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Embedding in a Direct Limit

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Abstract. Let $\mathbf{X}$ be a direct system of topological spaces and X its direct limit. We show that under certain conditions the direct limit of a "subsystem" of $\mathbf{X}$ embeds in a canonical way as a closed subspace of X.

1. Introduction

For any relation $\preceq$ on a set A and subset B of A, we shall let $\preceq_B$ denote the relation $\preceq$ restricted to B. Let $\mathbf{X} = (X_a, p_a^b, (A, \preceq))$ be a direct system (Section 2) of spaces and maps and $X = \text{dirlim}\,\mathbf{X}$. Suppose that $B \subset A$ and $(B, \preceq_B)$ is a directed set. We shall call $(B, \preceq_B)$ a *sub-directed set* of $(A, \preceq)$. Observe that $\mathbf{Y} = (X_a, p_a^b, (B, \preceq_B))$ is a direct system; we let $Y = \text{dirlim}\,\mathbf{Y}$. In this situation, there is a canonical map (Section 2) $\mathbf{f} : \mathbf{Y} \to \mathbf{X}$ of direct systems whose direct limit, $f = \text{dirlim}\,\mathbf{f} : Y \to X$, is a map. Our Main Results are Embedding Theorems 3.4 and 3.5. These show that if $\mathbf{Y}$ satisfies certain "Embedding Conditions" in $\mathbf{X}$, then the map f is a closed embedding. In Section 4 we shall provide an example that illustrates the effectiveness of Theorem 3.5.

2. Direct systems

A description of the notion of a *direct system* of topological spaces and its limit can be found in Appendix II of [1] where it is called a *direct spectrum* (see also [2]). We shall review the main ideas. A *direct system* $\mathbf{X} = (X_a, p_a^b, (A, \preceq))$ (only of topological spaces herein) consists of a directed set $(A, \preceq)$, topological spaces X_a, $a \in A$, and for each $a \preceq b$, a map $p_a^b : X_a \to X_b$ where $p_a^a : X_a \to X_a$ is always the identity map and whenever $a \preceq b \preceq c$, then $p_b^c \circ p_a^b = p_a^c$. The maps p_a^b are called *connecting maps*, and one refers to the spaces X_a as *coordinate spaces*. An equivalence relation $\sim$ is defined on $\sum\{X_a \,|\, a \in A\}$ as follows[1]. Let a, $b \in A$ and

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[1]As a set, $\sum\{X_a \,|\, a \in A\}$ equals $\{X_a \times \{a\} \,|\, a \in A\}$. This is done so that for $a \neq b$, $(X_a \times \{a\}) \cap (X_b \times \{b\}) = \emptyset$. By convention, one ignores the a-coordinate in $X_a \times \{a\}$ and just writes X_a. Because of this, some confusion can arise when speaking of an element x of X_a if also $x \in X_b$. When we write $x \in X_a$ we mean that $x \notin X_b$.

suppose that $x \in X_a$, $y \in X_b$. Then we shall say that $x \sim y$ if and only if there exists $c \in A$, $a \preceq c$, $b \preceq c$ such that $p_a^c(x) = p_b^c(y)$.

Let X be the set of equivalence classes of $\sim$, $p : \sum\{X_a \,|\, a \in A\} \to X$ the quotient function, and X have the quotient topology. Then X is referred to as the *direct limit* of $\mathbf{X}$, and we shall write $X = \operatorname{dirlim} \mathbf{X}$. For each $a \in A$, there is a map $p_a = p|X_a : X_a \to X$. A subset U of X is open (closed) if and only if $p_a^{-1}(U) = p^{-1}(U) \cap X_a$ is open (closed) in X_a for all $a \in A$. Note that whenever $a \preceq b$, then $p_a = p_b \circ p_a^b$.

Suppose that $(B, \preceq_B)$ is a sub-directed set of $(A, \preceq)$, and put

$$\mathbf{Y} = (X_a, p_a^b, (B, \preceq_B)).$$

Then $\mathbf{Y}$ is a direct system which we refer to as a *sub-direct system* of $\mathbf{X}$. Write $Y = \operatorname{dirlim} \mathbf{Y}$ and use $q : \sum\{Y_b \,|\, b \in B\} \to Y$ to designate the quotient map. The inclusion $j : B \to A$ determines a *canonical* map $\mathbf{f} = (f_b)_{b \in B} : \mathbf{Y} \to \mathbf{X}$ of direct systems given so that for each $b \in B$, $f_b : X_b \to X_b$ is the identity. This induces a *canonical* map $f = \operatorname{dirlim} \mathbf{f} : Y \to X$ which operates as follows. If $y \in Y$, then for some $b \in B$ and $t \in X_b$, $y = q(t)$. Then $f(y) = p(t)$.

3. Embedding theorems

We are now going to state "Embedding Conditions" which we shall see (Theorems 3.4 and 3.5) are related to the question of when the canonical map f indicated in Section 2 is a closed embedding.

Definition 3.1. Let $\mathbf{X} = (X_a, p_a^b, (A, \preceq))$ be a direct system and $(B, \preceq_B)$ a sub-directed set of $(A, \preceq)$. Define $\mathbf{Y} = (X_a, p_a^b, (B, \preceq_B))$. The *Embedding Conditions* of $\mathbf{Y}$ in $\mathbf{X}$ are as follows.

(1) if r, $s \in B$, $c \in A$, $r \preceq c$, $s \preceq c$, $y^* \in X_r$, $z^* \in X_s$, and $p_r^c(y^*) = p_s^c(z^*)$, then there exists $e \in B$ such that $r \preceq_B e$, $s \preceq_B e$, and $p_r^e(y^*) = p_s^e(z^*)$,

(2) if b, $d \in A$, $a \in B$, $a \preceq d$, $b \preceq d$, $w \in X_a$, $u \in X_b$, and $p_a^d(w) = p_b^d(u)$, then there exist $a^* \in B$ and $t \in X_{a^*}$ such that $a^* \preceq b$ and $p_{a^*}^b(t) = u$, and

(3) if $c \in A$, $R \subset B$, for each $r \in R$, $r \preceq c$, and $\mathcal{F} = \{p_r^c(X_r) \,|\, r \in R\}$, then $\bigcup \mathcal{F}$ is closed in X_c.

Lemma 3.2. *Suppose that* $\mathbf{X} = (X_a, p_a^b, (A, \preceq))$ *is a direct system,* $B \subset A$, *and* $(B, \preceq_B)$ *is a sub-directed set of* $(A, \preceq)$. *Let* $\mathbf{Y} = (X_a, p_a^b, (B, \preceq_B))$, $X = \operatorname{dirlim} \mathbf{X}$, *and* $Y = \operatorname{dirlim} \mathbf{Y}$. *If* $\mathbf{Y}$ *satisfies the Embedding Condition* (1) *in* $\mathbf{X}$, *then the canonical map* $f : Y \to X$ *is injective.*

Proof. We shall use p for the quotient projection from $\sum\{X_a \,|\, a \in A\}$ to X and q for the quotient projection from $\sum\{X_b \,|\, b \in B\}$ to Y. Suppose that y, $z \in Y$, and $f(y) = f(z)$. Let r, $s \in B$, $y^* \in X_r$, $z^* \in X_s$ be chosen so that $y = q(y^*)$, $z = q(z^*)$. Then by definition of $\mathbf{f}$, $p(y^*) = f(y) = f(z) = p(z^*)$. Hence there exists $c \in A$ such that $r \preceq c$, $s \preceq c$, and $p_r^c(y^*) = p_s^c(z^*)$. By (1) of Definition 3.1, there exists $e \in B$ so that $r \preceq_B e$, $s \preceq_B e$, and $p_r^e(y^*) = p_s^e(z^*)$. This shows that $y = q(y^*) = q(z^*) = z$, so f is injective. $\square$

Let us recall the next notion.

Definition 3.3. A directed set $(A, \preceq)$ is called *cofinite* if for each $a \in A$, $\{b \in A \mid b \preceq a\}$ is finite.

Here is our First Embedding Theorem.

Theorem 3.4. *Let* $\mathbf{X} = (X_a, p_a^b, (A, \preceq))$ *be a direct system with closed connecting maps where* $(A, \preceq)$ *is cofinite. Suppose that* $B \subset A$, $(B, \preceq_B)$ *is a sub-directed set of* $(A, \preceq)$, $\mathbf{Y} = (X_a, p_a^b, (B, \preceq_B))$, $X = \operatorname{dirlim} \mathbf{X}$, *and* $Y = \operatorname{dirlim} \mathbf{Y}$. *If* $\mathbf{Y}$ *satisfies the Embedding Conditions* (1) *and* (2) *in* $\mathbf{X}$, *then the canonical map* $f : Y \to X$ *is an embedding such that* $f(Y)$ *is closed in* X.

Proof. We shall use p for the quotient projection from $\sum\{X_a \mid a \in A\}$ to X and q for the quotient projection from $\sum\{X_b \mid b \in B\}$ to Y.
By Lemma 3.2, f is injective. We shall show that if D is a closed subset of Y, then $f(D)$ is a closed subset of X, and that will complete our proof. For each $b \in B$, let $D_b = q^{-1}(D) \cap X_b$. Then of course D_b is a closed subset of X_b, and $D = q(\bigcup\{D_b \mid b \in B\})$. Also notice that if $a \preceq_B b$, then

(A) $p_a^b(D_a) \subset D_b$, and

(B) if $v \in D_b$, $u \in X_a$, and $p_a^b(u) = v$, then $u \in D_a$.

For each $b \in A \setminus B$, let $\mathcal{S}_b = \{a \in B \mid a \preceq b\}$. Since $(A, \preceq)$ is cofinite, then $\mathcal{S}_b$ is finite. Put $D_b = \bigcup\{p_a^b(D_a) \mid a \in \mathcal{S}_b\}$. Note that p_a^b is a closed map, D_a is closed in X_a for all $a \in \mathcal{S}_b$, and $\mathcal{S}_b$ is finite; hence D_b is closed in X_b.
Define

(C) $D^* = \bigcup\{D_b \mid b \in A\}$.

We would like to show first that

(D) $f(D) = p(D^*)$.

By the definition of the map f as indicated in Section 2, it is certain that $f(D) \subset p(D^*)$. Let $s \in D^*$ and $x = p(s)$. Then for some $b \in A$, $s \in D_b$. If $b \in B$, put $y = q(s) \in D$, and we see that $x = f(y)$. Now suppose that $b \in A \setminus B$. By the definition of D_b, there exists $a \in \mathcal{S}_b$ and $u \in D_a$ such that $p_a^b(u) = s$. It follows that $p(u) = p(s) = x$. If we set $y = q(u)$, then $f(y) = x$. Therefore $p(D^*) \subset f(D)$, and we conclude that (D) is true.
To show that $f(D)$ is closed in X, we shall prove that $p^{-1}(f(D)) \cap X_b = D_b$ for all $b \in A$. From (D) $p^{-1}(f(D)) = p^{-1}(p(D^*))$, so (C) yields that $D_b \subset p^{-1}(f(D)) \cap X_b$. To prove the opposite inclusion, let $u \in p^{-1}(f(D)) \cap X_b$. Of course $p(u) \in p(D^*)$. So for some $c \in A$, there exists $v \in D_c$ with $p(u) = p(v)$. Choose $d \in A$ so that $b \preceq d$, $c \preceq d$, and $p_b^d(u) = p_c^d(v)$.
There exist $a \in \mathcal{S}_c$ and $w \in D_a$ so that $p_a^c(w) = v$. Observe that $a \preceq c$ and $c \preceq d$, so $a \preceq d$. We then have b, $d \in A$, $a \in B$, $a \preceq d$, $b \preceq d$, $w \in X_a$, $u \in X_b$, so that $p_a^d(w) = p_c^d \circ p_a^c(w) = p_c^d(v) = p_b^d(u)$. By (2) of Definition 3.1, there exist

$a^* \in B$ and $t \in X_{a^*}$ so that $a^* \preceq b$ and $p^b_{a^*}(t) = u$. So $p^d_{a^*}(t) = p^d_b \circ p^b_{a^*}(t) = p^d_b(u) = p^d_c(v) = p^d_a(w)$. Noting that a^*, $a \in B$, $d \in A$, $a^* \preceq d$, $a \preceq d$, $t \in X_{a^*}$, $w \in X_a$, and $p^d_{a^*}(t) = p^d_a(w)$, then according to (1) of Definition 3.1, there exists $e \in B$ such that $a^* \preceq_B e$, $a \preceq_B e$, and $p^e_{a^*}(t) = p^e_a(w)$. But $w \in D_a$, which by (A) implies that $p^e_a(w) \in D_e$. Then $p^e_{a^*}(t) \in D_e$, so by (B), $t \in D_{a^*}$. For this reason, $u = p^b_{a^*}(t) \in D_b$, and our proof is complete. □

The next is our Second Embedding Theorem.

Theorem 3.5. *Let $\mathbf{X} = (X_a, p^b_a, (A, \preceq))$ be a direct system. Suppose that $B \subset A$, $(B, \preceq_B)$ is a sub-directed set of $(A, \preceq)$, $\mathbf{Y} = (X_a, p^b_a, (B, \preceq_B))$, $X = \operatorname{dirlim} \mathbf{X}$, and $Y = \operatorname{dirlim} \mathbf{Y}$. If $\mathbf{Y}$ satisfies all three of the Embedding Conditions of Definition 3.1 in $\mathbf{X}$, then the canonical map $f : Y \to X$ is an embedding such that $f(Y)$ is closed in X.*

Proof. This proof is obtained from the proof of Theorem 3.4 by making exactly one alteration. In the definition of D_b for $b \in A \setminus B$, this time $\mathcal{S}_b = \{a \in B \mid a \preceq b\}$ need not be finite and of course the connecting maps need not be closed. But condition (3) of Definition 3.1 shows that, nevertheless, D_b is closed in X_b. □

4. Example

The following example applies Theorem 3.5. Let $c_0 = 1$ and for each $n \in \mathbb{N}$, $c_n = 1 - \dfrac{1}{n}$. Put $Q = \{c_0\} \cup \{c_n \mid n \in \mathbb{N}\}$. Thus Q is a compact subspace of $\mathbb{R}$ which has an order induced from $\mathbb{R}$.

Lemma 4.1. *Let $\mathcal{F}$ be a collection of closed subsets of Q at least one of which is infinite. Then $\bigcup \mathcal{F}$ is closed in Q.* □

We denote $Q^- = Q \setminus \{c_0\}$. A subset of Q^- will be called *co-infinite* if its complement in Q is infinite. Let $\mathcal{E}$ be the set of nonempty, co-infinite subsets of Q^-. For each $E \in \mathcal{E}$, let $\mathcal{I}(E)$ be the set of injective functions from E to $\mathbb{N}$. Put $A = \bigcup\{\mathcal{I}(E) \mid E \in \mathcal{E}\}$. For each $a \in A$, there is a unique element $E(a) \in \mathcal{E}$ such that $a \in \mathcal{I}(E(a))$. We denote $X_a = Q$.

Suppose that $a, b \in A$. Then we shall write $a \preceq b$ provided there is an embedding $p : X_a \to X_b$ such that:

(R1) $p(X_a \setminus E(a)) = X_b \setminus E(b)$,

(R2) p is order preserving on $X_a \setminus E(a)$, and

(R3) for each $x \in E(a)$, $a(x) = b \circ p(x)$.

Such p, if it exists, is unique. We shall write it p^b_a. Observe that p^a_a exists and equals the identity on X_a. If $a \preceq b$ and $b \preceq c$, then one sees that $p^c_b \circ p^b_a$ is a map of X_a to X_c which meets the requirements (R1)–(R3) with b replaced by c. So $a \preceq c$ and p^c_a equals $p^c_b \circ p^b_a$. It is not difficult to see that if a, $b \in A$, then there exists $c \in A$ with $a \preceq c$ and $b \preceq c$. Hence $(A, \preceq)$ is directed. This shows that $\mathbf{X} = (X_a, p^b_a, (A \preceq))$ is a direct system of compact 0-dimensional metrizable spaces and embeddings over an uncountable indexing set. Also the next lemma follows because of the "co-infinite" property.

Lemma 4.2. *If r, $c \in A$ and $r \preceq c$, then $p_r^c(X_r)$ is an infinite closed subset of X_c.* □

Put $B = \{a \in A \mid \text{card}(E(a)) < \infty\}$. It is not difficult to see that $(B, \preceq_B)$ is a sub-directed set of $(A, \preceq)$. Define $\mathbf{Y} = (X_a, \psi_a^b, (B, \preceq_B))$. Then it is routine to check that $\mathbf{Y}$ satisfies the Embedding Conditions (1), (2) of Definition 3.1 in $\mathbf{X}$. Applying Lemmas 4.1 and 4.2, one sees that $\mathbf{Y}$ satisfies the Embedding Condition (3) of Definition 3.1 in $\mathbf{X}$. Hence by Theorem 3.5, the canonical map $f : \text{dirlim}\,\mathbf{Y} \to \text{dirlim}\,\mathbf{X}$ is a closed embedding.

Note that in this example, $(B, \preceq)$ is not cofinal in $(A, \preceq)$. Hence, cofinality is not implied by the Embedding Conditions.

References

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