $\varphi-$conformally flat, $\varphi-$conharmonically flat, $\varphi-$projectively flat and $\varphi-$concircularly flat Lorentzian $\alpha-$Sasakian manifolds.

Let C be the Weyl conformal curvature tensor of M^n. Since at each point $p \in M^n$ the tangent space $T_p(M^n)$ can be decomposed into direct sum $T_p(M^n) = \varphi(T_p(M^n)) \oplus L(\xi_p)$, where $L(\xi_p)$ is a 1$-$dimensional linear subspace of $T_p(M^n)$ generated by ξ_p, we have map:

$$C : T_p(M^n) \times T_p(M^n) \times T_p(M^n) \to \varphi(T_p(M^n)) \oplus L(\xi_p).$$

It may natural to consider the following particular cases:

(1) $C : T_p(M^n) \times T_p(M^n) \times T_p(M^n) \to L(\xi_p)$, that is, the projection of the image of C in $\varphi(T_p(M^n))$ is zero.

(2) $C : T_p(M^n) \times T_p(M^n) \times T_p(M^n) \to \varphi(T_p(M^n))$, that is, the projection of the image of C in $L(\xi_p)$ is zero.

(3) $C : \varphi(T_p(M^n)) \times \varphi(T_p(M^n)) \times \varphi(T_p(M^n)) \to L(\xi_p)$, that is, when C is restricted to $T_p(M^n) \times \varphi(T_p(M^n)) \times \varphi(T_p(M^n))$,the projection of the image of C in $\varphi(T_p(M^n))$ is zero. This condition is equivalent to

$$\varphi^2 C(\varphi X, \varphi Y)\varphi Z = 0, \tag{3.1}$$

(see [4]).

Definition 3.1. A differentiable manifold (M^n, g), $n > 3$, satisfying the condition (3.1) is called $\varphi-$conformally flat.

The cases (1) and (2) were considered in [21] and [22], respectively. The case (3) was considered in [6] for the case M is a $K-$contact manifold and in [16] for the case M is a Lorentzian para-Sasakian manifold.

Furthermore in [1], Arslan, Murathan and Özgür studied $(k, \mu)-$contact metric manifolds satisfying (3.1). Now our aim is to find the characterization of Lorentzian $\alpha-$Sasakian manifolds satisfying the condition (3.1).

Theorem 3.2. *Let M^n be an $n-$dimensional, $(n > 3)$, $\varphi-$conformally flat Lorentzian $\alpha-$Sasakian manifold. Then M^n is an $\eta-$Einstein manifold.*

Proof. Suppose that (M^n, g), $n > 3$, is a $\varphi-$conformally flat Lorentzian $\alpha-$Sasakian manifold. It is easy to see that $\varphi^2 C(\varphi X, \varphi Y)\varphi Z = 0$ holds if and only if

$$g(C(\varphi X, \varphi Y)\varphi Z, \varphi W) = 0, \tag{3.2}$$

for any $X, Y, Z, W \in \chi(M^n)$. So by the use of (1.2) $\varphi-$conformally flat means

$$\begin{aligned} & g(R(\varphi X, \varphi Y)\varphi Z, \varphi W) \\ & \quad = \frac{1}{n-2}[g(\varphi Y, \varphi Z)S(\varphi X, \varphi W) - g(\varphi X, \varphi Z)S(\varphi Y, \varphi W) \\ & \qquad + g(\varphi X, \varphi W)S(\varphi Y, \varphi Z) - g(\varphi Y, \varphi W)S(\varphi X, \varphi Z)] \\ & \qquad - \frac{\tau}{(n-1)(n-2)}[g(\varphi Y, \varphi Z)g(\varphi X, \varphi W) - g(\varphi X, \varphi Z)g(\varphi Y, \varphi W)]. \end{aligned} \tag{3.3}$$

Let $\{e_1, ..., e_{n-1}, \xi\}$ be a local orthonormal basis of vector fields in M^n. Using that $\{\varphi e_1, ..., \varphi e_{n-1}, \xi\}$ is also a local orthonormal basis, if we put $X = W = e_i$ in (3.3) and sum up with respect to i, then

$$\begin{aligned} \sum_{i=1}^{n-1} & g(R(\varphi e_i, \varphi Y)\varphi Z, \varphi e_i) \\ &= \frac{1}{n-2}\sum_{i=1}^{n-1}[g(\varphi Y, \varphi Z)S(\varphi e_i, \varphi e_i) - g(\varphi e_i, \varphi Z)S(\varphi Y, \varphi e_i) \\ &+ g(\varphi e_i, \varphi e_i)S(\varphi Y, \varphi Z) - g(\varphi e_i, \varphi Y)S(\varphi e_i, \varphi Z)] \\ &- \frac{\tau}{(n-1)(n-2)}\sum_{i=1}^{n-1}[g(\varphi Y, \varphi Z)g(\varphi e_i, \varphi e_i) - g(\varphi e_i, \varphi Z)g(\varphi Y, \varphi e_i)]. \end{aligned} \tag{3.4}$$

It can be easily verify that

$$\sum_{i=1}^{n-1} g(R(\varphi e_i, \varphi Y)\varphi Z, \varphi e_i) = S(\varphi Y, \varphi Z) + g(\varphi Y, \varphi Z), \tag{3.5}$$

$$\sum_{i=1}^{n-1} S(\varphi e_i, \varphi e_i) = \tau - (n-1)\alpha^2, \tag{3.6}$$

$$\sum_{i=1}^{n-1} g(\varphi e_i, \varphi Z)S(\varphi Y, \varphi e_i) = S(\varphi Y, \varphi Z), \tag{3.7}$$

$$\sum_{i=1}^{n-1} g(\varphi e_i, \varphi e_i) = n-1, \tag{3.8}$$

and

$$\sum_{i=1}^{n-1} g(\varphi e_i, \varphi Z)g(\varphi Y, \varphi e_i) = g(\varphi Y, \varphi Z). \tag{3.9}$$

So by virtue of (3.5)-(3.9) the equation (3.4) can be written as

$$S(\varphi Y, \varphi Z) = [\frac{\tau}{n-1} - (n-1)\alpha^2 - (n-2)]g(\varphi Y, \varphi Z). \tag{3.10}$$

Then by making use of (2.3) and (2.14), the equation (3.10) takes the form

$$\begin{aligned} S(Y, Z) = [&\frac{\tau}{n-1} - (n-1)\alpha^2 - (n-2)]g(Y, Z) \\ &+ [\frac{\tau}{n-1} - (n-1)\alpha^2 - 2(n-2)]\eta(Y)\eta(Z), \end{aligned}$$

which implies that M^n is an $\eta-$Einstein manifold. This completes the proof of the theorem. □

Definition 3.3. A differentiable manifold (M^n, g), $n > 3$, satisfying the condition

$$\varphi^2 K(\varphi X, \varphi Y)\varphi Z = 0$$

is called $\varphi-$conharmonically flat.

Theorem 3.4. *Let M^n be an $n-$dimensional, $(n > 3)$, $\varphi-$conharmonically flat Lorentzian $\alpha-$Sasakian manifold. Then M^n is an $\eta-$Einstein manifold.*

Proof. Assume that (M^n, g), $n > 3$, is a $\varphi-$conharmonically flat Lorentzian $\alpha-$Sasakian manifold. It can be easily seen that $\varphi^2 K(\varphi X, \varphi Y)\varphi Z = 0$ holds if and only if

$$g(K(\varphi X, \varphi Y)\varphi Z, \varphi W) = 0, \tag{3.11}$$

for any $X, Y, Z, W \in \chi(M^n)$. Using (1.3) $\varphi-$conharmonically flat means

$$\begin{aligned} g(R(\varphi X, \varphi Y)\varphi Z, \varphi W) &= \frac{1}{n-2}[g(\varphi Y, \varphi Z)S(\varphi X, \varphi W) - g(\varphi X, \varphi Z)S(\varphi Y, \varphi W) \\ &+ g(\varphi X, \varphi W)S(\varphi Y, \varphi Z) - g(\varphi Y, \varphi W)S(\varphi X, \varphi Z)]. \end{aligned} \tag{3.12}$$

Similar the proof of Theorem 3.2, we can suppose that $\{e_1, ..., e_{n-1}, \xi\}$ is a local orthonormal basis of vector fields in M^n. By using the fact that $\{\varphi e_1, ..., \varphi e_{n-1}, \xi\}$ is also a local orthonormal basis, if we put $X = W = e_i$ in (3.12) and sum up with respect to i, then

$$\begin{aligned} &\sum_{i=1}^{n-1} g(R(\varphi e_i, \varphi Y)\varphi Z, \varphi e_i) \\ &= \frac{1}{n-2}\sum_{i=1}^{n-1}[g(\varphi Y, \varphi Z)S(\varphi e_i, \varphi e_i) - g(\varphi e_i, \varphi Z)S(\varphi Y, \varphi e_i) \\ &+ g(\varphi e_i, \varphi e_i)S(\varphi Y, \varphi Z) - g(\varphi e_i, \varphi Y)S(\varphi e_i, \varphi Z)]. \end{aligned} \tag{3.13}$$

So by use of the (3.5)-(3.8) the equation (3.13) turns into

$$S(\varphi Y, \varphi Z) = [\tau - (n-1)\alpha^2 - (n-2)]g(\varphi Y, \varphi Z). \tag{3.14}$$

Applying (2.3) and (2.14) into (3.14), we get

$$\begin{aligned} S(Y, Z) = [\tau - (n-1)\alpha^2 - (n-2)]g(Y, Z) \\ + [\tau - 2(n-1)\alpha^2 - (n-2)]\eta(Y)\eta(Z), \end{aligned}$$

which gives us M^n is an $\eta-$Einstein manifold. This completes the proof of the theorem. □

Definition 3.5. A differentiable manifold (M^n, g), $n > 3$, satisfying the condition

$$\varphi^2 P(\varphi X, \varphi Y)\varphi Z = 0$$

is called $\varphi-$projectively flat.

Theorem 3.6. *Let M^n be an $n-$dimensional, $(n > 3)$, $\varphi-$projectively flat Lorentzian $\alpha-$Sasakian manifold. Then M^n is an $\eta-$Einstein manifold.*

Proof. Assume that (M^n, g), $n > 3$, is a $\varphi-$projectively flat Lorentzian $\alpha-$Sasakian manifold. It can be easily seen that $\varphi^2 P(\varphi X, \varphi Y)\varphi Z = 0$ holds if and only if

$$g(P(\varphi X, \varphi Y)\varphi Z, \varphi W) = 0, \tag{3.15}$$

for any $X, Y, Z, W \in \chi(M^n)$. Using (1.4) $\varphi-$projectively flat means

$$\begin{aligned} &g(R(\varphi X, \varphi Y)\varphi Z, \varphi W) \\ &= \frac{1}{n-1}[g(\varphi Y, \varphi Z)S(\varphi X, \varphi W) - g(\varphi X, \varphi Z)S(\varphi Y, \varphi W)]. \end{aligned} \tag{3.16}$$

Similar the proofs of Theorem 3.2 and Theorem 3.4, we can suppose that $\{e_1, ..., e_{n-1}, \xi\}$ is a local orthonormal basis of vector fields in M^n. By using the fact that $\{\varphi e_1, ..., \varphi e_{n-1}, \xi\}$ is also a local orthonormal basis, if we put $X = W = e_i$ in (3.16) and sum up with respect to i, then

$$\begin{aligned} &\sum_{i=1}^{n-1} g(R(\varphi e_i, \varphi Y)\varphi Z, \varphi e_i) \\ &= \frac{1}{n-1}\sum_{i=1}^{n-1}[g(\varphi Y, \varphi Z)S(\varphi e_i, \varphi e_i) - g(\varphi e_i, \varphi Z)S(\varphi Y, \varphi e_i)]. \end{aligned} \tag{3.17}$$

By use of the (3.5)-(3.8) the equation (3.17) turns into

$$S(\varphi Y, \varphi Z) = (\frac{\tau - (n-1)\alpha^2 - (n-1)}{n})g(\varphi Y, \varphi Z). \tag{3.18}$$

Hence by virtue of (2.3) and (2.14),we obtain

$$\begin{aligned} S(Y, Z) = &(\frac{\tau - (n-1)\alpha^2 - (n-1)}{n})g(Y, Z) \\ &+ [\frac{\tau - (n-1)\alpha^2 - (n-1)}{n} - (n-1)\alpha^2]\eta(Y)\eta(Z), \end{aligned}$$

which gives us M^n is an $\eta-$Einstein manifold. This completes the proof of the theorem. □

Definition 3.7. A differentiable manifold (M^n, g), $n > 3$, satisfying the condition

$$\varphi^2 \tilde{C}(\varphi X, \varphi Y)\varphi Z = 0$$

is called $\varphi-$concircularly flat.

Theorem 3.8. *Let M^n be an $n-$dimensional, $(n > 3)$, $\varphi-$concircularly flat Lorentzian $\alpha-$Sasakian manifold. Then M^n is an $\eta-$Einstein manifold.*

Proof. Assume that $(M^n, g), n > 3$, is a $\varphi-$concircularly flat Lorentzian $\alpha-$Sasakian manifold. It can be easily seen that $\varphi^2\tilde{C}(\varphi X, \varphi Y)\varphi Z = 0$ holds if and only if

$$g(\tilde{C}(\varphi X, \varphi Y)\varphi Z, \varphi W = 0, \tag{3.19}$$

for any $X, Y, Z, W \in \chi(M^n)$. Using (1.5) $\varphi-$concircularly flat means

$$\begin{aligned} &g(R(\varphi X, \varphi Y)\varphi Z, \varphi W) \\ &\quad = \frac{\tau}{n(n-1)}[g(\varphi X, \varphi W)g(\varphi Y, \varphi Z) - g(\varphi Y, \varphi W)g(\varphi X, \varphi Z)]. \end{aligned} \tag{3.20}$$

Similar the proof of above Theorems, we can suppose that $\{e_1, ..., e_{n-1}, \xi\}$ is a local orthonormal basis of vector fields in M^n. By using the fact that $\{\varphi e_1, ..., \varphi e_{n-1}, \xi\}$ is also a local orthonormal basis, if we put $X = W = e_i$ in (3.20) and sum up with respect to i, then

$$\begin{aligned} &\sum_{i=1}^{n-1} g(R(\varphi e_i, \varphi Y)\varphi Z, \varphi e_i) \\ &\quad = \frac{\tau}{n(n-1)}\sum_{i=1}^{n-1}[g(\varphi e_i, \varphi e_i)g(\varphi Y, \varphi Z) - g(\varphi Y, \varphi e_i)g(\varphi e_i, \varphi Z)]. \end{aligned} \tag{3.21}$$

So by use of the (3.5)-(3.8) the equation (3.21) turns into

$$S(\varphi Y, \varphi Z) = (\frac{(\tau - n)(n-1)}{\tau + n(n-1)})g(\varphi Y, \varphi Z). \tag{3.22}$$

Applying (2.3) and (2.14) into (3.22), we get

$$S(Y, Z) = (\frac{(\tau - n)(n-1)}{\tau + n(n-1)})g(Y, Z) + (\frac{(\tau - n)(n-1)}{\tau + n(n-1)} - (n-1)\alpha^2)\eta(Y)\eta(Z),$$

which gives us M^n is an $\eta-$Einstein manifold. This completes the proof of the theorem. □

References

[1] Arslan K., Murathan C. and Özgür C., *On φ- Conformally flat contact metric manifolds*, Balkan J. Geom. Appl. (BJGA), **5(2)**(2000), 1-7.

[2] Blair D. E., Contact manifolds in Riemannian geometry, Lectures Notes in Mathematics, Springer-Verlag, Berlin, **509**(1976), 146.

[3] Blair D. E. and Oubina J. A., *Conformal and related changes of metric on the product of two almost contact metric manifolds*, Publications Matematiques, **34**(1990), 199-207.

[4] Cabrerizo J.L., Fernandez L.M., Fernandez M. and Zhen G., *The structure of a class of K-contact manifolds*, Acta Math., Hungar, **82(4)**(1999), 331-340.

[5] Chaki M. C., and Gupta B., *On Conformally Symmetric Spaces*, Indian J. Math., **5**(1963), 113-123.

[6] Chinea D. and Gonzales C., *Curvature relations in trans-Sasakian manifolds*, Proceedings of the XIIth Portuguese-Spanish Conference on Mathematics, Vol. II(Portuguese) (Braga, 1987), 564-571, Univ. Minho, Braga, 1987.

[7] De U. C. and Tripathi M. M., *Ricci Tensor in 3-dimensional Trans-Sasakian Manifolds*, Kyungpook Math. J., **43**(2003), 247-255.

[8] Gray A. and Hervella L. M., *The sixteen classes of almost Hermitian manifolds and their linear invariants*, Ann. Mat. Pura Appl., **123(4)**(1980), 35-58.

[9] Ishii Y., *On Conharmonic transformations,* Tensor N.S. **7**(1957), 73-80.

[10] Janssens D. and Vanhecke L., *Almost contact structures and curvature tensors*, Kodai Math. J., **4**(1981), 1-27.

[11] Kenmotsu K., *A class of almost contact Riemannian manifolds*, Tohoku Math. J., **24**(1972), 93-103.

[12] Kim J. S., Prasad R. and Tripathi M. M., *On generalized Ricci-recurrent trans-Sasakian manifolds*, J. Korean Math. Soc., **39(6)**(2002), 953-961.

[13] Marrero J. C., *The local structure of trans-Sasakian manifolds*, Ann. Mat. Pura Appl., **162(4)**(1992), 77-86. J.

[14] Marrero J. C. and Chinea D., *On trans-Sasakian manifolds*, Proceedings of the XIVth Spanish-Portuguese Conference on Mathematics, Vol. I-III (Spanish) (Puerto de la Cruz, 1989), 655-659, Univ. La Laguna, La Laguna, 1990.

[15] Özgür C. and De U. C., *On the quasi-conformal curvature tensor of a Kenmotsu manifold*, Mathematica Pannonica, 17/2, (2006), 221-228.

[16] Özgür C., *φ-conformally flat Lorentzian para-Sasakian manifolds*, Radovi Matematicki, Vol. 12, (2003), 99-106.

[17] Tanno S., *The automorphism groups of almost contact Riemannian manifolds*, Tohoku Math. J., **21**(1969), 21-38.

[18] Tripathi M. M., *Trans-Sasakian manifolds are generalized quasi-Sasakian*, Nepali Math. Sci. Rep., **18(1-2)**(1999-2000), 11-14.

[19] Yano K. and Kon M., Structures on Manifolds, Series in Pure Math. Vol 3. World Sci., (1984).

[20] Yıldız A. and Murathan C., *On Lorentzian α-Sasakian manifolds*, Kyungpook Math. J., **45**(2005), 95-103.

[21] Zhen G., *On conformal symmetric K-contact manifolds,* Chinese Quart. J. of Math., **7** (1992), 5-10.

[22] Zhen G., Cabrerizo J.L., Fernandez L.M. and Fernandez M., *On ξ-conformally flat contact metric manifolds*, Indian J. Pure Appl. Math., **28**(1997), 725-734.