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# Some Further Results on Entire Functions That Share Fixed-Points

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Abstract. In this paper, we study the uniqueness problem on entire functions sharing fixed points and prove some theorems which are related to one famous problem of Hayman.

## 1. Introduction and main results

In this paper, a meromorphic function will mean meromorphic in the whole complex plane. We say that two meromorphic functions $f(z)$ and $g(z)$ share a small function $a(z)$ IM (ignoring multiplicities) when $f-a$ and $g-a$ have the same zeros. If $f(z)$ and $g(z)$ have the same zeros with the same multiplicities, then we say that $f(z)$ and $g(z)$ share $a(z)$ CM (counting multiplicities). It is assumed that the reader is familiar with the standard symbols and fundamental results of Nevanlinna Theory, as found in [5], [8].

Let $p$ be a positive integer and $a \in \mathbb{C}$ . We denote by $N_p(r, \frac{1}{f-a})$ the counting function of the zeros of $f-a$ where an $m$-fold zero is counted $m$ times if $m \leq p$ and $p$ times if $m > p$. We denote by $\overline{N}_L(r, \frac{1}{f-1})$ the counting function for 1-points of both $f(z)$ and $g(z)$ about which $f(z)$ has larger multiplicity than $g(z)$, with multiplicity not being counted. We say that a finite value $z_0$ is a fixed point of $f(z)$ if $f(z_0) = z_0$, and we define

$$E_f = \{z \in \mathbb{C} : f(z) = z, \text{ counting multiplicities}\}.$$

In order to answer one famous question of Hayman [4], Fang and Hua [1] and Yang and Hua [7] obtained the following result.

**Theorem A.** *Let $f(z)$ and $g(z)$ be two nonconstant entire functions, and let $n \geq 6$*

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*be a positive integer. If* $f^n f'$ *and* $g^n g'$ *share 1 CM, then either* $f(z) = c_1 e^{cz}, g(z) = c_2 e^{-cz}$*, where* $c_1, c_2$ *and* $c$ *are three constants satisfying* $(c_1 c_2)^{n+1} c^2 = -1$ *or* $f = tg$ *for a constant* $t$ *such that* $t^{n+1} = 1$.

In [3], Fang also got the following results.

**Theorem B.** *Let* $f(z)$ *and* $g(z)$ *be two nonconstant entire functions, and let* $n$, $k$ *be two positive integers with* $n > 2k + 4$. *If* $(f^n)^{(k)}$ *and* $(g^n)^{(k)}$ *share 1 CM, then either* $f(z) = c_1 e^{cz}, g(z) = c_2 e^{-cz}$*, where* $c_1, c_2$ *and* $c$ *are three constants satisfying* $(-1)^k (c_1 c_2)^n (nc)^{2k} = 1$ *or* $f = tg$ *for a constant tsuch that* $t^n = 1$.

**Theorem C.** *Let* $f(z)$ *and* $g(z)$ *be two nonconstant entire functions, and let* $n, k$ *be two positive integers with* $n \geq 2k + 8$. *If* $(f^n(f-1))^{(k)}$ *and* $(g^n(g-1))^{(k)}$ *share 1 CM, then* $f = g$.

Corresponding to the above results, some authors considered the uniqueness problems of entire functions that have fixed points, see Fang and Qiu [2], Lin and Yi [6]. Recently, Zhang [11] proved the following results, which generalized some previous results.

**Theorem D.** *Let* $f(z)$ *and* $g(z)$ *be two nonconstant entire functions, and let* $n, k$ *be two positive integers with* $n > 2k + 4$. *If* $E_{(f^{(n)})^k} = E_{(g^{(n)})^k}$ *, then either* $(1) k = 1, f(z) = c_1 e^{cz^2}, g(z) = c_2 e^{-cz^2}$*, where* $c_1$, $c_2$ *and* $c$ *are three constants satisfying* $4(c_1 c_2)^n (nc)^2 = -1$ *or (2)* $f = tg$ *for a constant* $t$ *such that* $t^n = 1$.

**Theorem E.** *Let* $f(z)$ *and* $g(z)$ *be two nonconstant entire functions, and let* $n, k$ *be two positive integers with* $n \geq 2k + 6$. *If* $E_{(f^{(n)}(f-1))^k} = E_{(g^{(n)}(g-1))^k}$*, then* $f = g$.

Now it is natural to ask whether the CM sharing value can be replaced by the IM sharing value in Theorems D and E? In this paper, we give a positive answer to the above question by proving the following theorems.

**Theorem 1.** *Let* $f(z)$ *and* $g(z)$ *be two transcendental entire functions and let* $n$, $k$ *be two positive integers with* $n > 5k + 7$. *If* $(f^n)^{(k)}$ *and* $(g^n)^{(k)}$ *share* $z$ *IM, then* $f(z) = b_1 e^{bz^2}, g(z) = b_2 e^{-bz^2}$ *for three constants* $b_1$, $b_2$ *and* $b$ *that satisfy* $4(b_1 b_2)^n (nb)^2 = -1$ *or* $f = tg$ *for a constant* $t$ *such that* $t^n = 1$.

**Theorem 2.** *Let* $f(z)$ *and* $g(z)$ *be two transcendental entire functions and let* $n$, $k$ *be two positive integers with* $n > 5k + 11$. *If* $[f^n(f-1)]^{(k)}$ *and* $[g^n(g-1)]^{(k)}$ *share* $z$ *IM, then* $f = g$.

## 2. Some Lemmas

**Lemma 1**([8]). *Let* $f$ *be a nonconstant meromorphic function, and* $a_0$, $a_1$, $a_2$, $\cdots a_n$ *be small functions of* $f$ *such that* $a_n \not\equiv 0$. *Then*

$$T(r, a_n f^n + a_{n-1} f^{n-1} + \cdots a_1 f + a_0) = nT(r, f) + S(r, f).$$

**Lemma 2**([10]). *Let $f$ be a nonconstant meromorphic function, and $p$, $k$ be positive integers. Then*

$$N_p(r, \frac{1}{f^{(k)}}) \le T(r, f^{(k)}) - T(r, f) + N_{p+k}(r, \frac{1}{f}) + S(r, f), \tag{2.1}$$

$$N_p(r, \frac{1}{f^{(k)}}) \le k\overline{N}(r, f) + N_{p+k}(r, \frac{1}{f}) + S(r, f). \tag{2.2}$$

**Lemma 3**([9]). *Let*

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right), \tag{2.3}$$

*where $F$ and $G$ are two nonconstant meromorphic functions. If $F$ and $G$ share 1 IM and $H \not\equiv 0$, then*

$$\begin{aligned} T(r, F) + T(r, G) \le{}& 2(N_2(r, \frac{1}{F}) + N_2(r, \frac{1}{G}) + N_2(r, F) + N_2(r, G)) \\ &+ 3(\overline{N}_L(r, \frac{1}{F-1}) + \overline{N}_L(r, \frac{1}{G-1})) + S(r, F) + S(r, G). \end{aligned} \tag{2.4}$$

**Lemma 4**([11]). *Suppose that $f$ and $g$ are two nonconstant entire functions, and $n, k$ are two positive integers, and denote $F = (f^n)^{(k)}$ and $G = (g^n)^{(k)}$. If there exist two non-zero constants $a_1$ and $a_2$ such that $\overline{N}(r, \frac{1}{F-a_1}) = \overline{N}(r, \frac{1}{G})$ and $\overline{N}(r, \frac{1}{G-a_2}) = \overline{N}(r, \frac{1}{F})$, then $n \le 2k + 4$.*

**Lemma 5**([11]). *Suppose that $F$ and $G$ are given as in Lemma 4, if $n > 2k$ and $F = G$, then $f = tg$ for a constant $t$ such that $t^n = 1$.*

From the proof of Proposition 1 in [2] and Theorem 4 in [11], we get the following Lemma.

**Lemma 6**([2], [11]). *Suppose that $F$ and $G$ are given as in Lemma 4. If $n > 2k+4$ and $FG = z^2$, then $f(z) = b_1e^{bz^2}, g(z) = b_2e^{-bz^2}$ for three constants $b_1$, $b_2$ and $b$ that satisfy $4(b_1b_2)^n(nb)^2 = -1$.*

**Lemma 7**([11]). *Suppose that $f$ and $g$ are two nonconstant entire functions, and $n, k$ are two positive integers, and denote $F_1 = (f^n(f-1))^{(k)}$ and $G_1 = (g^n(g-1))^{(k)}$. If there exist two non-zero constants $a_1$ and $a_2$ such that $\overline{N}(r, \frac{1}{F-a_1}) = \overline{N}(r, \frac{1}{G})$ and $\overline{N}(r, \frac{1}{G-a_2}) = \overline{N}(r, \frac{1}{F})$, then $n \le 2k + 3$.*

**Lemma 8**([11]). *Suppose that $F_1$ and $G_1$ are given as in Lemma 7. If $n > 2k + 1$ and $F = G$, then $f = g$.*

**Lemma 9**([11]). *Suppose that $f$ is a transcendental meromorphic function with finite number of poles, $g$ is a transcendental entire function, and $n$, $k$ are two* *positive integers. If* $(f^n(f-1))^{(k)}(g^n(g-1))^{(k)} = z^2$*, then* $n \le k+2$.

## 3. Proof of Theorems

*Proof of Theorem 1.* Let

$$F = \frac{(f^n(z))^{(k)}}{z}, \quad G = \frac{(g^n(z))^{(k)}}{z}. \tag{3.1}$$

Then $F$ and $G$ are transcendental meromorphic functions that share 1 IM. Let $H$ be given by (2.3). If $H \not\equiv 0$, by Lemma 3, we know that (2.4) holds. From Lemma 1 and (2.1), we have

$$\begin{aligned} N_2(r, \frac{1}{F}) &\le N_2(r, \frac{1}{(f^n(z))^{(k)}}) + S(r,f) \\ &\le T(r, (f^n(z))^{(k)}) - nT(r,f) + N_{k+2}(r, \frac{1}{f^n(z)}) + S(r,f) \\ &= T(r,F) - nT(r,f) + N_{k+2}(r, \frac{1}{f^n(z)}) + S(r,f). \end{aligned} \tag{3.2}$$

Similarly, we have

$$N_2(r, \frac{1}{G}) \le T(r,G) - nT(r,g) + N_{k+2}(r, \frac{1}{g^n(z)}) + S(r,g). \tag{3.3}$$

From (3.2) and (3.3), we obtain

$$N_2(r, \frac{1}{F}) \le N_{k+2}(r, \frac{1}{f^n(z)}) + S(r,f), \tag{3.4}$$

and

$$N_2(r, \frac{1}{G}) \le N_{k+2}(r, \frac{1}{g^n(z)}) + S(r,g). \tag{3.5}$$

Again, from (3.2) and (3.3), we have

$$\begin{aligned} n(T(r,f) + T(r,g)) \le{}& T(r,F) + T(r,G) - N_2(r, \frac{1}{F}) - N_2(r, \frac{1}{G}) \\ &+ N_{k+2}(r, \frac{1}{f^n(z)}) + N_{k+2}(r, \frac{1}{g^n(z)}) + S(r,f) + S(r,g). \end{aligned}$$

Noting that

$$\overline{N}(r, \frac{1}{F}) \le \overline{N}(r, \frac{1}{(f^n)^{(k)}}) + S(r,f).$$

Combining with (2.2), we obtain

$$\begin{aligned} \overline{N}(r, \frac{1}{F}) &\le N_1(r, \frac{1}{(f^n)^{(k)}}) + S(r,f) \le k\overline{N}(r, f^n) + N_{k+1}(r, \frac{1}{f^n}) \\ &+ S(r,f) \le k\overline{N}(r,f) + (k+1)\overline{N}(r, \frac{1}{f}) + S(r,f). \end{aligned} \tag{3.6}$$

By the definition of $\overline{N}_L(r, \frac{1}{F-1})$ and (3.6), we get

$$\begin{aligned}\overline{N}_L(r, \frac{1}{F-1}) &\le N(r, \frac{1}{F-1}) - \overline{N}(r, \frac{1}{F-1}) \le N(r, \frac{F}{F'}) + S(r,f)\\ &\le N(r, \frac{F'}{F}) + S(r,f) \le \overline{N}(r, \frac{1}{F}) + \overline{N}(r,F) + S(r,f)\\ &\le (k+1)(\overline{N}(r,f) + \overline{N}(r, \frac{1}{f})) + S(r,f). \qquad (3.7)\end{aligned}$$

Similarly, we have

$$\overline{N}_L(r, \frac{1}{G-1}) \le (k+1)(\overline{N}(r,g) + \overline{N}(r, \frac{1}{g})) + S(r,g). \qquad (3.8)$$

From (2.4), (3.4)–(3.8), we obtain

$$\begin{aligned}n(T(r,f) + T(r,g)) \le 2(N_{k+2}(r, \frac{1}{f^n}) + N_{k+2}(r, \frac{1}{g^n}))\\ + 3(\overline{N}_L(r, \frac{1}{F-1}) + \overline{N}_L(r, \frac{1}{G-1})) + S(r,f) + S(r,g).\end{aligned}$$

i.e.,

$$\begin{aligned}n(T(r,f) + T(r,g)) \le (k+2)(\overline{N}(r, \frac{1}{f}) + \overline{N}(r, \frac{1}{g}))\\ + 3((k+1)\overline{N}(r, \frac{1}{f}) + (k+1)\overline{N}(r, \frac{1}{g})) + S(r,f) + S(r,g).\end{aligned}$$

That is

$$(n - (5k+7))(T(r,f) + T(r,g)) \le S(r,f) + S(r,g),$$

which is a contradiction as $n > 5k+7$. Therefore $H \equiv 0$. Integrating twice, we get from (2.3) that

$$\frac{1}{F-1} = \frac{A}{G-1} + B, \qquad (3.9)$$

where $A(\neq 0)$ and $B$ are constants. From (3.9), we have

$$F = \frac{(B+1)G + (A-B-1)}{BG + (A-B)}, \quad G = \frac{(B-A)F + (A-B-1)}{BF - (B+1)}. \qquad (3.10)$$

We consider the following three cases.

Case 1. Suppose that $B \neq 0, -1$. From (3.10) we have $\overline{N}(r, \frac{1}{F - \frac{B+1}{B}}) = \overline{N}(r, G)$. From the second fundamental theorem, we have

$$\begin{aligned}T(r,F) &\le \overline{N}(r, \frac{1}{F}) + \overline{N}(r, \frac{1}{F - \frac{B+1}{B}}) + S(r,F)\\ &= \overline{N}(r, \frac{1}{F}) + \overline{N}(r,G) + S(r,F) \le \overline{N}(r, \frac{1}{F}) + S(r,F). \qquad (3.11)\end{aligned}$$

By (2.1) and (3.11) , we obtain

$$T(r,F)\le N_1(r,\frac{1}{F})+S(r,f)$$
$$\le T(r,F)-nT(r,f)+N_{k+1}(r,\frac{1}{f^n(z)})+S(r,f).$$

Hence

$$nT(r,f)\le (k+1)\overline{N}(r,\frac{1}{f})+S(r,f)$$
$$\le (k+1)T(r,f)+S(r,f),$$

which contradicts $n>5k+7$.

Case 2. Suppose that $B=0$. From (3.10) we have

$$F=\frac{G+(A-1)}{A},\quad G=AF-(A-1). \tag{3.12}$$

If $A\neq 1$, we get from (3.12) that $\overline{N}(r,\frac{1}{F-\frac{A-1}{A}})=\overline{N}(r,\frac{1}{G})$ and $\overline{N}(r,\frac{1}{F})=\overline{N}(r,\frac{1}{G+(A-1)})$. By Lemma 4, we have $n\le 2k+4$. This contradicts the assumption that $n>5k+7$. Thus $A=1$ and $F=G$, that is, $(f^n)^{(k)}=(g^n)^{(k)}$. By Lemma 5, we get $f=tg$ for a constant $t$ such that $t^n=1$.

Case 3. Suppose that $B=-1$. From (3.10) we obtain

$$F=\frac{A}{-G+(A+1)},\quad G=\frac{(A+1)F-A}{F}. \tag{3.13}$$

If $A\neq -1$, we obtain from (3.13) that $\overline{N}(r,\frac{1}{F-\frac{A}{A+1}})=\overline{N}(r,\frac{1}{G})$, $\overline{N}(r,F)=\overline{N}(r,\frac{1}{G-A-1})$. By the same reasoning mentioned in Case 1 and Case 2, we get a contradiction. Hence $A=-1$, from (3.13), we have $FG=1$. That is $(f^n)^{(k)}(g^n)^{(k)}=z^2$, by Lemma 6, we obtain $f(z)=b_1e^{bz^2}, g(z)=b_2e^{-bz^2}$ for three constants $b_1$, $b_2$ and $b$ that satisfy $4(b_1b_2)^n(nb)^2=-1$. This completes the proof of Theorem 1. □

*Proof of Theorem 2.* Let

$$F=\frac{(f^n(f-1))^{(k)}}{z},\quad G=\frac{(g^n(g-1))^{(k)}}{z}. \tag{3.14}$$

Then $F$ and $G$ are transcendental meromorphic functions that share 1 IM. Let $H$ be given by (2.3). In the same manner as Theorem 1, we get

$$\overline{N}(r,\frac{1}{F})\le N_1(r,\frac{1}{(f^n(f-1))^{(k)}})+S(r,f)$$
$$\le k\overline{N}(r,f^n(f-1))+N_{k+1}(r,\frac{1}{f^n(f-1)})$$
$$+S(r,f)\le k\overline{N}(r,f)+(k+1)\overline{N}(r,\frac{1}{f})+N(r,\frac{1}{f-1})+S(r,f). \tag{3.15}$$

By the definition of $\overline{N}_L(r, \frac{1}{F-1})$ and (3.15), we have

$$\overline{N}_L(r, \frac{1}{F-1}) \leq (k+1)(\overline{N}(r,f) + \overline{N}(r, \frac{1}{f})) + N(r, \frac{1}{f-1}) + S(r,f). \tag{3.16}$$

Using the same argument in the proof of Theorem 1, we get $H \equiv 0$ and $F = G$ or $FG = 1$ when $n > 5k + 11$.
If $F = G$, then $(f^n(f-1))^{(k)} = (g^n(g-1))^{(k)}$. From Lemma 8, we obtain $f = g$.
If $FG = 1$, then $(f^n(f-1))^{(k)}(g^n(g-1))^{(k)} = z^2$. From Lemma 9, we get a contradiction. Theorem 2 follows. □

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