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# Extinction and Permanence of a Holling I Type Impulsive Predator-prey Model

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Abstract. We investigate the dynamical properties of a Holling type I predator-prey model, which harvests both prey and predator and stock predator impulsively. By using the Floquet theory and small amplitude perturbation method we prove that there exists a stable prey-extermination solution when the impulsive period is less than some critical value, which implies that the model could be extinct under some conditions. Moreover, we give a sufficient condition for the permanence of the model.

## 1. Introduction

One important component of the predator-prey relationship is the predator's rate of feeding on prey, i.e., the so-called predator's functional response. Functional response refers to the change in the density of prey attached per unit time per predator as the prey density changes. Based on experiments, Holling [6] gave three different kinds of functional response for different kinds of species to model the phenomena of predation. The basic model we considered is based on the following predator-prey model with Holling type I.

$$\begin{aligned} x'(t) &= ax(t)(1-\frac{x(t)}{K}) - \phi(x(t))y(t), \\ y'(t) &= -Dy(t) + b\phi(x(t))y(t), \\ (x(0^+), y(0^+)) &= (x_0, y_0) = \mathbf{x}_0, \end{aligned} \tag{1.1}$$

with

$$\phi(x(t)) = \begin{cases} cx(t), x \le \nu, \\ c\nu, x > \nu, \end{cases} \tag{1.2}$$

where $x(t), y(t)$ denote, respectively, the prey and predator densities. Here, $a, b, D, K, \nu$ are positive constants and $K$ represents the environmental capacity

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and $a$ a intrinsic birth rate, $D$ denotes the death rate of the predator, $b$ is the rate of conversion of a consumed prey to a predator, $\phi(x(t))$ is the capture rate of prey per predator or functional response of a predator and $\nu$ is a constant characterizing the threshold of prey concentration above which the predation rate is constant and under which the predation rate is similar to the Lotka-Volterra one.

The theory of impulsive differential equations is much richer than the corresponding theory of differential equations without impulse effects [1, 2, 3, 5, 7]. Thus, with the idea of periodic forcing and impulsive perturbations, we considered the following predator-prey model.

$$\begin{cases} \left.\begin{array}{l} x'(t) = ax(t)(1-\frac{x(t)}{K}) - \phi(x(t))y(t), \\ y'(t) = -Dy(t) + b\phi(x(t))y(t), \end{array}\right\} t \neq nT, \\ \left.\begin{array}{l} \Delta x(t) = -p_1 x(t), \\ \Delta y(t) = -p_2 y(t) + q, \end{array}\right\} t = nT, \\ (x(0^+), y(0^+)) = (x_0, y_0) = \mathbf{x}_0, \end{cases} \tag{1.3}$$

where $\Delta x(t) = x(t^+) - x(t), \Delta y(t) = y(t^+) - y(t)$ and $0 \le p_1, p_2 < 1$. $T$ is the period of the impulsive immigration or stock of the predator, $q$ is the size of immigration or stock of the predator.

## 2. Preliminaries

Firstly, we give some notations, definitions and Lemmas which will be useful for our main results.

Let $\mathbb{R}_+ = [0, \infty)$ and $\mathbb{R}_+^2 = \{\mathbf{x} = (x(t), y(t)) \in \mathbb{R}^2 : x(t), y(t) \ge 0\}$. Denote $\mathbb{N}$ the set of all of nonnegative integers and $f = (f_1, f_2)^T$ the right hand of (1.3). Let $V : \mathbb{R}_+ \times \mathbb{R}_+^2 \to \mathbb{R}_+$, then $V$ is said to be in a class $V_0$ if

(1) $V$ is continuous on$(nT, (n+1)T] \times \mathbb{R}_+^2$, and $\lim_{\substack{(t,\mathbf{y})\to(nT,\mathbf{x}) \\ t>nT}} V(t, \mathbf{y}) = V(nT^+, \mathbf{x})$ exists.

(2) $V$ is locally Lipschitzian in $\mathbf{x}$.

**Definition 2.1.** Let $V \in V_0, (t, \mathbf{x}) \in (nT, (n+1)T] \times \mathbb{R}_+^2$. The upper right derivatives of $V(t, \mathbf{x})$ with respect to the impulsive differential system (1.3) is defined as

$$D^+V(t, \mathbf{x}) = \limsup_{h\to 0+} \frac{1}{h}[V(t+h, \mathbf{x} + hf(t, \mathbf{x})) - V(t, \mathbf{x})].$$

**Remark 2.2.** (1) The solution of the system (1.3) is a piecewise continuous function $\mathbf{x} : \mathbb{R}_+ \to \mathbb{R}_+^2$, $\mathbf{x}(t)$ is continuous on $(nT, (n+1)T], n \in \mathbb{N}$ and $\mathbf{x}(nT^+) = \lim_{t\to nT^+} \mathbf{x}(t)$ exists.
(2) The smoothness properties of $f$ guarantee the global existence and uniqueness of solution of the system (1.3) (see [7] for the details).

Since $\frac{dx}{dt} = \frac{dy}{dt} = 0$ whenever $x(t) = y(t) = 0, t \neq nT$ and $x(nT^+) = (1 - p_1)x(nT)$, $y(nT^+) = (1 - p_2)y(nT) + q(0 \leq p_i < 1, i = 1, 2, q \geq 0)$. We have the following lemma.

**Lemma 2.3.** *Let* $\mathbf{x}(t) = (x(t), y(t))$ *be a solution of* (1.3). *Then we have the following assertions.*
(1) *If* $\mathbf{x}(0^+) \geq 0$ *then* $\mathbf{x}(t) \geq 0$ *for all* $t \geq 0$.
(2) *If* $\mathbf{x}(0^+) > 0$ *then* $\mathbf{x}(t) > 0$ *for all* $t \geq 0$.

We show that all solutions of (1.3) are uniformly ultimately bounded.

**Lemma 2.4.** *There is an* $M > 0$ *such that* $x(t), y(t) \leq M$ *for all* $t$ *large enough, where* $(x(t), y(t))$ *is a solution of* (1.3).

*Proof.* Let $\mathbf{x}(t) = (x(t), y(t))$ be a solution of (1.3) and let $V(t, \mathbf{x}) = bx(t) + y(t)$. Then $V \in V_0$, if $t \neq nT$

$$D^+V + kV = -\frac{ba}{K}x^2(t) + b(a + k)x(t) + c(k - D)y(t). \tag{2.1}$$

Clearly, the right hand of (2.1), is bounded by $M_0 = \frac{b(a + k)^2K^2}{4ak}$ when $0 < k < D$. When $t = nT$, $V(nT^+) = bx(nT^+) + y(nT^+) = (1 - p_1)bx(nT) + (1 - p_2)y(nT) + q \leq V(nT) + q$. So we can choose $0 < k_0 < D$ and $M_0 > 0$ such that

$$\begin{cases} D^+V \leq -k_0V + M_0, t \neq nT, \\ V(n\tau^+) \leq V(n\tau) + q, t = nT. \end{cases} \tag{2.2}$$

From Lemma 2.2 of [4], we can obtain that

$$\begin{aligned} V(t) \leq &(V(0^+) - \frac{M_0}{k_0}) \exp(-k_0t) \\ &+ \frac{q(1 - \exp(-(n + 1)k_0T))}{1 - \exp(-k_0T)} \exp(-k_0(t - nT)) + \frac{M_0}{k_0} \end{aligned} \tag{2.3}$$

for $t \in (nT, (n + 1)T]$. Therefore, $V(t)$ is bounded by $M = \frac{q \exp(k_0T)}{\exp(k_0T) - 1}$ for sufficiently large $t$. Hence there is an $M > 0$ such that $x(t) \leq M, y(t) \leq M$ for a solution $(x(t), y(t))$ with all $t$ large enough. □

Now, we give the basic properties of the following impulsive differential equation.

$$\begin{cases} y'(t) = -Dy(t), \ t \neq nT, \\ y(t^+) = (1 - p_2)y(t) + q, \ t = nT, \\ y(0^+) = y_0. \end{cases} \tag{2.4}$$

Then we can easily obtain the following results.

**Lemma 2.6.** (1) $y^*(t) = \dfrac{q\exp(-D(t-nT))}{1-(1-p_2)\exp(-DT)}$, *$t \in (nT, (n+1)T]$, $n \in \mathbb{N}$ and* $y^*(0^+) = \dfrac{q}{1-(1-p_2)\exp(-DT)}$ *is a positive periodic solution of* (2.4).
(2) $y(t) = (1-p_2)^{n+1}\Big(y(0^+) - \dfrac{q\exp(-DT)}{1-(1-p_2)\exp(-DT)}\Big)\exp(-Dt) + y^*(t)$ *is the solution of* (2.4) *with $y_0 \geq 0$, $t \in (nT, (n+1)T]$ and $n \in \mathbb{N}$.*
(3) *All solutions $y(t)$ of* (1.3) *with $y_0 \geq 0$ tend to $y^*(t)$. i.e., $|y(t) - y^*(t)| \to 0$ as $t \to \infty$.*

## 3. Extinction and Permanence

Now, we present a condition which guarantees locally asymptotical stability of the prey-free periodic solution $(0, y^*(t))$.

**Theorem 3.1.** *The solution $(0, y^*(t))$ is locally asymptotically stable if*

$$aT < \frac{cq(1-\exp(-DT))}{bD(1-(1-p_2)\exp(-DT))} + \ln\frac{1}{1-p_2}.$$

*Proof.* The local stability of the periodic solution $(0, y^*(t))$ of (1.3) may be determined by considering the behavior of small amplitude perturbations of the solution. So in this case we can take $\phi(x(t)) = cx(t)$. Let $(x(t), y(t))$ be any solution of (1.3). Define $x(t) = u(t), y(t) = y^*(t) + v(t)$. Then they may be written as

$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \Phi(t)\begin{pmatrix} u(0) \\ v(0) \end{pmatrix}, 0 \leq t \leq T, \tag{3.1}$$

where $\Phi(t)$ satisfies

$$\frac{d\Phi}{dt} = \begin{pmatrix} a - cy^*(t) & 0 \\ bcy^*(t) & -D \end{pmatrix}\Phi(t) \tag{3.2}$$

and $\Phi(0) = I$, the identity matrix. The linearization of the third and fourth equation of (1.3) becomes

$$\begin{pmatrix} u(nT^+) \\ v(nT^+) \end{pmatrix} = \begin{pmatrix} 1-p_1 & 0 \\ 0 & 1-p_2 \end{pmatrix}\begin{pmatrix} u(nT) \\ v(nT) \end{pmatrix}. \tag{3.3}$$

Note that all eigenvalues of $S = \begin{pmatrix} 1-p_1 & 0 \\ 0 & 1-p_2 \end{pmatrix}\Phi(T)$ are $\mu_1 = \exp(-dT) < 1$ and $\mu_2 = (1-p_2)\exp(\int_0^T a - cy^*(t)dt)$. Since

$$\int_0^T y^*(t)dt = \frac{q(1-\exp(-DT))}{D(1-(1-p_2)\exp(-DT))},$$

we have

$$\mu_2 = (1-p_2)\exp\Big(aT - \frac{cq(1-\exp(-DT))}{D(1-(1-p_2)\exp(-DT))}\Big).$$

By Floquet Theory ([4]), $(0, y^*(t))$ is locally asymptotically stable if $|\mu_2| < 1$.i.e.,

$$aT < \frac{cq(1-\exp(-DT))}{D(1-(1-p_2)\exp(-DT))} + \ln\frac{1}{1-p_2}. \qquad \square$$

**Definition 3.2.** The system (1.3) is permanent if there exist $M \geq m > 0$ such that, for any solution $(x(t), y(t))$ of (1.3) with $\mathbf{x}_0 > 0$,

$$m \leq \lim_{t\to\infty}\inf x(t) \leq \lim_{t\to\infty}\sup x(t) \leq M \text{ and } m \leq \lim_{t\to\infty}\inf y(t) \leq \lim_{t\to\infty}\sup y(t) \leq M.$$

**Theorem 3.3.** *The system* (1.3) *is permanent if*

$$aT > \frac{cq(1-\exp(-DT))}{D(1-(1-p_2)\exp(-DT))} + \ln\frac{1}{1-p_2}.$$

*Proof.* Let $(x(t), y(t))$ be any solution of (1.3) with $\mathbf{x}_0 > 0$. From Lemma 2.4, we may assume that $x(t) \leq M$, $y(t) \leq M$ , $t \geq 0$ and $M > \frac{a}{c}$. Let $m_2 = \frac{q\exp(-DT)}{1-(1-p_2)\exp(-DT)} - \epsilon_2$, $\epsilon_2 > 0$. From Lemma 2.5, clearly we have $y(t) \geq m_2$ for all $t$ large enough. Now we shall find an $m_1 > 0$ such that $x(t) \geq m_1$ for all $t$ large enough. We will do this in the following two steps.

(Step 1) Since

$$aT > \frac{cq(1-\exp(-DT))}{D(1-(1-p_2)\exp(-DT))} + \ln\frac{1}{1-p_2},$$

we can choose $m_3 > 0$, $\epsilon_1 > 0$ small enough such that $0 < m_3 < \min\{\frac{D}{bc}, \nu\}$ and $R = (1-p_2)\exp\Big(aT - \frac{a}{K}Tm_3 - \frac{cq(1-\exp(-DT))}{D(1-(1-p_2)\exp(-DT))} - c\epsilon_1 T\Big) > 1$. Suppose that $x(t) < m_3$ for all $t$. Then we get $y'(t) \leq y(t)(-D+\delta)$, where $\delta = bcm_3$.
By Lemma 2.2 of [4], we have $y(t) \leq u(t)$ and $u(t) \to u^*(t)$, $t \to \infty$, where $u(t)$ is the solution of

$$\begin{cases} u'(t) = (-D+\delta)u(t), \ t \neq nT, \\ u(t^+) = (1-p_2)u(t) + q, \ t = nT, \\ u(0^+) = y_0, \end{cases} \tag{3.4}$$

and $u^*(t) = \frac{q\exp((-D+\delta)(t-nT))}{1-(1-p_2)\exp((-D+\delta)T)}$, $t \in (nT, (n+1)T]$. Then there exists $T_1 > 0$ such that $y(t) \leq u(t) \leq u^*(t) + \epsilon_1$ and

$$\begin{aligned} x'(t) &= x(t)(a - \frac{a}{K}x(t)) - cx(t)y(t) \\ &\geq x(t)\big(a - \frac{a}{K}m_3 - c(u^*(t)+\epsilon_1)\big) \text{ for } t \geq T_1. \end{aligned}$$

Let $N_1 \in \mathbb{N}$ and $N_1 T \geq T_1$. We have, for $n \geq N_1$

$$\begin{cases} x'(t) & \geq x(t)(a - \frac{a}{K}m_3 - c(u^*(t) + \epsilon_1)), t \neq nT, \\ x(t^+) & = (1-p)x(t), t = nT. \end{cases} \tag{3.5}$$

Integrating (3.5) on $(nT, (n+1)T](n \geq N_1)$, we obtain

$$x((n+1)T) \geq x(nT^+)\exp(\int_{nT}^{(n+1)T} a - \frac{a}{K}m_3 - c(u^*(t) + \epsilon_1)dt) = x(nT)R.$$

Then $x((N_1+k)T) \geq x(N_1 T)R^k \to \infty$ as $k \to \infty$ which is a contradiction. Hence there exists a $t_1 > 0$ such that $x(t_1) \geq m_3$.

(Step 2) If $x(t) \geq m_3$ for all $t \geq t_1$, then we are done. If not, we may let $t^* = \inf_{t > t_1}\{x(t) < m_3\}$.Then $x(t) \geq m_3$ for $t \in [t_1, t^*]$ and, by the continuity of $x(t)$, we have $x(t^*) = m_3$. In this step, we have only to consider two possible cases.

Case 1) $t^* = n_1 T$ for some $n_1 \in \mathbb{N}$. Then $(1-p_1)m_3 \leq x(t^{*+}) = (1-p_1)x(t^*) < m_3$. Select $n_2, n_3 \in \mathbb{N}$ such that $(n_2 - 1)T > \dfrac{\ln(\frac{\epsilon_1}{M+q})}{-d+\delta}$ and $(1-p_1)^{n_2}R^{n_3}\exp(n_2\sigma T) > (1-p_1)^{n_2}R^{n_3}\exp((n_2+1)\sigma T) > 1$, where $\sigma = a - \dfrac{a}{K}m_3 - cM < 0$. Let $T' = n_2 T + n_3 T$. In this case we will show that there exists $t_2 \in (t^*, t^* + T']$ such that $x(t_2) \geq m_3$. Otherwise, by (3.4) with $u(t^{*+}) = y(t^{*+})$, we have

$$u(t) - u^*(t) = $$
$$(1-p_2)^{n_1+1}\left(u(t^{*+}) - \frac{q\exp((-D+\delta)T)}{1-(1-p_2)\exp((-D+\delta)T)}\right)\exp((-D+\delta)(t-t^*))$$

for $(n-1)T < t \leq nT$ and $n_1 + 1 \leq n \leq n_1 + 1 + n_2 + n_3$. So we get $|u(t) - u^*(t)| \leq (M+q)\exp((-D+\delta)(t-t^*)) < \epsilon_1$ and $y(t) \leq u(t) \leq u^*(t) + \epsilon_1$ for $t^* + n_2 T \leq t \leq t^* + T'$. Also we get to know that

$$\begin{cases} x'(t) \geq x(t)\big(a - \frac{a}{K}m_3 - c(u^* + \epsilon_1)\big), t \neq nT, \\ x(t^+) = (1-p_1)x(t), t = nT, \end{cases} \tag{3.6}$$

for $t \in [t^* + n_2 T, t^* + T']$. As in step 1, we have

$$x(t^* + T') \geq x(t^* + n_2 T)R^{n_3}.$$

Since $y(t) \leq M$, we have

$$\begin{cases} x'(t) \geq x(t)\big(a - \frac{a}{K}m_3 - cM\big) = \sigma x(t), t \neq nT, \\ x(t^+) = (1-p_1)x(t), t = nT, \end{cases} \tag{3.7}$$

for $t \in [t^*, t^* + n_2T]$. Integrating (3.7) on $[t^*, t^* + n_2T]$ we have

$$\begin{aligned} x((t^* + n_2T)) &\geq m_3 \exp(\sigma n_2 T) \\ &\geq m_3(1 - p_1)^{n_2} \exp(\sigma n_2 T) > m_3. \end{aligned}$$

Thus $x(t^* + T') \geq m_3(1-p_1)^{n_2} \exp(\sigma n_2 T) R^{n_3} > m_3$ which is a contradiction. Now, let $\bar{t} = \inf_{t>t^*}\{x(t) \geq m_3\}$. Then $x(t) \leq m_3$ for $t^* \leq t < \bar{t}$ and $x(\bar{t}) = m_3$. For $t \in [t^*, \bar{t})$, suppose $t \in (t^* + (k-1)T, t^* + kT]$, $k \in \mathbb{N}$ and $k \leq n_2 + n_3$So, we have, for $t \in [t^*, \bar{t})$, from (3.7) we obtain $x(t) \geq x(t^{*+})(1 - p_1)^{k-1} \exp((k-1)\sigma T) \exp(\sigma(t - (t^* + (k-1)T))) \geq m_3(1-p_1)^k \exp(k\sigma T) \leq m_3(1-p_1)^{n_2+n_3} \exp(\sigma(n_2+n_3)T) \equiv m_1'$.

Case (2) $t^* \neq nT, n \in \mathbb{N}$. Then $x(t) \geq m_3$ for $t \in [t_1, t^*)$ and $x(t^*) = m_3$. Suppose that $t^* \in (n_1'T, (n_1' + 1)T)$ for some $n_1' \in \mathbb{N}$. There are two possible cases.

Case(2(a)) $x(t) < m_3$ for all $t \in (t^*, (n_1' + 1)T]$. In this case we will show that there exists $t_2 \in [(n_1' + 1)T, (n_1' + 1)T + T']$ such that $x_2(t_2) \geq m_3$. Suppose not. i.e., $x(t) < m_3$, for all $t \in [(n_1' + 1)T, (n_1' + 1 + n_2 + n_3)T]$. Then $x(t) < m_3$ for all $t \in (t^*, (n_1' + 1 + n_2 + n_3)T]$. By (3.4) with $u((n_1' + 1)T^+) = y((n_1' + 1)T^+)$, we have

$$\begin{aligned} &u(t) - u^*(t) = \\ &\Big(u((n_1' + 1)T^+) - \frac{q \exp(-D + \delta)}{1 - (1 - p_2) \exp(-D + \delta)}\Big) \exp((-D + \delta)(t - (n_1' + 1)T)) \end{aligned}$$

for $t \in (nT, (n + 1)T], n_1' + 1 \leq n \leq n_1' + n_2 + n_3$. A similar argument as in (step 1), we have

$$x((n_1' + 1 + n_2 + n_3)T) \geq x_2((n_1' + 1 + n_2)T)R^{n_3}.$$

It follows from (3.7) that

$$x((n_1' + 1 + n_2)T) \geq m_3(1 - p)^{n_2+1} \exp(\sigma(n_2 + 1)T).$$

Thus $x((n_1' + 1 + n_2 + n_3)T) \geq m_3(1 - p)^{n_2+1} \exp(\sigma(n_2 + 1)T)R^{n_3} > m_3$ which is a contradiction. Now, let $\bar{t} = \inf_{t>t^*}\{x(t) \geq m_3\}$. Then $x(t) \leq m_3$ for $t^* \leq t < \bar{t}$ and $x(\bar{t}) = m_3$. For $t \in [t^*, \bar{t})$, suppose $t \in (n_1'T + (k' + 1)T, n_1'T + k'T]$, $k' \in \mathbb{N}$, $k' \leq 1 + n_2 + n_2$, we have $x(t) \geq m_3(1 - p)^{1+n_2+n_3} \exp(\sigma(1 + n_2 + n_3)T) \equiv m_1$. Since $m_1 < m_1'$, so $x(t) \geq m_1$ for $t \in (t^*, \bar{t})$.

Case (2(b)) There is a $t' \in (t^*, (n_1' + 1)T]$ such that $x_2(t') \geq m_3$. Let $\hat{t} = \inf_{t>t^*}\{x(t) \geq m_3\}$. Then $x(t) \leq m_3$ for $t \in [t^*, \hat{t})$ and $x(\hat{t}) = m_3$. Also, (3.7) holds for $t \in [t^*, \hat{t})$. Integrating the equation on $[t^*, t)(t^* \leq t \leq \hat{t})$, we can get that $x(t) \geq x(t^*) \exp(\sigma(t - t^*)) \geq m_3 \exp(\sigma T) \geq m_1$. Thus in both case the similar argument can be continued since $x(t) \geq m_1$ for some $t > t_1$. This completes the proof. □

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