Some Properties of Harmonic Functions Defined by Convolution

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ABSTRACT. In this paper, we introduce and study a comprehensive family of harmonic univalent functions which contains various well-known classes of harmonic univalent functions as well as many new ones. Also, we improve some results obtained by Frasin [3] and obtain coefficient bounds, distortion bounds and extreme points, convolution conditions and convex combination are also determined for functions in this family. It is worth mentioning that many of our results are either extensions or new approaches to those corresponding previously known results.

1. Introduction

A continuous complex-valued function $f = u + iv$ is said to be harmonic in a simply connected domain D if both u and v are real harmonic in D. In any simply connected domain we can write $f = h + \overline{g}$, where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in D is that $\left|h'(z)\right| > \left|g'(z)\right|, z \in D$. See Clunie and Sheil-Small [2].

Let S_H denote the class of functions $f = h + \overline{g}$ that are harmonic univalent and sense-preserving in the unit disk $U = \{z : |z| < 1\}$ for which $f(0) = f_z(0) - 1 = 0$. Then for $f = h + \overline{g} \in S_H$ we may express the analytic functions h and g as

$$(1.1) \qquad h(z) = z + \sum_{n=2}^{\infty} a_n z^n, g(z) = \sum_{n=1}^{\infty} b_n z^n, |b_1| < 1.$$

Note that S_H reduces to class S of normalized analytic univalent functions if the co-analytic part of its member is zero. In fact Clunie and Sheil-Small [2] investigated the class S_H.

Recently, Frasin [3]defined the class $S_H(\phi, \psi; \alpha)$ the subclass of S_H consisting of functions $f = h + \overline{g} \in S_H$ that satisfy the condition

$$(1.2) \qquad Re\left\{\frac{h(z) * \phi(z) - \overline{g(z) * \psi(z)}}{h(z) + \overline{g(z)}}\right\} > \alpha,$$

where $(0 \leq \alpha < 1)$, $\phi(z) = z + \sum\limits_{n=2}^{\infty} \lambda_n z^n$ and $\psi(z) = z + \sum\limits_{n=2}^{\infty} \mu_n z^n$ are analytic in U with the conditions $\lambda_n \geq 0$, $\mu_n \geq 0$. The operator "*" stands for the convolution of two power series and $TS_H(\phi, \psi; \alpha)$ denote the subclass of $S_H(\phi, \psi; \alpha)$ consisting of functions $f = h + \overline{g}$ such that h and g are of the form

$$(1.3) \qquad h(z) = z - \sum_{n=2}^{\infty} |a_n| \, z^n, g(z) = \sum_{n=1}^{\infty} |b_n| \, z^n$$

We let $S_H(\phi, \psi; \alpha; \lambda)$ denote the subclass of S_H consisting of functions $f = h + \overline{g} \in S_H$ that satisfy the condition

$$(1.4) \qquad Re \left\{ \frac{h(z) * \phi(z) - \overline{g(z) * \psi(z)}}{\lambda \left(h(z) * \phi(z) - \overline{g(z) * \psi(z)} \right) + (1 - \lambda) \left(h(z) + \overline{g(z)} \right)} \right\} > \alpha,$$

for some $\alpha \, (0 \leq \alpha < 1)$, $\lambda \, (0 \leq \lambda < 1)$ and for all $z \in U$.

We further let $TS_H(\phi, \psi; \alpha; \lambda)$ denote the subclass of $S_H(\phi, \psi; \alpha; \lambda)$ consisting of functions $f = h + \overline{g} \in S_H$ such that h and g are of the form (1.3).

We note that by specializing the parameters we obtain the following known subclasses studied by various authors.

(i) $S_H(\phi, \psi; \alpha; 0) \equiv S_H(\phi, \psi; \alpha)$ and $TS_H(\phi, \psi; \alpha; 0) \equiv TS_H(\phi, \psi; \alpha)$ the subclasses of S_H studied by Frasin [3].

(ii) $TS_H \left(\dfrac{z}{(1-z)^2}, \dfrac{z}{(1-z)^2}; \alpha; 0 \right) \equiv HS(\alpha)$ the class of sense preserving harmonic univalent functions f which are starlike of order α in U studied by Jahangiri [4].

(iii) $S_H \left(\dfrac{z}{(1-z)^2}, \dfrac{z}{(1-z)^2}; \alpha; \lambda \right) \equiv S_H^*(\lambda, \alpha)$ and

$TS_H \left(\dfrac{z}{(1-z)^2}, \dfrac{z}{(1-z)^2}; \alpha; \lambda \right) \equiv TS_H^*(\lambda, \alpha)$ convex subclass of harmonic starlike functions studied by Metin Öztürk et al. [5].

Remark. (i) For the harmonic functions f of the form (1.1) with $b_1 = 0$, Avci and Zlotkiewicz [1] showed that if $\sum\limits_{k=2}^{\infty} k \left(|a_k| + |b_k| \right) \leq 1$ then f belong to the class HS(0), and Silverman [7] proved that the above coefficient condition is also necessary if $f = h + \overline{g}$ has negative coefficients. Later, Silverman and Silvia [8] improved the results of [1], [7] to the case b_1 not necessarily zero.

(ii) For $g \equiv 0$, the class S_H reduces to class of analytic univalent functions S. We note that for $g \equiv 0$, $S_H(\phi, \psi; \alpha; \lambda) \equiv S(\phi, \alpha; \lambda)$ and $TS_H(\phi, \psi; \alpha; \lambda) \equiv TS(\phi, \alpha; \lambda)$.

Noting that by suitable choice of ϕ and λ we obtain the following subclasses studied in literature.

$$TS\left(\frac{z}{(1-z)^2}, \alpha; 0\right) \equiv TS^*(\alpha) \text{ and } TS\left(\frac{z+z^2}{(1-z)^3}, \alpha; 0\right) \equiv TK(\alpha) \text{ studied by}$$

Silverman [6].

In this note, we extend the above results to the families $S_H(\phi, \psi; \alpha; \lambda)$ and $TS_H(\phi, \psi; \alpha; \lambda)$. We also obtain extreme points, distortion bounds, convolution conditions and convex combinations for the class $TS_H(\phi, \psi; \alpha; \lambda)$. Study of the present note is of special interest because we improve number of Theorems of Frasin [3].

2. Main results

We begin with a sufficient coefficient condition for functions in $S_H(\phi, \psi; \alpha; \lambda)$.

Theorem 2.1. *Let the function $f = h + \overline{g}$ to be so that h and g are given by (1.1). Furthermore, let*

$$(2.1) \quad \sum_{n=2}^{\infty} \frac{\lambda_n(1-\alpha\lambda) - \alpha(1-\lambda)}{1-\alpha}|a_n| + \sum_{n=1}^{\infty} \frac{\mu_n(1-\alpha\lambda) + \alpha(1-\lambda)}{1-\alpha}|b_n| \leq 1,$$

where $0 \leq \alpha < 1$, $0 \leq \lambda < 1$, $n(1-\alpha) \leq \lambda_n(1-\alpha\lambda) - \alpha(1-\lambda)$ and $n(1-\alpha) \leq \mu_n(1-\alpha\lambda) + \alpha(1-\lambda)$ for $n \geq 2$ then f is sense-preserving, harmonic univalent in U and $f \in S_H(\phi, \psi; \alpha; \lambda)$.

Proof. First we note that f is locally univalent and sense-preserving in U. This is because

$$|h'(z)| \geq 1 - \sum_{n=2}^{\infty} n|a_n||z|^{n-1} > 1 - \sum_{n=2}^{\infty} n|a_n|$$

$$\geq 1 - \sum_{n=2}^{\infty} \frac{\lambda_n(1-\alpha\lambda) - \alpha(1-\lambda)}{1-\alpha}|a_n| \geq \sum_{n=1}^{\infty} \frac{\mu_n(1-\alpha\lambda) + \alpha(1-\lambda)}{1-\alpha}|b_n|$$

$$\geq \sum_{n=1}^{\infty} n|b_n| > \sum_{n=1}^{\infty} n|b_n||z|^{n-1} \geq |g'(z)|.$$

To show that f is univalent in U, suppose $z_1, z_2 \in U$ so that $z_1 \neq z_2$, then

$$\left| \frac{f(z_1) - f(z_2)}{h(z_1) - h(z_2)} \right| \geq 1 - \left| \frac{g(z_1) - g(z_2)}{h(z_1) - h(z_2)} \right|$$

$$= 1 - \left| \frac{\displaystyle\sum_{n=1}^{\infty} b_n (z_1^n - z_2^n)}{z_1 - z_2 + \displaystyle\sum_{n=2}^{\infty} a_n (z_1^n - z_2^n)} \right|$$

$$> 1 - \frac{\displaystyle\sum_{n=1}^{\infty} n|b_n|}{1 - \displaystyle\sum_{n=2}^{\infty} n|a_n|}$$

$$\geq 1 - \frac{\left(\displaystyle\sum_{n=1}^{\infty} \frac{\mu_n (1 - \alpha\lambda) + \alpha (1 - \lambda)}{1 - \alpha} |b_n| \right)}{1 - \displaystyle\sum_{n=2}^{\infty} \frac{\lambda_n (1 - \alpha\lambda) - \alpha (1 - \lambda)}{1 - \alpha} |a_n|}$$

$$\geq 0.$$

Now, we show that $f \in S_H (\phi, \psi; \alpha; \lambda)$. Using the fact that $Re\,\omega > \alpha$ if and only if $|1 - \alpha + \omega| > |1 + \alpha - \omega|$, it suffices to show that

(2.2) $$|A(z) + (1 - \alpha) B(z)| - |A(z) - (1 + \alpha) B(z)| \geq 0,$$

where $A(z) = h(z) * \phi(z) - \overline{g(z) * \psi(z)}$ and

$$B(z) = \lambda \left(h(z) * \phi(z) - \overline{g(z) * \psi(z)} \right) + (1 - \lambda) \left(h(z) + \overline{g(z)} \right).$$

Substituting for $A(z)$ and $B(z)$ in L. H. S. of (2.2) and making use of (2.1) we obtain

$$|A(z) + (1 - \alpha)B(z)| - |A(z) - (1 + \alpha)B(z)|$$
$$= |h(z) * \phi(z) - \overline{g(z) * \psi(z)} + (1 - \alpha)\{\lambda(h(z) * \phi(z) - \overline{g(z) * \psi(z)})$$
$$+ (1 - \lambda)(h(z) + \overline{g(z)})\}| - |h(z) * \phi(z) - \overline{g(z) * \psi(z)} - (1 + \alpha)\{\lambda(h(z) * \phi(z)$$
$$- \overline{g(z) * \psi(z)}) + (1 - \lambda)(h(z) + \overline{g(z)})\}|$$

$$= |(2-\alpha)z + \sum_{n=2}^{\infty} [\lambda_n + (1-\alpha)\lambda\lambda_n + (1-\alpha)(1-\lambda)]a_n z^n$$

$$- \sum_{n=1}^{\infty} \overline{(\mu_n + (1-\alpha)\lambda\mu_n - (1-\alpha)(1-\lambda))b_n z^n}|$$

$$- | -\alpha z + \sum_{n=2}^{\infty} [\lambda_n - (1+\alpha)\lambda\lambda_n - (1+\alpha)(1-\lambda)]a_n z^n$$

$$- \sum_{n=1}^{\infty} \overline{(\mu_n - (1+\alpha)\lambda\mu_n + (1+\alpha)(1-\lambda))b_n z^n}|$$

$$\geq (2-\alpha)|z| - \sum_{n=2}^{\infty} (\lambda_n + (1-\alpha)\lambda\lambda_n + (1-\alpha)(1-\lambda))|a_n||z|^n$$

$$- \sum_{n=1}^{\infty} (\mu_n + (1-\alpha)\lambda\mu_n - (1-\alpha)(1-\lambda))|b_n||z|^n$$

$$- \alpha|z| - \sum_{n=2}^{\infty} (\lambda_n - (1+\alpha)\lambda\lambda_n - (1+\alpha)(1-\lambda))|a_n||z|^n$$

$$- \sum_{n=1}^{\infty} (\mu_n - (1+\alpha)\lambda\mu_n + (1+\alpha)(1-\lambda))|b_n||z|^n$$

$$= 2(1-\alpha)|z| \left\{ 1 - \sum_{n=2}^{\infty} \frac{\lambda_n(1-\alpha\lambda) - \alpha(1-\lambda)}{1-\alpha} |a_n||z|^{n-1} \right.$$

$$\left. - \sum_{n=1}^{\infty} \frac{\mu_n(1-\alpha\lambda) + \alpha(1-\lambda)}{1-\alpha} |b_n||z|^{n-1} \right\}$$

$$\geq 2(1-\alpha)|z| \left\{ 1 - \sum_{n=2}^{\infty} \frac{\lambda_n(1-\alpha\lambda) - \alpha(1-\lambda)}{1-\alpha} |a_n| \right.$$

$$\left. - \sum_{n=1}^{\infty} \frac{\mu_n(1-\alpha\lambda) + \alpha(1-\lambda)}{1-\alpha} |b_n| \right\}$$

$$\geq 0.$$

The Coefficient bound (2.1) is sharp for the function
(2.3)
$$f(z) = z + \sum_{n=2}^{\infty} \frac{1-\alpha}{\lambda_n(1-\alpha\lambda) - \alpha(1-\lambda)} x_n z^n + \sum_{n=1}^{\infty} \frac{1-\alpha}{\mu_n(1-\alpha\lambda) + \alpha(1-\lambda)} \overline{y_n z^n},$$

where $\sum_{n=2}^{\infty} |x_n| + \sum_{n=1}^{\infty} |y_n| = 1.$ □

If we put $\lambda = 0$ the class $S_H(\phi, \psi; \alpha; \lambda)$ reduces to the class $S_H(\phi, \psi; \alpha)$ studied

by Frasin [3]. Thus, from Theorem 2.1, we have

Corollary 2.2. *Let the function* $f = h + \overline{g}$ *to be so that* h *and* g *are given by* (1.1). *Furthermore, let*

$$(2.4) \qquad \sum_{n=2}^{\infty} \frac{\lambda_n - \alpha}{1 - \alpha} |a_n| + \sum_{n=1}^{\infty} \frac{\mu_n + \alpha}{1 - \alpha} |b_n| \leq 1$$

where $0 \leq \alpha < 1$, $n(1 - \alpha) \leq \lambda_n - \alpha$ *and* $n(1 - \alpha) \leq \mu_n + \alpha$ *for* $n \geq 2$ *then* f *is sense-preserving, harmonic univalent in* U *and* $f \in S_H(\phi, \psi; \alpha)$.

Remark 1. Frasin has shown that if condition (2.4) and $n(1 - \alpha) \leq \lambda_n - \alpha \leq \mu_n + \alpha$ are satisfied then f is sense-preserving harmonic univalent and $f \in S_H(\phi, \psi; \alpha)$, whereas we have shown that if condition (2.4) in conjunction with $n(1 - \alpha) \leq \lambda_n - \alpha$ and $n(1 - \alpha) \leq \mu_n + \alpha$ are satisfied. Thus our result is the improvement of Theorem 2.1 of [3].

In the following theorem, it is proved that the condition (2.1) is also necessary for functions $f = h + \overline{g}$, where h and g are of the form (1.3).

Theorem 2.3. *Let the functions* $f = h + \overline{g}$ *be so that* h *and* g *are given by* (1.3). *Then* $f \in TS_H(\phi, \psi; \alpha; \lambda)$ *if and only if*

$$(2.5) \qquad \sum_{n=2}^{\infty} \frac{\lambda_n(1 - \alpha\lambda) - \alpha(1 - \lambda)}{1 - \alpha} |a_n| + \sum_{n=1}^{\infty} \frac{\mu_n(1 - \alpha\lambda) + \alpha(1 - \lambda)}{1 - \alpha} |b_n| \leq 1,$$

where $0 \leq \alpha < 1$, $0 \leq \lambda < 1$, $n(1 - \alpha) \leq \lambda_n(1 - \alpha\lambda) - \alpha(1 - \lambda)$ *and* $n(1 - \alpha) \leq \mu_n(1 - \alpha\lambda) + \alpha(1 - \lambda)$ *for* $n \geq 2$.

Proof. The if part, follows from Theorem 2.1. To prove the only if part, let $f \in TS_H(\phi, \psi; \alpha; \lambda)$ then from (1.4), we have

$$\text{Re}\left\{ \frac{h(z) * \phi(z) - \overline{g(z) * \psi(z)}}{\lambda\left(h(z) * \phi(z) - \overline{g(z) * \psi(z)}\right) + (1 - \lambda)\left(h(z) + \overline{g(z)}\right)} - \alpha \right\} \geq 0$$

is equivalent to

$$\text{Re}\left\{ \frac{(1 - \alpha)z - \displaystyle\sum_{n=2}^{\infty}[\lambda_n(1 - \alpha\lambda) - \alpha(1 - \lambda)]|a_n|z^n - \displaystyle\sum_{n=1}^{\infty}[\mu_n(1 - \alpha\lambda) + \alpha(1 - \lambda)]|b_n|\bar{z}^n}{z - \displaystyle\sum_{n=2}^{\infty}[\lambda\lambda_n + 1 - \lambda]|a_n|z^n + \displaystyle\sum_{n=1}^{\infty}[\lambda\mu_n - 1 + \lambda]|b_n|\bar{z}^n} \right\} \geq 0.$$

If we choose z to be real and $z \to 1^-$, we get

$$\left\{(1-\alpha) - \sum_{n=2}^{\infty} \left(\lambda_n \left(1-\alpha\lambda\right) - \alpha \left(1-\lambda\right)\right) |a_n| - \sum_{n=1}^{\infty} \left(\mu_n \left(1-\alpha\lambda\right) + \alpha \left(1-\lambda\right)\right) |b_n|\right\}$$

$$\times \left\{1 - \sum_{n=2}^{\infty} \left(\lambda\lambda_n + 1 - \lambda\right) |a_n| + \sum_{n=1}^{\infty} \left(\lambda\mu_n - 1 + \lambda\right) |b_n|\right\}^{-1} \geq 0.$$

or equivalently

$$\sum_{n=2}^{\infty} \frac{\lambda_n \left(1-\alpha\lambda\right) - \alpha \left(1-\lambda\right)}{1-\alpha} |a_n| + \sum_{n=1}^{\infty} \frac{\mu_n \left(1-\alpha\lambda\right) + \alpha \left(1-\lambda\right)}{1-\alpha} |b_n| \leq 1$$

which is the required condition. $\square$

Taking different choices of $\phi\left(z\right)$, $\psi\left(z\right)$ and λ in Theorem 2.3 we obtain the following corollaries obtained by Frasin [3].

Corollary 2.4. *Let the functions $f = h + \overline{g}$ be so that h and g are given by (1.3). Then $f \in TS_H\left(\phi, \psi; \alpha; 0\right)$ if and only if*

$$\sum_{n=2}^{\infty} \frac{\lambda_n - \alpha}{1-\alpha} |a_n| + \sum_{n=1}^{\infty} \frac{\mu_n + \alpha}{1-\alpha} |b_n| \leq 1.$$

where $0 \leq \alpha < 1, n\left(1-\alpha\right) \leq \lambda_n - \alpha$ and $n\left(1-\alpha\right) \leq \mu_n + \alpha$ for $n \geq 2$.

Remark 2. We note that $TS_H\left(\phi, \psi; \alpha; 0\right) \equiv TS_H\left(\phi, \psi; \alpha\right)$, the class defined by Frasin [3]. We easily seen that the above corollary improves the Theorem 2.2 of [3] because our result holds for the condition $n\left(1-\alpha\right) \leq \lambda_n - \alpha$ and $n\left(1-\alpha\right) \leq \mu_n + \alpha$ for $n \geq 2$ whereas Frasin's result holds only for $n\left(1-\alpha\right) \leq \lambda_n - \alpha \leq \mu_n + \alpha, \left(n \geq 2\right)$.

Corollary 2.5. *Let the function $f = h + \overline{g}$ be so that h and g are given by (1.3). Then*

$$f \in TS_H\left(\frac{z}{\left(1-z\right)^2}, \frac{z}{\left(1-z\right)^2}; \alpha; 0\right)$$

if and only if

$$\sum_{n=2}^{\infty} \frac{n - \alpha}{1-\alpha} |a_n| + \sum_{n=1}^{\infty} \frac{n + \alpha}{1-\alpha} |b_n| \leq 1,$$

where $0 \leq \alpha < 1$.

The above result is obtained by Jahangiri [4].

Corollary 2.6. *Let the function $f = h + \overline{g}$ be so that h and g are given by (1.3). Then*

$$f \in TS_H\left(\frac{z + z^2}{\left(1-z\right)^3}, \frac{z}{\left(1-z\right)^2}; \alpha; 0\right)$$

if and only if

$$\sum_{n=2}^{\infty} \frac{n^2 - \alpha}{1 - \alpha} |a_n| + \sum_{n=1}^{\infty} \frac{n + \alpha}{1 - \alpha} |b_n| \leq 1,$$

where $0 \leq \alpha < 1$.

The following theorem gives the bounds for functions in $TS_H(\phi, \psi; \alpha; \lambda)$, which yields a covering result for this family.

Theorem 2.7. *Let* $f \in TS_H(\phi, \psi; \alpha; \lambda)$ *and*

$$A \leq \lambda_n (1 - \alpha\lambda) - \alpha (1 - \lambda), A \leq \mu_n (1 - \alpha\lambda) + \alpha (1 - \lambda)$$

for $n \geq 2$. *Then for* $|z| = r < 1$ *we have*

$$|f(z)| \leq (1 + |b_1|) r + \left(\frac{1 - \alpha}{A} - \frac{\mu_1 (1 - \alpha\lambda) + \alpha (1 - \lambda)}{A} |b_1| \right) r^2, |z| = r < 1$$

and

$$|f(z)| \geq (1 - |b_1|) r - \left(\frac{1 - \alpha}{A} - \frac{\mu_1 (1 - \alpha\lambda) + \alpha (1 - \lambda)}{A} |b_1| \right) r^2, |z| = r < 1,$$

where $A = \min \{\lambda_2 (1 - \alpha\lambda) - \alpha (1 - \lambda), \mu_2 (1 - \alpha\lambda) + \alpha (1 - \lambda)\}$.

Proof. We only prove the right hand inequality. The proof for the left hand inequality is similar and will be omitted. Let $f \in TS_H(\phi, \psi; \alpha; \lambda)$. Taking the absolute value of f we have

$$|f(z)| \leq (1 + |b_1|)r + \sum_{n=2}^{\infty} (|a_n| + |b_n|) r^n$$

$$\leq (1 + |b_1|)r + \sum_{n=2}^{\infty} (|a_n| + |b_n|) r^2$$

$$= (1 + |b_1|)r + \frac{1 - \alpha}{A} \sum_{n=2}^{\infty} \left(\frac{A}{1 - \alpha} |a_n| + \frac{A}{1 - \alpha} |b_n| \right) r^2$$

$$\leq (1 + |b_1|)r + \frac{(1 - \alpha)}{A} \sum_{n=2}^{\infty} \left(\frac{\lambda_n (1 - \alpha\lambda) - \alpha(1 - \lambda)}{1 - \alpha} |a_n| \right.$$

$$\left. + \frac{\mu_n (1 - \alpha\lambda) + \alpha(1 - \lambda)}{1 - \alpha} |b_n| \right) r^2$$

$$\leq (1 + |b_1|)r + \frac{1 - \alpha}{A} \left(1 - \frac{\mu_1 (1 - \alpha\lambda) + \alpha(1 - \lambda)}{1 - \alpha} |b_1| \right) r^2.$$

$$= (1 + |b_1|)r + \left(\frac{1 - \alpha}{A} - \frac{\mu_1 (1 - \alpha\lambda) + \alpha(1 - \lambda)}{A} |b_1| \right) r^2, |z| = r < 1$$

This completes the proof of Theorem 2.7. □

The following covering result follows from the left hand inequality in Theorem 2.7.

Corollary 2.8. *Let* $f \in TS_H(\phi, \psi; \alpha; \lambda)$ *and* $A \leq \lambda_n (1 - \alpha\lambda) - \alpha (1 - \lambda)$, $A \leq \mu_n (1 - \alpha\lambda) + \alpha (1 - \lambda)$ *for* $n \geq 2$ *where*

$$A = \min \{\lambda_2 (1 - \alpha\lambda) - \alpha (1 - \lambda), \mu_2 (1 - \alpha\lambda) + \alpha (1 - \lambda)\}.$$

Then we have

$$\left\{ \omega : |\omega| < \left(\frac{A - 1 + \alpha}{A} - \frac{A - \mu_1 (1 - \alpha\lambda) - \alpha (1 - \lambda)}{A} |b_1| \right) \right\} \subset f(U).$$

Remark 3. If we put $\phi(z) = \dfrac{z}{(1 - z)^2}$, $\psi(z) = \dfrac{z}{(1 - z)^2}$; $\lambda = 0$ in Corollary 2.8 we have the result of Jahangiri [4] and if $\phi(z) = \dfrac{z}{(1 - z)^2}$, $\psi(z) = \dfrac{z}{(1 - z)^2}$ we have the result of Öztürk et al. [5].

3. Extreme points

In this section, we determine the extreme points of $TS_H(\phi, \psi; \alpha; \lambda)$.

Theorem 3.1. *Let*

$$h_1(z) = z, \ h_n(z) = z - \frac{1 - \alpha}{\lambda_n (1 - \alpha\lambda) - \alpha (1 - \lambda)} z^n \ (n \geq 2)$$

and

$$g_n(z) = z + \frac{1 - \alpha}{\mu_n (1 - \alpha\lambda) + \alpha (1 - \lambda)} \bar{z}^n \ (n \geq 1).$$

Then $f \in TS_H(\phi, \psi; \alpha; \lambda)$ *if and only if it can be expressed as*

(3.1) $$f(z) = \sum_{n=1}^{\infty} (x_n h_n(z) + y_n g_n(z))$$

where $x_n \geq 0$ *and* $y_n \geq 0$, $\sum_{n=1}^{\infty} (x_n + y_n) = 1$. *In particular the extreme points of* $TS_H(\phi, \psi; \alpha; \lambda)$ *are* $\{h_n\}$ *and* $\{g_n\}$.

Proof. The proof of Theorem 3.1 is similar to those of Theorem 4.1 of Frasin [3], therefore we omit details involved. □

4. Convolution and convex Combination

In this section we show that the class $TS_H(\phi, \psi; \alpha; \lambda)$ is closed under convolution and convex combinations.

We need the following definition of convolution of two harmonic functions.

Let the function $f(z)$ be defined by

$$f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n + \sum_{n=1}^{\infty} |b_n| \bar{z}^n$$

and the function $F(z)$ be defined by

$$F(z) = z - \sum_{n=2}^{\infty} |A_n| z^n + \sum_{n=1}^{\infty} |B_n| \bar{z}^n$$

we define the convolution of two harmonic functions f and F as

$$(f * F)(z) = f(z) * F(z) = z - \sum_{n=2}^{\infty} |a_n A_n| z^n + \sum_{n=1}^{\infty} |b_n B_n| \bar{z}^n.$$

Theorem 4.1. *If $f \in TS_H(\phi, \psi; \alpha; \lambda)$ and $F \in TS_H(\phi, \psi; \alpha; \lambda)$ then*

$$f * F \in TS_H(\phi, \psi; \alpha; \lambda).$$

Theorem 4.2. *The class $TS_H(\phi, \psi; \alpha; \lambda)$ is closed under convex combinations.*

Proof. The proofs of the above Theorems are analogues to the corresponding similar Theorems proved in [3] and therefore we omit details involved.   □

References

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