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An Application of Furuta Inequality to Linear Operator Equations

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Abstract. A class of Hermitian operators B admitting a positive semidefinite solution of the linear operator equation $\sum_{j=1}^{n} A^{n-j}XA^{j-1} = B$ for a fixed positive definite operator A is given via the Furuta inequality.

1. Introduction

The main concern of this paper is to study the linear operator equation (a general form of the Lyapunov equation $AX + XA = B$)

$$\sum_{j=1}^{n} A^{n-j}XA^{j-1} = B \tag{1.1}$$

where $A > 0$ (positive definite) and B is a Hermitian operator on a Hilbert space $\mathcal{H}$, and positive semidefinite solution X is sought.

In [1], Bhatia and Uchiyama obtained an explicit form of the unique solution of (1.1) (when no point of the spectrum of A is on the negative real axis) by

$$\begin{aligned} X &= \frac{\sin \pi/n}{\pi} \int_0^\infty (t + A^n)^{-1} B (t + A^n)^{-1} t^{1/n} dt \\ &= \frac{1/n}{\Gamma(1 - 1/n)} \int_0^\infty \left[\int_0^t e^{-sA^n} B e^{-(t-s)A^n} ds \right] t^{-(1/n+1)} dt. \end{aligned}$$

This implies in particular that if A and B are positive semidefinite, then so is X, an analogue of one of the important facts for the Lyapunov equation. However for positive definite A but for general Hermitian operator B, it is non-trivial to determine the positive semidefiniteness of the solution X. The main purpose of this paper is to find a class of Hermitian operators B that assure the positive semefinitness of the solution for a fixed positive definite operator A.

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In [2], Chan and Kwong have established the existence of positive semidefinite solution of $A^2X + XA^2 = A(AB + BA)A$, equivalently,

$$AX + XA = A^{1/2}(A^{1/2}B + BA^{1/2})A^{1/2} \tag{1.2}$$

for positive definite matrix A and positive semidefinite matrix B via the inequality

$$(BA^2B)^{1/2} \geq B^2, \quad A \geq B \geq 0 \text{ (positive semidefinite)}, \tag{1.3}$$

a special type of the Furuta inequality:

$$(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{1}{q}} \geq B^{\frac{r+p}{q}}, \quad A \geq B \geq 0$$

for any $(r,p) \in \mathcal{D}_q := \{(r,\alpha) \in [0,\infty) \times [0,\infty) : (1+r)q \geq r + \alpha\},\ q \geq 1$. Also, see [5] for the matrix equation

$$AX + XA = f(A)B + Bf(A) \tag{1.4}$$

where f is a matrix monotone function. These results are quite non-trivial since the right-hand side of (1.2) or (1.4) fails to be non-negative, in general.

Recently T. Furuta obtained the existence of semidefinite solution of (1.1) for a special B as follows:

Theorem A[4]. *Let A be a positive definite operator and B be a positive semidefinite operator. Let m and n be natural numbers. There exists positive semidefinite operator solution X of the following operator equation:*

$$\sum_{j=1}^{n} A^{n-j}XA^{j-1} = A^{\frac{nr}{2(m+r)}}\left(\sum_{j=1}^{m} A^{\frac{n(m-j)}{m+r}}BA^{\frac{n(j-1)}{m+r}}\right)A^{\frac{nr}{2(m+r)}}$$

$$\textit{for } r \textit{ such that } \begin{cases} r \geq 0 & \text{if } n \geq m & \text{(i)} \\ r \geq \frac{m-n}{n-1} & \text{if } m \geq n \geq 2 & \text{(ii)} \end{cases}$$

This includes the case $r = n = m = 2$, yielding the equation (1.2).

In this paper we generalize the result of Furuta by finding a general class of Hermitian operators B that assure the existence of positive semidefinite solution of (1.1). For $A > 0$ and $\alpha \geq 0$, we consider the set $\mathcal{F}_A^{(\alpha)}$ of all (Hermitian operator) derivatives at $t = 0$

$$\frac{d}{dt}\Big|_{t=0} f(t)^\alpha$$

where $f : (-\epsilon, \epsilon) \to \mathrm{Herm}(\mathcal{H})$ varies over differentiable functions defined near $t = 0$ satisfying

$$f(t) \geq f(0) = A, \quad \text{for all } t \geq 0. \tag{1.5}$$

One directly sees that the map defined by $f(t) = A + tB$ with $A > 0$ and $B \geq 0$ satisfies (1.5) and eventually yields $\sum_{j=1}^{m} A^{m-j}BA^{j-1} \in \mathcal{F}_A^{(m)}$.

The main result of this paper is the following Theorem 1.1 which is further extension of Theorem A.

Theorem 1.1. *Let $A > 0$. Then the operator equation*

$$\sum_{j=1}^{n} A^{n-j} X A^{j-1} = A^{\frac{rn}{2(r+\alpha)}} B A^{\frac{rn}{2(r+\alpha)}} \tag{1.6}$$

has a (unique) positive semidefinite solution for any $B \in \mathcal{F}^{(\alpha)}_{A^{\frac{n}{r+\alpha}}}$ and $(r, \alpha) \in \mathcal{D}_n$.

2. Properties of $\mathcal{F}^{(\alpha)}_A$

Let $\mathcal{H}$ be a Hilbert space and let $\mathrm{Herm}(\mathcal{H})$ be the Banach space of all Hermitian operators on $\mathcal{H}$. Let $\mathrm{P}(\mathcal{H})$ be the open convex cone of all positive definite operators on $\mathcal{H}$.

Definition 2.1. For $\epsilon > 0$, we denote $\mathcal{F}_\epsilon$ by the set of differentiable functions $f : (-\epsilon, \epsilon) \to \mathrm{Herm}(\mathcal{H})$ satisfying

$$f(t) \geq f(0) > 0 \quad \forall t \geq 0.$$

We also define

$$\mathcal{F} = \bigcup_{\epsilon > 0} \mathcal{F}_\epsilon, \qquad \mathcal{F}_A = \{f \in \mathcal{F} : f(0) = A\}$$

Remark 2.2. Let $f \in \mathcal{F}_A$ defined on $(-\epsilon, \epsilon)$. From $A > 0$ and continuity of f we can find a small $\epsilon' < \epsilon$ so that $f(t) > 0$ for all $t \in (-\epsilon', \epsilon')$.

Example 2.3. (1) Let $\mu, \nu : (-\epsilon, \epsilon) \to R$ be increasing and differentiable functions such that $\mu(0) = 1$ and $\nu(0) = 0$. Then for $A > 0$ and $B \geq 0$, $f(t) = \mu(t)A + \nu(t)B$ belongs to $\mathcal{F}_A$.

(2) Let $B \geq A > 0$, and let $0 < \epsilon < 1$. Let $\mu, \nu : (-\epsilon, \epsilon) \to (0, \infty)$ be increasing and differentiable functions with $\mu(0) = 1$. Then $f(t) = \mu(t)A\#_t\nu(t)B$ belongs to $\mathcal{F}_A$, where $X\#_tY = X^{1/2}(X^{-1/2}YX^{-1/2})^tX^{1/2}$ denotes the weighted geometric mean of X and Y. Indeed for $t \geq 0$,

$$f(t) = \mu(t)A\#_t\nu(t)B \geq \mu(t)A\#_t\nu(t)A = \mu(t)^{1-t}\nu(t)^tA \geq \mu(0)A(= f(0)) = A$$

where the first inequality follows from the Löwner-Heinz inequality:

$$A\#_tB \leq A'\#_tB', t \in [0, 1], \quad 0 \leq A \leq A', 0 \leq B \leq B'.$$

(3) The set $\mathcal{F}_A$ is closed under the geometric and arithmetic mean operations; if $f_1, f_2 \in \mathcal{F}_A$ defined on $(-\epsilon_1, \epsilon_1)$ and $(-\epsilon_2, \epsilon_2)$ respectively, then

$$f_1\#f_2, \frac{f_1 + f_2}{2} \in \mathcal{F}_A$$

where the functions are on a small interval $(-\epsilon', \epsilon'), \epsilon' < \min\{\epsilon_1, \epsilon_2\}$. We may assume that f_1 and f_2 have positive definite values on $(-\epsilon', \epsilon')$. It then follows from the Löwner-Heinz inequality that

$$(f_1 \# f_2)(t) = f_1(t) \# f_2(t) \geq f_1(0) \# f_2(0) (= (f_1 \# f_2)(0)) = A \# A = A, t \geq 0$$

and similarly for the arithmetic mean $\frac{f_1+f_2}{2}$.

(4) For $\alpha \geq 0$, we consider the power map $\hat{\alpha}(X) = X^\alpha$ on the convex cone of positive definite operators. For $f \in \mathcal{F}_A$, one can find $\epsilon > 0$ such that $f(t) > 0$ for all $t \in (-\epsilon, \epsilon)$ by Remark 2.2. Define $\hat{\alpha}(f) := \hat{\alpha} \circ f$ on $(-\epsilon, \epsilon)$. Then $\hat{\alpha}(\mathcal{F}_A) \subset \mathcal{F}_{A^\alpha}$ for any $\alpha \in [0, 1]$ from the Löwner-Heinz inequality; if $f \in \mathcal{F}_A$, then $f(t) \geq f(0) = A$ and hence

$$(\hat{\alpha} \circ f) = f(t)^\alpha \geq f(0)^\alpha = (\hat{\alpha} \circ f)(0), \ t \geq 0.$$

Proposition 2.4(Invariancy under congruence transformations). *Let $A > 0$ and let M be an invertible operator on $\mathcal{H}$. Then*

$$\Gamma_M(\mathcal{F}_A) = \mathcal{F}_{\Gamma_M(A)},$$

where $\Gamma_M(X) = MXM^*$.

Proof. Let $f : (-\epsilon, \epsilon) \to \mathrm{Herm}(\mathcal{H})$ be a member of $\mathcal{F}_A$. Then $(\Gamma_M \circ f)(t) = Mf(t)M^*$ for all $t \in (-\epsilon, \epsilon)$ and

$$(\Gamma_M \circ f)(t) \quad = \quad Mf(t)M^* \geq Mf(0)M^* = (\Gamma_M \circ f)(0) > 0$$

and hence $\Gamma_M \circ f \in \mathcal{F}_{\Gamma_M(A)}$. This implies that $\Gamma_M(\mathcal{F}_A) \subset \mathcal{F}_{\Gamma_M(A)}$. Since M is invertible and $\Gamma_M^{-1} = \Gamma_{M^{-1}}$, the equality $\Gamma_M(\mathcal{F}_A) = \mathcal{F}_{\Gamma_M(A)}$ holds. □

For $f \in \mathcal{F}_A$, the map $t \mapsto f(t)^\alpha$ composed by the power map $\hat{\alpha}(X) = X^\alpha$ is differentiable on an appropriate interval $(-\epsilon', \epsilon')$, in particular at $t = 0$ (Remark 2.2).

Definition 2.5. For $A > 0$ and $\alpha \geq 0$, we define

$$\mathcal{F}_A^{(\alpha)} = \{\frac{d}{dt}\Big|_{t=0} f(t)^\alpha : f \in \mathcal{F}_A\}.$$

Note that $\mathcal{F}_A^{(\alpha)}$ consists of Hermitian operators and $\mathcal{F}_A^{(0)} = \{0\}$.

Example 2.6. (1) Let $A > 0, B \geq 0$ and $m \in N$. For differentiable and increasing functions $\mu, \nu : (-\epsilon, \epsilon) \to R$ with $\mu(0) = 1$ and $\nu(0) = 0$ (see Examples 2.3 (1)),

$$m\mu'(0)A^m + \nu'(0) \sum_{j=1}^{m} A^{m-j}BA^{j-1} \in \mathcal{F}_A^{(m)}.$$

This follows by taking $f(t) = \mu(t)A + \nu(t)B$. In particular, $\sum_{j=1}^m A^{m-j}BA^{j-1} \in \mathcal{F}_A^{(m)}$ with $\mu(t) = 1$ and $\nu(t) = t$. Furthermore, $\mathcal{F}_A^{(1)}$ contains all positive semidefinite operators.

(2) For $A, B > 0$ with $AB = BA$ and $B \geq A$,

$$A(\log B - \log A) \in \mathcal{F}_A^{(1)}$$

by taking $f(t) = A\#_t B = A^{1-t}B^t$.

(3) The case $A > 0$ and $0 < \alpha < 1$: We consider the power map $\hat{\alpha}(X) = X^\alpha$ defined on the convex cone of positive definite operators. Then the derivative of $\hat{\alpha}$ at A is a linear map whose action is given by ([1])

$$D\hat{\alpha}(A)(X) = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty \left[\int_0^t e^{-sA} X e^{-(t-s)A} ds\right] t^{-(r+1)} dt.$$

In this case,

$$\mathcal{F}_A^{(\alpha)} = \left\{ \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty \left[\int_0^t e^{-sA} f'(0) e^{-(t-s)A} ds\right] t^{-(r+1)} dt : f \in \mathcal{F}_A \right\}.$$

Lemma 2.7(Unitary invariancy)**.** *For any unitary operator U,*

$$\Gamma_U(\mathcal{F}_A^{(\alpha)}) = \mathcal{F}_{\Gamma_U(A)}^{(\alpha)}.$$

Proof. This follows from that

$$\begin{aligned}
\Gamma_U(\mathcal{F}_A^{(\alpha)}) &= \{U \left[\frac{d}{dt}\Big|_{t=0} f(t)^\alpha\right] U^* : f \in \mathcal{F}_A\} \\
&= \{\frac{d}{dt}\Big|_{t=0} [U f(t)^\alpha U^*] : f \in \mathcal{F}_A\} \\
&= \{\frac{d}{dt}\Big|_{t=0} (U f(t) U^*)^\alpha : f \in \mathcal{F}_A\} \\
&= \mathcal{F}_{UAU^*}^{(\alpha)}.
\end{aligned}$$

□

3. Proof of the main result

Definition 3.1. For $q \geq 1$,

$$\mathcal{D}_q := \{(r,p) \in [0,\infty) \times [0,\infty) : (1+r)q \geq r + p\}.$$

We note that the domain $\mathcal{D}_q$ coincides with that of parameters satisfying the Furuta inequality:

Theorem 3.2(Furuta, [3])**.** *If $A \geq B \geq 0$ then*

$$(B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{1}{q}} \geq B^{\frac{r+p}{q}}$$

for any $(r,p) \in \mathcal{D}_q$.

We consider the operator equation

$$\sum_{j=1}^{n} A^{n-j} X A^{j-1} = A^{\frac{rn}{2(r+\alpha)}} B A^{\frac{rn}{2(r+\alpha)}}, A > 0, B \in \mathrm{Herm}(\mathcal{H}). \tag{3.1}$$

Here we assume $\frac{rn}{2(r+\alpha)} = 0$ for $r = \alpha = 0$.

Proof of Theorem 1.1. If $(r, \alpha) = (0, 0)$, then the right hand side of (1.6) is the zero matrix from $\mathcal{F}_A^{(0)} = \{0\}$ and hence the equation has the trivial solution $X = 0$. Suppose that $(r, \alpha) \neq (0, 0)$. Let $B \in \mathcal{F}_A^{(\alpha)}$. Let $f \in \mathcal{F}_A$ and let $\epsilon > 0$ such that $f(t) > 0$ for all $t \in (-\epsilon, \epsilon)$ and $B = \frac{d}{dt}\big|_{t=0} f(t)^\alpha$. We consider the curve

$$Y(t) := \left(A^{\frac{r}{2}} f(t)^\alpha A^{\frac{r}{2}}\right)^{\frac{1}{n}}, \quad t \in (-\epsilon, \epsilon).$$

Then $Y(\cdot)$ is differentiable and $Y(0) = \left(A^{\frac{r}{2}} A^\alpha A^{\frac{r}{2}}\right)^{\frac{1}{n}} = A^{\frac{r+\alpha}{n}}$. By definition and the Furuta inequality, $f(t) \geq f(0) = A$ and

$$Y(t) = \left(A^{\frac{r}{2}} f(t)^\alpha A^{\frac{r}{2}}\right)^{\frac{1}{n}} \geq A^{\frac{r+\alpha}{n}} = Y(0)$$

for all $t \in [0, \epsilon)$. This implies that

$$X := Y'(0) = \lim_{t \downarrow 0} \frac{Y(t) - Y(0)}{t} \geq 0.$$

By differentiating $Y^n(t) = A^{\frac{r}{2}} f(t)^\alpha A^{\frac{r}{2}}$ at $t = 0$, we have

$$\begin{aligned}
\sum_{j=1}^{n} A^{\frac{(r+\alpha)(n-j)}{n}} X A^{\frac{(r+\alpha)(j-1)}{n}} &= \sum_{j=1}^{n} Y(0)^{n-j} X Y(0)^{j-1} \\
&= \frac{d}{dt}\Big|_{t=0} [A^{\frac{r}{2}} f(t)^\alpha A^{\frac{r}{2}}] \\
&= A^{\frac{r}{2}} [\frac{d}{dt}\Big|_{t=0} f(t)^\alpha] A^{\frac{r}{2}} = A^{\frac{r}{2}} B A^{\frac{r}{2}}
\end{aligned}$$

Replacing A to $A^{\frac{n}{r+\alpha}}$ yields $B \in \mathcal{F}^{(\alpha)}_{A^{\frac{n}{r+\alpha}}}$ and

$$\sum_{j=1}^{n} A^{n-j} X A^{j-1} = A^{\frac{rn}{2(r+\alpha)}} B A^{\frac{rn}{2(r+\alpha)}}.$$

This completes the proof of Theorem 1.1. □

Remark 3.3. (1) It is natural to see whether Theorem 1.1 holds for positive semidefinite operator A or not. We don't have an answer yet.

(2) The (right) differentiability of $Y(t)$ at $t = 0$ is enough in the proof of Theorem 1.1 and so more general f can be thought; $f : [0, \epsilon) \to \mathrm{Herm}(\mathcal{H})$ is (right) differentiable at $t = 0$ and $f(t) \geq f(0) > 0$ for all $t \in [0, \epsilon)$.

Remark 3.4. If $A = \mathrm{diag}(a_1, a_2, \ldots, a_l) > 0$, then the (unique) solution of

$$\sum_{j=1}^{n} A^{n-j} X A^{j-1} = B$$

is given by $X = \left(\frac{a_i - a_j}{a_i^n - a_j^n} b_{ij} \right)_{l \times l}$, where $B = \left(b_{ij} \right)_{l \times l}$. When $a_i = a_j$ the quotient $\frac{a_i - a_j}{a_i^n - a_j^n}$ is interpreted to mean $\frac{1}{n a_i^{n-1}}$ (See [1]). Our result shows that if $B \in A^{\frac{rn}{2(r+\alpha)}} \mathcal{F}^{(\alpha)}_{A^{\frac{n}{r+\alpha}}} A^{\frac{rn}{2(r+\alpha)}}$ with $(r, \alpha) \in \mathcal{D}_n$ then the matrix X with entries $x_{ij} = \frac{a_i - a_j}{a_i^n - a_j^n} b_{ij}$ is positive semidefinite. This provides a construction of positive semidefinite matrices depending on the set $\mathcal{F}^{(\alpha)}_{A^{\frac{n}{r+\alpha}}}$ of Hermitian matrices and $(r, \alpha) \in \mathcal{D}_n$. For instance, if $(r, m) \in \mathcal{D}_n$ with $m \in N$, then $\sum_{j=1}^{m} A^{m-j} B A^{j-1} \in \mathcal{F}^{(m)}_A$ for any $B \geq 0$ by Example 2.6(1) and hence the unique semidefinite solution of

$$\sum_{j=1}^{n} A^{n-j} X A^{j-1} = A^{\frac{rn}{2(r+m)}} \left(\sum_{j=1}^{m} A^{\frac{n(m-j)}{r+m}} B A^{\frac{n(j-1)}{r+m}} \right) A^{\frac{rn}{2(r+m)}}$$

is given by

$$X = \left((a_i a_j)^{\frac{rn}{2(r+m)}} \frac{a_i^{\frac{nm}{r+m}} - a_j^{\frac{nm}{r+m}}}{a_i^n - a_j^n} b_{ij} \right)_{l \times l}. \tag{3.2}$$

Conversely, the $l \times l$ matrix X in (3.2) is positive semidefinite for any $B = (b_{ij})_{l \times l} \geq 0$ and $(r, m) \in \mathcal{D}_n$.

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References

[1] R. Bhatia and M. Uchiyama, *The operator equation* $\sum_{i=0}^{n} A^{n-i} X B^{i} = Y$, Expo. Math., **27**(2009), 251-255.

[2] N. Chan and M. Kwong, *Hermitian matrix inequalities and a conjecture*, Amer. Math. Monthly, **92**(1985), 533-541.

[3] T. Furuta, *$A \geq B \geq 0$ assures $(B^r A^p B^r)^{1/q} \geq B^{(p+2r)/q}$ for $r \geq 0, p \geq 0, q \geq 1$ with $(1+2r)q \geq p+2r$*, Proc. Amer. Math. Soc., **101**(1987), 85-88.

[4] T. Furuta, *Positive semidefinite solution of the operator equation $\sum_{j=1}^{n} A^{n-j} X A^{j-1} = B$*, Linear Alg. and Its Appl., **432**(2010), 949-955.

[5] M. Kwong, *Some results on matrix monotone functions*, Linear Alg. and Its Appl., **118**(1989), 129-153.