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# On a Reverse of the Slightly Sharper Hilbert-type Inequality

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Abstract. In this paper, by introducing parameters $\lambda$, $\alpha$ and two pairs of conjugate exponents $(p,q)$, $(r,s)$ and applying the improved Euler-Maclaurin's summation formula, we establish a reverse of the slightly sharper Hilbert-type inequality. As applications, the strengthened version and the equivalent form are given.

## 1. Introduction

Suppose $(p,q)$ is one pair of conjugate exponents $(\frac{1}{p}+\frac{1}{q}=1)$, and $p>1$, $a_n, b_n \geq 0$, $0<\sum_{n=1}^{\infty} a_n^p<\infty$ and $0<\sum_{n=1}^{\infty} b_n^q<\infty$, then we have the slightly sharper Hilbert's inequality as (see [1]):

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_m b_n}{m+n+1}<\frac{\pi}{\sin(\pi/p)}(\sum_{n=0}^{\infty}a_n^p)^{1/p}(\sum_{n=0}^{\infty}b_n^q)^{1/q}, \tag{1.1}$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible. Inequality (1.1) is important in analysis and its applications (see [2]). In recent years, some best extensions and a new applications are given for inequality (1.1) by introducing a parameter $\lambda$ and the $\beta$ function (see [3]-[5]). In 2005, Yang gave the following extended form of (1.1) with several parameters (see [6], (3.1)):

If $(p,q),(r,s)$ are two pairs of conjugate exponents, and $p>1, r>1, 0<\lambda\leq \min\{r,s\}$ $a_n, b_n\geq 0$, then

$$\begin{aligned}&\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_m b_n}{(m+n+1)^{\lambda}}\\ &<B(\frac{\lambda}{r},\frac{\lambda}{s})\{\sum_{n=0}^{\infty}(n+\frac{1}{2})^{p(1-\frac{\lambda}{r})-1}a_n^p\}^{\frac{1}{p}}\{\sum_{n=0}^{\infty}(n+\frac{1}{2})^{q(1-\frac{\lambda}{s})-1}b_n^q\}^{\frac{1}{q}},\end{aligned} \tag{1.2}$$

where the constant factor $B(\frac{\lambda}{r},\frac{\lambda}{s})$ is the best possible. Inequality (1.2) turns into

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(1.1) when $\lambda = 1, s = p, r = q$.

Under the same condition of (1.1), we have the Hilbert-type inequality (see [1], Th. 319, Th. 341) similar to (1.1) as:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m,n\}} < pq(\sum_{n=1}^{\infty}a_n^p)^{1/p}(\sum_{n=1}^{\infty}b_n^q)^{1/q}, \tag{1.3}$$

where the constant factor $pq$ is the best possible.

Recently, Yang (see [7]) gave the following form which is shaper than inequality (1.3) by introducing a parameter $\alpha \geq \frac{3}{4}$:

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_m b_n}{\max\{m,n\}+\alpha} < pq(\sum_{n=0}^{\infty}a_n^p)^{1/p}(\sum_{n=0}^{\infty}b_n^q)^{1/q}, \tag{1.4}$$

where the constant factor $pq$ is also the best possible. Obviously, inequality (1.4) reduces to (1.3) when $\alpha = 1$.

It is a difficult problem to discuss the reverse forms of inequalities (1.1) and (1.3) before introducing some parameters and the $\beta$ function. In 2004, Yang (see [8]) gave the reverse form of Hilbert's double series inequality as follows: Suppose $0 < p < 1, \frac{1}{p} + \frac{1}{q} = 1$, then

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_m b_n}{(m+n+1)^2} > 2\{\sum_{n=0}^{\infty}[1-\frac{1}{4(n+1)}]\frac{a_n^p}{2n+1}\}^{\frac{1}{p}}\{\sum_{n=0}^{\infty}\frac{b_n^q}{2n+1}\}^{\frac{1}{q}}, \tag{1.5}$$

where the constant factor 2 is the best possible.

In 2006, Yang (see [9]) established the estimate value of reminder for the Euler-Maclaurin summation formula in mild conditions and gave the reversed version of Hilbert-type inequality as follows:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m^2+n^2} > \frac{\pi}{2}\{\sum_{n=1}^{\infty}[1-\frac{3}{2\pi n}]\frac{a_n^p}{n}\}^{1/p}\{\sum_{n=1}^{\infty}\frac{b_n^q}{n}\}^{1/q}, \tag{1.6}$$

where the constant factor $\frac{\pi}{2}$ is the best possible. And Yang also gave a strengthened version of inequality (1.6) as:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m^2+n^2} > \frac{\pi}{2}\{\sum_{n=1}^{\infty}[1-\frac{3}{2\pi n}]\frac{a_n^p}{n}\}^{1/p}\{\sum_{n=1}^{\infty}(1-\frac{1}{2\pi n})\frac{b_n^q}{n}\}^{1/q}. \tag{1.7}$$

In this paper, by introducing two parameters $\lambda, \alpha$ and two pairs of conjugate exponents, and estimating the weight coefficient, we establish a reverse version of the extended form of (1.4). As applications, we give its strengthened version and the equivalent form as well.

Hence we will use the improved Euler-Maclaurin's summation formula as follows (see [10], [11]):

Suppose the function $f(x)$ is a smooth piecemeal on $[0,\infty)$, $(-1)^i f^{(i)}(x) \geq 0, f^{(i)}(\infty) = 0 (i = 0,1,2,3)$ and $\int_0^\infty f(x)dx < \infty$, then

$$\int_0^\infty f(x)dx + \frac{1}{2}f(0) \leq \sum_{n=0}^\infty f(n) \leq \int_0^\infty f(x)dx + \frac{1}{2}f(0) - \frac{1}{12}f'(0). \tag{1.8}$$

Inequality (1.8) takes the form of strict inequality as $f(x)$ satisfies $(-1)^i f^{(i)}(x) > 0 (i = 0,1,2,3)$.

## 2. Some lemmas

**Lemma 2.1.** *Suppose $r > 1$, then*

(1) *when $\lambda > 0, \alpha \geq \frac{2}{3}\lambda + \frac{1}{12}$, we have*

$$F(\lambda,\alpha) := \frac{1}{\lambda}\alpha^2 - \frac{1}{2}\alpha - \frac{1}{12} \geq \frac{5\lambda}{48} > 0, \frac{5\lambda^2}{48r\alpha^2} < 1 \textit{ and } \frac{\lambda}{2s\alpha} < 1; \tag{2.1}$$

(2) *when $\lambda > 1, \alpha \geq \frac{2}{3}\lambda + \frac{1}{12}$, we get*

$$2\alpha - 1 > \frac{5\lambda^2}{24\alpha}. \tag{2.2}$$

*Proof.* By the condition of (1), we find

$$\begin{aligned} F(\lambda,\alpha) &= \frac{-\lambda - 6\lambda\alpha + 12\alpha^2}{12\lambda} = \frac{6\alpha(2\alpha-\lambda) - \lambda}{12\lambda} \\ &\geq \frac{(4\lambda + \frac{1}{2})(\frac{1}{3}\lambda + \frac{1}{6}) - \lambda}{12\lambda} = \frac{15\lambda^2 + (\lambda-1)^2}{144\lambda} \geq \frac{5\lambda}{48} > 0; \end{aligned}$$

at the same time, we obtain

$$48r\alpha^2 > 48(\frac{2}{3}\lambda + \frac{1}{12})^2 = \frac{64}{3}\lambda^2 + \frac{16}{3}\lambda + \frac{1}{3} > 5\lambda^2,$$

$$2s\alpha > 2(\frac{2}{3}\lambda + \frac{1}{12}) > \lambda.$$

Hence we get inequality (2.1).

By the condition of (2), we find

$$\begin{aligned} 24\alpha(2\alpha - 1) &\geq (16\lambda + 2)(\frac{4}{3}\lambda - \frac{5}{6}) \\ &= 9\lambda^2 + 7(\lambda^2 - 1) + \frac{16}{3}(\lambda - 1)^2 > 5\lambda^2. \end{aligned}$$

So we complete to prove inequality (2.2). □

Suppose $(r,s)$ is one pair of conjugate exponents ($\frac{1}{r}+\frac{1}{s}=1$ and $r>1$). Define the weight coefficient $\omega_m(s,\lambda,\alpha)$ as:

$$\omega_m(s,\lambda,\alpha) := \sum_{n=0}^{\infty}\frac{(m+\alpha)^{\lambda/r}}{(\max\{m,n\}+\alpha)^{\lambda}}\left(\frac{1}{n+\alpha}\right)^{1-\lambda/s}\ (m\in N_0). \tag{2.3}$$

**Lemma 2.2.** *Suppose $r>1, \frac{1}{r}+\frac{1}{s}=1, 0<\lambda\le\min\{r,s\}, \alpha\ge\frac{2}{3}\lambda+\frac{1}{12}, m\in N_0$. Then we get the following bilateral inequality:*

$$0<\frac{rs}{\lambda}\left[1-\frac{\theta_1(\lambda,s,\alpha)}{(m+\alpha)^{\lambda/s}}\right]<\omega_m(s,\lambda,\alpha)<\frac{rs}{\lambda}\left[1-\frac{\theta_2(\lambda,s,\alpha)}{(m+\alpha)^{\lambda/s}}\right]<\frac{rs}{\lambda}. \tag{2.4}$$

*where* $0<\theta_1(\lambda,s,\alpha)=(1-\frac{\lambda}{2s\alpha})\alpha^{\frac{\lambda}{s}}<\alpha^{\frac{\lambda}{s}}, \theta_2(\lambda,s,\alpha)=\frac{5\lambda^2}{48r}\alpha^{\frac{\lambda}{s}-2}$

*Proof.* Setting $f_m(x)=\frac{1}{(\max\{x,m\}+\alpha)^{\lambda}}(\frac{1}{x+\alpha})^{1-\lambda/s}$ $(m\in N_0), x\in[0,\infty)$. First for $m\in N_+$, when $\lambda<s$, we find

$$f_m(x)=\begin{cases}(x+\alpha)^{-1-\lambda/r}, & x\ge m,\\ (m+\alpha)^{-\lambda}(x+\alpha)^{-1+\lambda/s}, & x<m.\end{cases} \tag{2.5}$$

When $\lambda=s$, we have

$$f_m(x)=\frac{1}{(\max\{x,m\}+\alpha)^{\lambda}}=\begin{cases}(x+\alpha)^{-s}, & x\ge m,\\ (m+\alpha)^{-s}, & x<m.\end{cases} \tag{2.6}$$

By (2.5) and (2.6), it follows

$$f_m(0)=\frac{\alpha^{-1+\lambda/s}}{(m+\alpha)^{\lambda}}\ (\lambda\le s),$$

$$f'_m(0)=\begin{cases}(-1+\frac{\lambda}{s})\alpha^{-2+\lambda/s}(m+\alpha)^{-\lambda}, & \lambda<s,\\ 0, & \lambda=s,\end{cases}\ (m\ge 1). \tag{2.7}$$

When $\lambda<s$, by (2.5), we have

$$\begin{aligned}\int_0^{\infty}f_m(x)dx &= \int_m^{\infty}(x+\alpha)^{-1-\lambda/r}dx+(m+\alpha)^{-\lambda}\int_0^m(x+\alpha)^{-1+\lambda/s}dx\\ &= \frac{r}{\lambda}(m+\alpha)^{-\lambda/r}+\frac{s}{\lambda}(m+\alpha)^{-\lambda/r}-\frac{s}{\lambda}\alpha^{\lambda/s}(m+\alpha)^{-\lambda}\end{aligned} \tag{2.8}$$

When $\lambda=s$, by (2.6),

$$\begin{aligned}\int_0^{\infty}f_m(x)dx &= \int_m^{\infty}(x+\alpha)^{-s}dx+(m+\alpha)^{-s}\int_0^m dx\\ &= (r-1)(m+\alpha)^{1-s}+m(m+\alpha)^{-s}.\end{aligned} \tag{2.9}$$

It is obvious that (2.5) and (2.6) satisfy the condition of (1.8). In particular, (1.8) takes strictly inequality when $\lambda < s, 0 < \lambda \leq \min\{r, s\}$. Thus we obtain the following inequality by (1.8), (2.3) (2.7), (2.8) and (2.1):

$$\begin{aligned}
\omega_m(s,\lambda,\alpha) &= \sum_{n=0}^{\infty} \frac{(m+\alpha)^{\lambda/r}}{(\max\{m,n\}+\alpha)^{\lambda}}(\frac{1}{n+\alpha})^{1-\lambda/s} \\
&= (m+\alpha)^{\lambda/r}\sum_{n=0}^{\infty} f_m(n) \\
&< (m+\alpha)^{\lambda/r}\int_0^{\infty} f_m(x)dx + \frac{1}{2}(m+\alpha)^{\lambda/r}f(0) - \frac{1}{12}(m+\alpha)^{\lambda/r}f'(0) \\
&= \frac{rs}{\lambda} + \alpha^{-2+\lambda/s}[-\frac{s}{\lambda}\alpha^2 + \frac{\alpha}{2} + \frac{1}{12}(1-\frac{\lambda}{s})](m+\alpha)^{-\lambda/s} \\
&< \frac{rs}{\lambda} - \frac{1}{\alpha^2}(\frac{s}{\lambda}\alpha^2 - \frac{\alpha}{2} - \frac{1}{12})(\frac{\alpha}{m+\alpha})^{\lambda/s} \\
&< \frac{rs}{\lambda}[1 - \frac{\lambda}{r\alpha^2}(\frac{1}{\lambda}\alpha^2 - \frac{\alpha}{2} - \frac{1}{12})(\frac{\alpha}{m+\alpha})^{\lambda/s}] \\
&\leq \frac{rs}{\lambda}[1 - \frac{5\lambda^2}{48r\alpha^2}(\frac{\alpha}{m+\alpha})^{\lambda/s}] = \frac{rs}{\lambda}[1 - \frac{\theta_2(\lambda,s,\alpha)}{(m+\alpha)^{\lambda/s}}] < \frac{rs}{\lambda}. \qquad (2.10)
\end{aligned}$$

At the same time, we have

$$\begin{aligned}
\omega_m(s,\lambda,\alpha) &= (m+\alpha)^{\lambda/r}\sum_{n=0}^{\infty} f_m(n) \\
&> (m+\alpha)^{\lambda/r}\int_0^{\infty} f_m(x)dx + \frac{1}{2}(m+\alpha)^{\lambda/r}f_m(0) \\
&= \frac{rs}{\lambda} - (\frac{s}{\lambda} - \frac{1}{2\alpha})(\frac{\alpha}{m+\alpha})^{\lambda/s} = \frac{rs}{\lambda}[1 - \frac{1}{r}(1-\frac{\lambda}{2s\alpha})(\frac{\alpha}{m+\alpha})^{\lambda/s}] \\
&> \frac{rs}{\lambda}[1 - \frac{(1-\frac{\lambda}{2s\alpha})\alpha^{\lambda/s}}{(m+\alpha)^{\lambda/s}}] = \frac{rs}{\lambda}[1 - \frac{\theta_1(\lambda,s,\alpha)}{(m+\alpha)^{\lambda/s}}]. \qquad (2.11)
\end{aligned}$$

When $\lambda = s \leq r$, by (1.8), (2.3), (2.7), (2.9), (2.2) and $\alpha \geq \frac{2}{3}\lambda + \frac{1}{12} > \frac{3}{4}$, we get

$$\begin{aligned}
\omega_m(s,\lambda,\alpha) &= \sum_{n=0}^{\infty} \frac{(m+\alpha)^{s/r}}{(\max\{m,n\}+\alpha)^{s}} = (m+\alpha)^{s-1}\sum_{n=0}^{\infty} f_m(n) \\
&= (m+\alpha)^{s-1}\int_0^{\infty} f_m(x)dx + \frac{f_m(0)}{2}(m+\alpha)^{s-1} \\
&= r - 1 + \frac{2m+1}{2(m+\alpha)} = r - \frac{2\alpha-1}{2(m+\alpha)} \\
&< r - \frac{5\lambda^2}{48\alpha(m+\alpha)} = r[1 - \frac{5\lambda^2}{48r\alpha(m+\alpha)}] < r. \qquad (2.12)
\end{aligned}$$

At the same time,

$$\begin{aligned}\omega_m(s,\lambda,\alpha) &= (m+\alpha)^{s-1}\sum_{n=0}^{\infty} f_m(n) = r - \frac{2\alpha-1}{2(m+\alpha)} \\ &= r[1-\frac{2\alpha-1}{2r(m+\alpha)}] > r[1-\frac{2\alpha-1}{2(m+\alpha)}] = r[1-(1-\frac{1}{2\alpha})\frac{\alpha}{(m+\alpha)}]. \quad (2.13)\end{aligned}$$

Hence (2.4) is correct by (2.10), (2.11), (2.12), (2.13) when $m \in N_+$.

Then we consider the condition of $m=0$, we get $f_0(x) = (x+\alpha)^{-1-\frac{\lambda}{r}}$ for $0 < \lambda \le \min\{r,s\}, x \in [0,+\infty\}$. Obviously, $f_0(0) = \alpha^{-1-\frac{\lambda}{r}}$, $f_0'(0) = -(1+\frac{\lambda}{r})\alpha^{-2-\frac{\lambda}{r}}$. Hence by (1.8) and (2.3), we have

$$\begin{aligned}\alpha^{\lambda/r}\int_0^{\infty}(x+\alpha)^{-1-\frac{\lambda}{r}}dx + \frac{1}{2\alpha} &< \omega_0(s,\lambda,\alpha) = \alpha^{\lambda/r}\sum_{n=0}^{\infty} f_0(n) \\ &< \alpha^{\lambda/r}\int_0^{\infty}(x+\alpha)^{-1-\frac{\lambda}{r}}dx + \frac{1}{2\alpha} + \frac{1}{12\alpha^2}(1+\frac{\lambda}{r}),\end{aligned}$$

i.e.,

$$\frac{r}{\lambda} + \frac{1}{2\alpha} < \omega_0(s,\lambda,\alpha) < \frac{r}{\lambda} + \frac{1}{2\alpha} + \frac{1}{12\alpha^2}(1+\frac{\lambda}{r}).$$

And by $2\alpha \ge \frac{4}{3}\lambda + \frac{1}{6} > \lambda > \lambda\cdot\frac{r-1}{r}$, we obtain $\frac{r}{\lambda} > \frac{r-1}{2\alpha}$, i.e., $\frac{r}{\lambda} + \frac{1}{2\alpha} > \frac{r}{2\alpha} = \frac{rs}{\lambda}[1-(1-\frac{\lambda}{2s\alpha})]$. By (2.1), we obtain

$$\begin{aligned}&\frac{rs}{\lambda}(1-\frac{5\lambda^2}{48r\alpha^2}) - \frac{r}{\lambda} - \frac{1}{2\alpha} - \frac{1}{12\alpha^2}(1+\frac{\lambda}{r}) \\ &= \frac{1}{\alpha^2}(\frac{s}{\lambda}\alpha^2 - \frac{\alpha}{2} - \frac{1}{12} - \frac{5s\lambda}{48} - \frac{\lambda}{12r}) \\ &= \frac{1}{\lambda\alpha^2}(\frac{s}{r}\alpha^2 + \lambda F(\lambda,\alpha) - \frac{5s\lambda^2}{48} - \frac{\lambda^2}{12r}) \\ &\ge \frac{1}{\lambda\alpha^2}(\frac{s}{r}\alpha^2 + \frac{5(1-s)\lambda^2}{48} - \frac{\lambda^2}{12r}) \\ &= \frac{s-1}{\lambda\alpha^2}(\alpha^2 - \frac{5\lambda^2}{48} - \frac{\lambda^2}{12s}) \\ &\ge \frac{s-1}{\lambda\alpha^2}[(\frac{2}{3}\lambda + \frac{1}{12})^2 - \frac{5\lambda^2}{48} - \frac{\lambda^2}{12s}] \\ &= \frac{(s-1)[(49s-12)\lambda^2 + s(16\lambda+1)]}{144\lambda s\alpha^2} > 0\end{aligned}$$

i.e.,

$$\frac{r}{\lambda} + \frac{1}{2\alpha} + \frac{1}{12\alpha^2}(1+\frac{\lambda}{r}) < \frac{rs}{\lambda}(1-\frac{5\lambda^2}{48r\alpha^2}).$$

Hence (2.4) is correct for $m=0$. The lemma is proved. □

**Remark.** By symmetry, (2.4) is still correct as $s$ and $m$ are replaced by $r$ and $n$ respectively.

**Lemma 2.3.** *Suppose* $(p,q),(r,s)$ *are two pairs of conjugate exponents, and* $0<p<1, r>1, 0<\lambda\le\min\{r,s\}, \alpha\ge\frac{2}{3}\lambda+\frac{1}{12}$, *then we get the following inequality for* $0<\varepsilon<\frac{p\lambda}{r}$ :

$$\begin{aligned}\tilde{I} &:= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(m+\alpha)^{-1-\frac{\varepsilon}{p}+\frac{\lambda}{r}}(n+\alpha)^{-1-\frac{\varepsilon}{q}+\frac{\lambda}{s}}}{(\max\{m,n\}+\alpha)^{\lambda}}\\ &< \frac{\lambda}{(-\frac{\varepsilon}{p}+\frac{\lambda}{r})(\frac{\varepsilon}{p}+\frac{\lambda}{s})}\sum_{n=1}^{\infty}\frac{1}{(n+\alpha)^{1+\varepsilon}}. \qquad (2.14)\end{aligned}$$

*Proof.* Setting $g(x)=\frac{1}{(\max\{x,n\}+\alpha)^{\lambda}}(x+\alpha)^{-1-\frac{\varepsilon}{p}+\frac{\lambda}{r}}$. Obviously, $g(x)$ is decreasing in $(-\alpha,\infty)$, thus

$$\begin{aligned}\sum_{m=1}^{\infty}g(m) &< \int_{-\alpha}^{\infty}g(x)dx=\int_{-\alpha}^{\infty}\frac{(x+\alpha)^{-1-\frac{\varepsilon}{p}+\frac{\lambda}{r}}}{(\max\{x,n\}+\alpha)^{\lambda}}dx\\ &= \int_{-\alpha}^{n}\frac{(x+\alpha)^{-1-\frac{\varepsilon}{p}+\frac{\lambda}{r}}}{(n+\alpha)^{\lambda}}dx+\int_{n}^{\infty}\frac{(x+\alpha)^{-1-\frac{\varepsilon}{p}+\frac{\lambda}{r}}}{(x+\alpha)^{\lambda}}dx\\ &= \frac{\lambda(n+\alpha)^{-\frac{\varepsilon}{p}-\frac{\lambda}{s}}}{(-\frac{\varepsilon}{p}+\frac{\lambda}{r})(\frac{\varepsilon}{p}+\frac{\lambda}{s})}. \qquad (2.15)\end{aligned}$$

By (2.15), we find

$$\begin{aligned}\tilde{I} &:= \sum_{n=1}^{\infty}(n+\alpha)^{-1-\frac{\varepsilon}{q}+\frac{\lambda}{s}}\sum_{m=1}^{\infty}\frac{(m+\alpha)^{-1-\frac{\varepsilon}{p}+\frac{\lambda}{r}}}{(\max\{m,n\}+\alpha)^{\lambda}}\\ &= \sum_{n=1}^{\infty}(n+\alpha)^{-1-\frac{\varepsilon}{q}+\frac{\lambda}{s}}\sum_{m=1}^{\infty}g(m)\\ &< \sum_{n=1}^{\infty}(n+\alpha)^{-1-\frac{\varepsilon}{q}+\frac{\lambda}{s}}[\frac{\lambda(n+\alpha)^{-\frac{\varepsilon}{p}-\frac{\lambda}{s}}}{(-\frac{\varepsilon}{p}+\frac{\lambda}{r})(\frac{\varepsilon}{p}+\frac{\lambda}{s})}]\\ &= \frac{\lambda}{(-\frac{\varepsilon}{p}+\frac{\lambda}{r})(\frac{\varepsilon}{p}+\frac{\lambda}{s})}\sum_{n=1}^{\infty}\frac{1}{(n+\alpha)^{1+\varepsilon}}.\end{aligned}$$

Hence (2.14) is true. The lemma is proved. □

## 3. Main results and applications

**Theorem 3.1.** *Suppose* $(p,q),(r,s)$ *are two pairs of conjugate exponents, and* $0<$

$p < 1$, $r > 1, 0 < \lambda \leq \min\{r, s\}, \alpha \geq \frac{2}{3}\lambda + \frac{1}{12}$, $a_n, b_n \geq 0$, *such that* $0 < \sum_{n=0}^{\infty}(n + \alpha)^{p(1-\frac{\lambda}{r})-1}a_n^p < \infty$ *and* $0 < \sum_{n=0}^{\infty}(n+\alpha)^{q(1-\frac{\lambda}{s})-1}b_n^q < \infty$*, then*

$$\begin{aligned} I &:= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_m b_n}{(\max\{m,n\}+\alpha)^\lambda} \\ &> \frac{rs}{\lambda}\{\sum_{n=0}^{\infty}[1-\frac{\theta_1(\lambda,s,\alpha)}{(n+\alpha)^{\frac{\lambda}{s}}}](n+\alpha)^{p(1-\frac{\lambda}{r})-1}a_n^p\}^{\frac{1}{p}}\{\sum_{n=0}^{\infty}(n+\alpha)^{q(1-\frac{\lambda}{s})-1}b_n^q\}^{\frac{1}{q}}, \end{aligned} \tag{3.1}$$

*where* $0 < \theta_1(\lambda, s, \alpha) = (1 - \frac{\lambda}{2s\alpha})\alpha^{\frac{\lambda}{s}} < \alpha^{\frac{\lambda}{s}}$ *and the constant factor* $\frac{rs}{\lambda}$ *is the best possible. And* (3.1) *can be strengthened as*

$$\begin{aligned} I &> \frac{rs}{\lambda}\{\sum_{n=0}^{\infty}[1-\frac{\theta_1(\lambda,s,\alpha)}{(n+\alpha)^{\frac{\lambda}{s}}}](n+\alpha)^{p(1-\frac{\lambda}{r})-1}a_n^p\}^{\frac{1}{p}} \\ &\quad \times\{\sum_{n=0}^{\infty}[1-\frac{\theta_2(\lambda,s,\alpha)}{(n+\alpha)^{\frac{\lambda}{s}}}](n+\alpha)^{q(1-\frac{\lambda}{s})-1}b_n^q\}^{\frac{1}{q}}, \end{aligned} \tag{3.2}$$

*where* $\theta_2(\lambda, s, \alpha) = \frac{5\lambda^2}{48r}\alpha^{\frac{\lambda}{s}-2}$.

*Proof.* Applying Hölder's inequality (see [12]) and (2.3), we have

$$\begin{aligned} I &:= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_m b_n}{(\max\{m,n\}+\alpha)^\lambda} \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{1}{(\max\{m,n\}+\alpha)^\lambda}[\frac{(m+\alpha)^{(1-\frac{\lambda}{r})/q}}{(n+\alpha)^{(1-\frac{\lambda}{s})/p}}a_m][\frac{(n+\alpha)^{(1-\frac{\lambda}{s})/p}}{(m+\alpha)^{(1-\frac{\lambda}{r})/q}}b_n] \\ &\geq \{\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(m+\alpha)^{(p-1)(1-\frac{\lambda}{r})}}{(\max\{m,n\}+\alpha)^\lambda}(\frac{1}{n+\alpha})^{1-\frac{\lambda}{s}}a_m^p\}^{\frac{1}{p}} \\ &\quad \times\{\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(n+\alpha)^{(q-1)(1-\frac{\lambda}{s})}}{(\max\{m,n\}+\alpha)^\lambda}(\frac{1}{m+\alpha})^{1-\frac{\lambda}{r}}b_n^q\}^{\frac{1}{q}} \\ &= \{\sum_{m=0}^{\infty}\omega_m(s,\lambda,\alpha)(m+\alpha)^{p(1-\frac{\lambda}{r})-1}a_m^p\}^{\frac{1}{p}}\{\sum_{n=0}^{\infty}\omega_n(r,\lambda,\alpha)(n+\alpha)^{q(1-\frac{\lambda}{s})-1}b_n^q\}^{\frac{1}{q}}. \end{aligned} \tag{3.3}$$

Since $0 < p < 1, q < 0$, by (2.4) and the remark of Lemma 2, (3.1) can be seen. And by the right hand side of (2.4), we get (3.2). For $q < 0$, we have

$$\{\frac{rs}{\lambda}\sum_{n=0}^{\infty}[1-\frac{\theta_2(\lambda,s,\alpha)}{(n+\alpha)^{\frac{\lambda}{s}}}](n+\alpha)^{q(1-\frac{\lambda}{s})-1}b_n^q\}^{\frac{1}{q}} > \{\sum_{n=0}^{\infty}\omega_n(r,\lambda,\alpha)(n+\alpha)^{q(1-\frac{\lambda}{s})-1}b_n^q\}^{\frac{1}{q}}.$$

(3.1) can be deduced from (3.2), hence (3.2) is the strengthened version of (3.1).

If the constant factor $\frac{rs}{\lambda}$ in (3.1) is not the best possible, then there exists a positive constant $k$ (with $k < \frac{rs}{\lambda}$). For $0 < \varepsilon < \frac{p\lambda}{r}$, in particular, setting $\tilde{a}_0 = 0, \tilde{b}_0 = 0$; $\tilde{a}_n = (n+\alpha)^{-1-\frac{\varepsilon}{p}+\frac{\lambda}{r}}, \tilde{b}_n = (n+\alpha)^{-1-\frac{\varepsilon}{q}+\frac{\lambda}{s}}$, $n \in N$, by the assumption, we find

$$\begin{aligned}
\tilde{I} &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\tilde{a}_m\tilde{b}_n}{(\max\{m,n\}+\alpha)^{\lambda}} \\
&> k\{\sum_{n=1}^{\infty}[1-\frac{\theta_1(\lambda,s,\alpha)}{(n+\alpha)^{\frac{\lambda}{s}}}](n+\alpha)^{p(1-\frac{\lambda}{r})-1}\tilde{a}_n^p\}^{\frac{1}{p}}\{\sum_{n=1}^{\infty}(n+\alpha)^{q(1-\frac{\lambda}{s})-1}\tilde{b}_n^q\}^{\frac{1}{q}} \\
&= k\{\sum_{n=1}^{\infty}[1-\frac{\theta_1(\lambda,s,\alpha)}{(n+\alpha)^{\frac{\lambda}{s}}}]\frac{1}{(n+\alpha)^{1+\varepsilon}}\}^{\frac{1}{p}}\{\sum_{n=1}^{\infty}\frac{1}{(n+\alpha)^{1+\varepsilon}}\}^{\frac{1}{q}} \\
&= k[1-o(1)]^{1/p}\sum_{n=1}^{\infty}\frac{1}{(n+\alpha)^{1+\varepsilon}}. \qquad (3.4)
\end{aligned}$$

In view of (2.14) and (3.4), we get

$$\frac{\lambda}{(-\frac{\varepsilon}{p}+\frac{\lambda}{r})(\frac{\varepsilon}{p}+\frac{\lambda}{s})}\sum_{n=1}^{\infty}\frac{1}{(n+\alpha)^{1+\varepsilon}} > k[1-o(1)]^{1/p}\sum_{n=1}^{\infty}\frac{1}{(n+\alpha)^{1+\varepsilon}},$$

i.e., $\frac{\lambda}{(-\frac{\varepsilon}{p}+\frac{\lambda}{r})(\frac{\varepsilon}{p}+\frac{\lambda}{s})} > k[1-o(1)]^{1/p}$ for $\varepsilon \to 0^+$, it follows that $k \le \frac{rs}{\lambda}$, which contradicts the fact that $k > \frac{rs}{\lambda}$. Hence the constant factor $\frac{rs}{\lambda}$ in (3.1) is the best possible. The theorem is proved. □

**Theorem 3.2.** *Suppose* $(p,q),(r,s)$ *are two pairs of conjugate exponents, and* $0 < p < 1$, $r > 1, 0 < \lambda \le \min\{r,s\}, \alpha \ge \frac{2}{3}\lambda + \frac{1}{12}$, $a_n, b_n \ge 0$ *such that* $0 < \sum_{n=0}^{\infty}(n+\alpha)^{p(1-\frac{\lambda}{r})-1}a_n^p < \infty$, *then*

$$\begin{aligned}
J &:= \sum_{n=0}^{\infty}(n+\alpha)^{\frac{p\lambda}{s}-1}[\sum_{m=0}^{\infty}\frac{a_m}{(\max\{m,n\}+\alpha)^{\lambda}}]^p \\
&> (\frac{rs}{\lambda})^p\sum_{n=0}^{\infty}[1-\frac{\theta_1(\lambda,s,\alpha)}{(n+\alpha)^{\frac{\lambda}{s}}}](n+\alpha)^{p(1-\frac{\lambda}{r})-1}a_n^p, \qquad (3.5)
\end{aligned}$$

*where* $0 < \theta_1(\lambda,s,\alpha) = (1-\frac{\lambda}{2s\alpha})\alpha^{\frac{\lambda}{s}} < \alpha^{\frac{\lambda}{s}}$, *and the constant factor* $(\frac{rs}{\lambda})^p$ *is the best possible. Inequality* (3.5) *is equivalent to* (3.1).

*Proof.* Setting $b_n$ as

$$b_n := (n+\alpha)^{\frac{p\lambda}{s}-1}[\sum_{m=0}^{\infty}\frac{a_m}{(\max\{m,n\}+\alpha)^{\lambda}}]^{p-1}\ (n\in N_0)$$

then $b_n > 0$ and $\sum_{n=0}^{\infty}(n+\alpha)^{q(1-\frac{\lambda}{s})-1}b_n^q = J > 0$. If $J = \infty$, then (3.5) is naturally valid; if $0 < J < \infty$, then by (3.1), we find

$$\sum_{n=0}^{\infty}(n+\alpha)^{q(1-\frac{\lambda}{s})-1}b_n^q = J = I$$

$$(3.6)\quad > \frac{rs}{\lambda}\{\sum_{n=0}^{\infty}[1-\frac{\theta_1(\lambda,s,\alpha)}{(n+\alpha)^{\frac{\lambda}{s}}}](n+\alpha)^{p(1-\frac{\lambda}{r})-1}a_n^p\}^{\frac{1}{p}}\{\sum_{n=0}^{\infty}(n+\alpha)^{q(1-\frac{\lambda}{s})-1}b_n^q\}^{\frac{1}{q}}.$$

Thus

$$(3.7)\quad \sum_{n=0}^{\infty}(n+\alpha)^{q(1-\frac{\lambda}{s})-1}b_n^q = J > (\frac{rs}{\lambda})^p\sum_{n=0}^{\infty}[1-\frac{\theta_1(\lambda,s,\alpha)}{(n+\alpha)^{\frac{\lambda}{s}}}](n+\alpha)^{p(1-\frac{\lambda}{r})-1}a_n^p,$$

and (3.5) is correct.

On the other hand, suppose (3.5) is valid. By Hölder's inequality, we find

$$\begin{aligned} I &= \sum_{n=0}^{\infty}\{(n+\alpha)^{\frac{\lambda}{s}-\frac{1}{p}}\sum_{m=0}^{\infty}\frac{a_m}{(\max\{m,n\}+\alpha)^{\lambda}}\}\{(n+\alpha)^{\frac{1}{p}-\frac{\lambda}{s}}b_n\} \\ (3.8)\ &\geq \{\sum_{n=0}^{\infty}(n+\alpha)^{\frac{p\lambda}{s}-1}[\sum_{m=0}^{\infty}\frac{a_m}{(\max\{m,n\}+\alpha)^{\lambda}}]^p\}^{\frac{1}{p}}\{\sum_{n=0}^{\infty}(n+\alpha)^{q(1-\frac{\lambda}{s})-1}b_n^q\}^{\frac{1}{q}}. \end{aligned}$$

By (3.5) and $0 < p < 1$, we get (3.1). Hence (3.1) is equivalent to (3.5). If the constant factor $(\frac{rs}{\lambda})^p$ in (3.5) is not the best possible, then by (3.8), it contradicts the fact that $\frac{rs}{\lambda}$ in (3.1) is not the best possible too. Thus the constant factor $(\frac{rs}{\lambda})^p$ in (3.5) is the best possible. The theorem is proved. □

**Remark.** (i) By Th. 3.1, for $\lambda = 1, \alpha \geq \frac{2}{3}+\frac{1}{12}=\frac{3}{4}$, (3.1) reduces to

(3.9)
$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_m b_n}{\max\{m,n\}+\alpha} > rs\{\sum_{n=0}^{\infty}[1-\frac{\theta_1(s,\alpha)}{(n+\alpha)^{\frac{1}{s}}}](n+\alpha)^{\frac{p}{s}-1}a_n^p\}^{\frac{1}{p}}\{\sum_{n=0}^{\infty}(n+\alpha)^{\frac{q}{r}-1}b_n^q\}^{\frac{1}{q}},$$

where $\theta_1(s,\alpha) = (1-\frac{1}{2s\alpha})\alpha^{\frac{1}{s}}$, and (3.9) can be strengthened as

$$(3.10)\quad I > rs\{\sum_{n=0}^{\infty}[1-\frac{\theta_1(s,\alpha)}{(n+\alpha)^{\frac{1}{s}}}](n+\alpha)^{\frac{p}{s}-1}a_n^p\}^{\frac{1}{p}}\{\sum_{n=0}^{\infty}[1-\frac{\theta_2(s,\alpha)}{(n+\alpha)^{\frac{1}{s}}}](n+\alpha)^{\frac{q}{r}-1}b_n^q\}^{\frac{1}{q}},$$

where $\theta_2(s,\alpha) = \frac{5}{48r}\alpha^{\frac{1}{s}-2}$. By the characteristics of the constant factor of (1.4) and (3.9), it shows that the problem of reverse version of (1.4) can be solved by introducing two pairs of conjugate exponents independent. Thus (3.9) is the reverse version of (1.4).

When $\lambda = 1, \alpha \geq \frac{3}{4}, r = \frac{1}{p}, s = \frac{1}{1-p}, 0 < p < 1$, (3.9) becomes to

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_m b_n}{\max\{m,n\}+\alpha}$$
$$> \frac{1}{p(1-p)}\{\sum_{n=0}^{\infty}[1-\frac{\theta_1(p,\alpha)}{(n+\alpha)^{1-p}}](n+\alpha)^{(1-p)p-1}a_n^p\}^{\frac{1}{p}}\{\sum_{n=0}^{\infty}(n+\alpha)^{pq-1}b_n^q\}^{\frac{1}{q}}. \tag{3.11}$$

And by (3.5), we obtain the equivalent form of (3.11) as

$$\sum_{n=0}^{\infty}(n+\alpha)^{(1-p)p-1}[\sum_{m=0}^{\infty}\frac{a_m}{\max\{m,n\}+\alpha}]^p$$
$$> [\frac{1}{p(1-p)}]^p\sum_{n=0}^{\infty}[1-\frac{\theta_1(p,\alpha)}{(n+\alpha)^{1-p}}](n+\alpha)^{(1-p)p-1}a_n^p, \tag{3.12}$$

where $\theta_1(p,\alpha) = (1-\frac{1-p}{2\alpha})\alpha^{1-p}$.

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