KYUNGPOOK Math. J. 49(2009), 701-712

Approximation of Common Fixed Points for a Family of Non-Lipschitzian Mappings

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Abstract. In this paper, we first introduce a family $\mathcal{S} = \{S_n : C \to C\}$ of non-Lipschitzian mappings, called *total asymptotically nonexpansive* (briefly, TAN) on a nonempty closed convex subset C of a real Banach space X, and next give necessary and sufficient conditions for strong convergence of the sequence $\{x_n\}$ defined recursively by the algorithm $x_{n+1} = S_n x_n,\ n \geq 1$, starting from an initial guess $x_1 \in C$, to a common fixed point for such a continuous TAN family $\mathcal{S}$ in Banach spaces. Finally, some applications to a finite family of TAN self mappings are also added.

1. Introduction

Let C be a nonempty closed convex subset of a real Banach space X and let $T : C \to C$ be a mapping. Then T is said to be a *Lipschitzian* mapping if, for each $n \geq 1$, there exists a constant $k_n > 0$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\| \tag{1.1}$$

for all $x, y \in C$ (we may assume that all $k_n \geq 1$). A Lipschitzian mapping T is called *uniformly k-Lipschitzian* if $k_n = k$ for all $n \geq 1$, *nonexpansive* if $k_n = 1$ for all $n \geq 1$, and *asymptotically nonexpansive* [4] if $\lim_{n\to\infty} k_n = 1$, respectively. The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [4] as a generalization of the class of nonexpansive mappings. They proved that if C is a nonempty bounded closed convex subset of a uniformly convex Banach space X, then every asymptotically nonexpanisve mapping $T : C \to C$ has a fixed point.

On the other hand, as the classes of non-Lipschitzian mappings, there appear in the literature two definitions, one is due to Kirk who says that T is a mapping

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Received September 25, 2009; accepted November 16, 2009.
2000 Mathematics Subject Classification: Primary 47H09, Secondary 65J15.
Key words and phrases: Common fixed points, non-Lipschitzian mappings, total asymptotically nonexpansive mappings, strong convergence.

of *asymptotically nonexpansive type* [7] if for each $x \in C$,

$$\limsup_{n\to\infty} \sup_{y\in C}(\|T^n x - T^n y\| - \|x-y\|) \le 0 \tag{1.2}$$

and T^N is continuous for some $N \ge 1$. The other is the stronger concept due to Bruck, Kuczumov and Reich [2]. They say that T is *asymptotically nonexpansive in the intermediate sense* if T is uniformly continuous and

$$\limsup_{n\to\infty} \sup_{x,y\in C}(\|T^n x - T^n y\| - \|x-y\|) \le 0 \tag{1.3}$$

In this case, observe that if we define

$$\delta_n := \sup_{x,y\in C}(\|T^n x - T^n y\| - \|x-y\|) \vee 0,$$

where $a \vee b := \max\{a,b\}$, then $\delta_n \to 0$ and (1.3) immediately reduces to

$$\|T^n x - T^n y\| - \|x-y\| \le \|x-y\| + \delta_n \tag{1.4}$$

for all x, $y \in C$ and $n \ge 1$.

Recently, Alber et al. [1] introduced the wider class of total asymptotically nonexpansive mappings to unify various definitions of classes of nonlinear mappings associated with the class of asymptotically nonexpansive mappings; see also Definition 1 of [3]. They say that a mapping $T : C \to C$ is said to be *total asymptotically nonexpansive* (TAN, in brief) [1] if there exist sequences $\{c_n\}$ and $\{d_n\}$ of nonnegative real numbers with c_n, $d_n \to 0$ as $n \to \infty$ and $\phi \in \Gamma(\mathbb{R}^+)$ such that

$$\|T^n x - T^n y\| \le \|x-y\| + c_n\,\phi(\|x-y\|) + d_n, \tag{1.5}$$

for all x, $y \in C$ and $n \ge 1$, where $\mathbb{R}^+ := [0,\infty)$ and

$$\phi \in \Gamma(\mathbb{R}^+) \Leftrightarrow \phi \text{ is strictly increasing, continuous on } \mathbb{R}^+ \text{ and } \phi(0) = 0.$$

Remark 1.1. (i) Note that if T is continuous, the property (1.5) with $c_n = 0$ for all $n \ge 1$ is equivalent to (1.4) with $d_n = \delta_n$, and also that a mapping satisfying the property (1.3) is non-Lipschitzian; see [6].

(ii) Also, if we take $\phi(t) = t$ for all $t \ge 0$ and $d_n = 0$ for all $n \ge 1$ in (1.5), it can be reduced to the concept of asymptotically nonexpansive mapping. Furthermore, in addition, taking $c_n = 0$ for all $n \ge 1$, it is *nonexpansive*, that is,

$$\|Tx - Ty\| \le \|x-y\|$$

for all $x, y \in C$.

A point $x \in C$ is a *fixed point* of T provided $Tx = x$. Denote by $F(T) := \{x \in C : Tx = x\}$ the set of all fixed points of T.

Let $\{T_i\}_{i=1}^N$ be a finite family of mappings from C into itself. Then we denote $T_{n \bmod N}$ by $T_{[n]}$, namely, the *mod function* takes values in the set $\{1, 2, \cdots, N\}$ as

$$T_{[n]} := \begin{cases} T_N, & \text{if } r = 0; \\ T_r, & \text{if } 0 < r < N \end{cases}$$

for $n = kN + r$ for some integers $j \geq 0$ and $0 \leq r < N$. In this case, setting

$$k(n) := \begin{cases} k, & \text{if } r = 0; \\ k+1, & \text{if } 0 < r < N \end{cases} \tag{1.6}$$

for each $n \geq 1$, it is not hard to see that $k(n) \to \infty$ as $n \to \infty$,

$$k(n-N) = k(n) - 1, \text{ and } T_{[n-N]} = T_{[n]}, \quad n \geq N. \tag{1.7}$$

We begin with the following simple observation.

Proposition 1.2. *Let C be a nonempty closed convex subset of a real Banach space X, $N \geq 1$ a positive integer, and let $\{T_i\}_{i=1}^N$ be a finite family of TAN mappings from C into itself. Then there exist sequences $\{c_n\}$ and $\{d_n\}$ of nonnegative real numbers with c_n, $d_n \to 0$ and $\phi \in \Gamma(\mathbb{R}^+)$ such that*

$$\|A_n x - A_n y\| \leq \|x - y\| + c_n\, \phi(\|x - y\|) + d_n,$$

for all x, $y \in C$ and $n \geq 1$, where

$$A_n \text{ is either } T_{[n]}^n \text{ or } \sum_{i=1}^N \lambda_i^{(n)} T_i^{(n)},$$

for all the $\lambda_i^{(n)} \in [0,1]$ with $\sum_{i=1}^N \lambda_i^{(n)} = 1$. In particular, if $k(n)$ is given as in (1.6)*, then*

$$\|T_{[n]}^{k(n)} x - T_{[n]}^{k(n)} y\| \leq \|x - y\| + c_n\, \phi(\|x - y\|) + d_n.$$

Definition 1.3. Let C be a nonempty closed convex subset of a real Banach space X. A discrete family $\mathcal{S} = \{S_n : C \to C\}$ is said to be *TAN* on C if there exist sequences $\{c_n\}$ and $\{d_n\}$ of nonnegative real numbers converging to zero and $\phi \in \Gamma(\mathbb{R}^+)$ such that

$$\|S_n x - S_n y\| \leq \|x - y\| + c_n\, \phi(\|x - y\|) + d_n \tag{1.8}$$

for all $x, y \in C$ and $n \geq 1$. Furthermore, we say that $\mathcal{S}$ is *continuous* on C provided each $S_n \in \mathcal{S}$ is continuous on C.

Example 1.4. The discrete families of $\{A_n\}_{n=1}^\infty$ and $\{T_{[n]}^{k(n)}\}_{n=1}^\infty$ in Proposition 1.2 are obviously TAN on C.

Example 1.5. Let $X = \mathbb{R}$, $C = [0, \infty)$ and, for each $n \geq 1$, define

$$S_n x = \Big(1 + \frac{1}{n}\Big) x + \frac{1}{n} \tan^{-1} x, \quad x \in C.$$

Then the family $\mathcal{S} = \{S_n : C \to C\}$ is continuous TAN on C. In fact, use $|\tan^{-1} x| < \frac{\pi}{2}$ to get

$$|S_n x - S_n y| \leq \Big(1 + \frac{1}{n}\Big)|x - y| + \frac{\pi}{n}$$

for all $x, y \in C$ and $n \geq 1$, where $\phi(t) = t$, $c_n = \frac{1}{n}$ and $d_n = \frac{\pi}{n}$.

Moreover, we have the following

Example 1.6. Let C be a nonempty closed convex subset of a real Banach space X. Let two families $\Im_i = \{T_n^{(i)} : C \to C\}$ be continuous TAN on C satisfying the property (1.8) with $c_n^{(i)} \in [0, 1]$, $d_n^{(i)} \equiv 0$ and $\phi_i \in \Gamma(\mathbb{R}^+)$ for $i = 1, 2$, respectively. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0, 1]$. Then the family $\mathcal{S} = \{S_n : C \to C\}$ defined by

$$S_n = (1 - \alpha_n)I + \alpha_n T_n^{(1)}[(1 - \beta_n)I + \beta_n T_n^{(2)}]$$

for each $n \geq 1$ is also continuous TAN on C.

Proof. Putting $U_n := (1 - \beta_n)I + \beta_n T_n^{(2)}$ and using (1.8) yield

$$\begin{aligned} \|U_n x - U_n y\| &\leq (1 - \beta_n)\|x - y\| + \beta_n \|T_n^{(2)} x - T_n^{(2)} y\| \\ &\leq (1 - \beta_n)\|x - y\| + \beta_n [\|x - y\| + c_n^{(2)}\, \phi_2(\|x - y\|)] \\ &\leq \|x - y\| + c_n^{(2)}\, \phi_2(\|x - y\|) \end{aligned}$$

for all $x, y \in C$. Then, we can also compute

$$\begin{aligned} \|S_n x - S_n y\| &\leq (1 - \alpha_n)\|x - y\| + \alpha_n \|T_n^{(1)}(U_n x) - T_n^{(1)}(U_n y)\| \\ &\leq (1 - \alpha_n)\|x - y\| + \alpha_n [\|U_n x - U_n y\| + c_n^{(1)}\, \phi_1(\|U_n x - U_n y\|)] \\ &\leq \|x - y\| + c_n^{(2)} \phi_2(\|x - y\|) + c_n^{(1)} \phi_1\big(\|x - y\| + M\phi_2(\|x - y\|)\big) \\ &= \|x - y\| + c_n \psi(\|x - y\|), \end{aligned}$$

where $\psi(t) := \phi_2(t) + \phi_1\big(t + M\phi_2(t)\big)$ for all $t \geq 0$, $M := \sup_{n \geq 1} c_n^{(2)}$, and $c_n := \max\{c_n^{(1)}, c_n^{(2)}\}$. Therefore, the family $\mathcal{S} = \{S_n : C \to C\}$ is continuous TAN on C with c_n and $\psi \in \Gamma(\mathbb{R}^+)$. □

Example 1.7. Let C be a nonempty closed convex subset of a real Banach space X. Let families $\Im_i = \{T_n^{(i)} : C \to C\}$ be continuous TAN on C, equipped with $c_n^{(i)} \in [0, 1]$, $d_n^{(i)}$ and $\phi_i \in \Gamma(\mathbb{R}^+)$ as in (1.8) for $i = 1, 2, \cdots, N$, respectively, such that the following two properties hold:

($\widetilde{C}1$) $\exists\, \alpha_0,\ \beta > 0$ such that $\phi_i(t) \leq \alpha_0 t$ for all $t \geq \beta$, $1 \leq i \leq N$;

$(\widetilde{C}2)$ $\sum_{n=1}^{\infty} c_n^{(i)} < \infty$ and $\sum_{n=1}^{\infty} d_n^{(i)} < \infty$, $1 \le i \le N$.

Let $\{\alpha_n^{(i)}\}$ be sequences in $[0,1]$ for $1 \le i \le N$. Then the family $\mathcal{S} = \{S_n : C \to C\}$ defined by

$$\begin{aligned} S_n \;=\; & (1-\alpha_n^{(1)})I + \alpha_n^{(1)} T_n^{(1)} \Big[(1-\alpha_n^{(2)})I + \alpha_n^{(2)} T_n^{(2)} \big[(1-\alpha_n^{(3)})I + \\ & \alpha_n^{(3)} T_n^{(3)} \big(\cdots + \alpha_n^{(N-1)} T_n^{(N-1)} \big((1-\alpha_n^{(N)})I + \alpha_n^{(N)} T_n^{(N)} \big) \cdots \big) \Big] \end{aligned}$$

is also continuous TAN on C, namely, there exist $\{c_n\}, \{d_n\}$ and $\phi \in \Gamma(\mathbb{R}^+)$ such that

$$\|S_n x - S_n y\| \le \|x-y\| + c_n \phi(\|x-y\|) + d_n, \quad x, y \in C.$$

Furthermore, the following properties are also satisfied:

$(\widetilde{C}1)'$ $\exists\, \alpha\, (\ge \alpha_0)$, $\beta > 0$ such that $\phi(t) \le \alpha t$ for all $t \ge \beta$.

$(\widetilde{C}2)'$ $\sum_{n=1}^{\infty} c_n < \infty$ and $\sum_{n=1}^{\infty} d_n < \infty$.

Proof. First, we claim that the family $\mathcal{S} = \{S_n : C \to C\}$ is continuous TAN on C for $N = 2$, that is, where

$$S_n = (1-\alpha_n^{(1)})I + \alpha_n^{(1)} T_n^{(1)} \big[(1-\alpha_n^{(2)})I + \alpha_n^{(2)} T_n^{(2)}\big].$$

Indeed, putting $U_n := (1-\alpha_n^{(2)})I + \alpha_n^{(2)} T_n^{(2)}$ simply and using (1.8) yield

$$\begin{aligned} (1.9) \quad \|U_n x - U_n y\| &\le (1-\alpha_n^{(2)})\|x-y\| + \alpha_n^{(2)} \|T_n^{(2)} x - T_n^{(2)} y\| \\ &\le (1-\alpha_n^{(2)})\|x-y\| + \alpha_n^{(2)} [\|x-y\| + c_n^{(2)}\, \phi_2(\|x-y\|)] + d_n^{(2)}] \\ &\le \|x-y\| + c_n^{(2)}\, \phi_2(\|x-y\|) + d_n^{(2)} \end{aligned}$$

for all $x, y \in C$. Then, we can also have

$$\begin{aligned} (1.10) \quad \|S_n x - S_n y\| \;&\le\; (1-\alpha_n^{(1)})\|x-y\| + \alpha_n^{(1)} \|T_n^{(1)}(U_n x) - T_n^{(1)}(U_n y)\| \\ &\le\; (1-\alpha_n^{(1)})\|x-y\| + \alpha_n^{(1)} [\|U_n x - U_n y\| \\ &\quad + c_n^{(1)}\, \phi_1(\|U_n x - U_n y\|) + d_n^{(1)}]. \end{aligned}$$

Using $(\widetilde{C}1)$ and the strictly increasing property of ϕ_i, we easily see

$$\phi_i(t) \le \phi_i(\beta) + \alpha t$$

for all $t \ge 0$ and $1 \le i \le N$. In particular,

$$(1.11) \qquad \phi_1(\|U_n x - U_n y\|) \le \phi_1(\beta) + \alpha \|U_n x - U_n y\|.$$

Now substituting (1.11) combined with (1.9) into (1.10) and simplifying, we get

$$\begin{aligned} \|S_n x - S_n y\| \;&\le\; \|x-y\| + c_n^{(2)} \phi_2(\|x-y\|) + c_n^{(1)} [\alpha(\|x-y\| + c_n^{(2)} \phi_2(\|x-y\|)] \\ &\quad + d_n^{(1)} + (1 + \alpha c_n^{(1)}) d_n^{(2)} + \phi_1(\beta) c_n^{(1)} \\ &\le\; \|x-y\| + c_n \psi(\|x-y\|) + d_n, \end{aligned}$$

where $\phi(t) := \phi_2(t) + \phi_1\big(\alpha(t + M\phi_2(t))\big)$ for all $t \geq 0$, $M := \sup_{n\geq 1} c_n^{(2)}$, $c_n := \max\{c_n^{(1)}, c_n^{(2)}\}$ and $d_n := d_n^{(1)} + (1 + \alpha c_n^{(1)})d_n^{(2)} + \phi_1(\beta)c_n^{(1)}$. Therefore, the family $\mathcal{S} = \{S_n : C \to C\}$ is continuous TAN on C with c_n, d_n and $\phi \in \Gamma(\mathbb{R}^+)$ for $N = 2$. Obviously,

$$\begin{aligned} \phi(t) &= \phi_2(t) + \phi_1\big(\alpha(t + M\phi_2(t))\big) \\ &\leq \alpha[t + \alpha(t + M\phi_2(t))] \\ &\leq \alpha(1 + \alpha + M)t := \tilde{\alpha}t, \quad t \geq \beta \end{aligned}$$

and also $\sum_{n=1}^{\infty} c_n < \infty$, $\sum_{n=1}^{\infty} d_n < \infty$.

Now use the mathematical induction to complete the proof. □

Remark 1.8. Note that S_n in Example 1.7 is rewritten as the following recursive form:

$$\tag{1.12} \begin{cases} S_n = (1 - \alpha_n^{(1)})I + \alpha_n^{(1)} T_n^{(1)} U_n^{(1)}, \\ U_n^{(1)} = (1 - \alpha_n^{(2)})I + \alpha_n^{(2)} T_n^{(2)} U_n^{(2)}, \\ \quad \vdots \\ U_n^{(N-2)} = (1 - \alpha_n^{(N-1)})I + \alpha_n^{(N-1)} T_n^{(N-1)} U_n^{(N-1)}, \\ U_n^{(N-1)} = (1 - \alpha_n^{(N)})I + \alpha_n^{(N)} T_n^{(N)}, \quad n \geq 1, \end{cases}$$

First let us consider a brief history of strong convergence problems for a single non-Lipschitzian mapping $T : C \to C$ which is both completely continuous and asymptotically nonexpansive in the intermediate sense as in (1.3) with $F(T) \neq \emptyset$.

Theorem 1.9([6]). *Suppose that a mapping $T : C \to C$ is both completely continuous and AN in the intermediate sense with $F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence defined by*

$$\begin{cases} x_1 \in C \text{ chosen arbitrarily}, \\ x_{n+1} = \alpha_n x_n + \beta_n T^n y_n + \gamma_n u_n, \\ y_n = \alpha'_n x_n + \beta'_n T^n x_n + \gamma'_n v_n, \qquad n \geq 1, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$,$\{\gamma_n\}$, $\{\alpha'_n\}$, $\{\beta'_n\}$,$\{\gamma'_n\}$ are real sequences in [0,1] and $\{u_n\}_{n=1}^{\infty}$, $\{v_n\}_{n=1}^{\infty}$ are two bounded sequences in C such that

(i) *$\{\alpha_n\}$ is bounded away from 0, $\{\beta'_n\}$ is bounded away from 1, and $\{\beta_n\}$ is bounded away from both 0 and 1.*

(ii) *$\alpha_n + \beta_n + \gamma_n = \alpha'_n + \beta'_n + \gamma'_n = 1$ for all $n \geq 1$,*

(iii) *$\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \gamma'_n < \infty$ and $\sum_{n=1}^{\infty} \delta_n < \infty$, where δ_n is given as in* (1.4)*. Then, $\{x_n\}$ converges strongly to a fixed point of T.*

Recently, Chidume and Ofoedu [3] established the following necessary and sufficient condition for strong convergence for a finite family of TAN self mappings defined on a nonempty closed convex subset in real Banach spaces.

Theorem 1.10([3]). *Let C be a nonempty closed convex subset of a real Banach space X and let $\{T_i\}_{i=1}^N$ be a finite family of TAN mappings from C into itself with $F_N := \cap_{i=1}^N F(T_i) \neq \emptyset$. Let $\{x_n\}$ be the sequence defined explicitly by either*

$$\begin{cases} x_1 \in C \text{ chosen arbitrarily,} \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T_1^n x_n, \quad n \geq 1 \end{cases}$$

for $N = 1$ or

$$\begin{cases} x_1 \in C \text{ chosen arbitrarily,} \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T_1^n y_{1n}, \\ y_{1n} = (1-\alpha_n)x_n + \alpha_n T_2^n y_{2n}, \\ \quad \vdots \\ y_{(N-2)n} = (1-\alpha_n)x_n + \alpha_n T_{N-1}^n y_{(N-1)n}, \\ y_{(N-1)n} = (1-\alpha_n)x_n + \alpha_n T_N^n x_n, \quad n \geq 1 \end{cases} \tag{1.13}$$

for $N \geq 2$, where $\{\alpha_n\}$ is a sequence in $[0,1]$. Assume that $\{c_n^{(i)}\}$, $\{d_n^{(i)}\}$ and ϕ_i, $1 \leq i \leq N$, satisfy the following properties:

(C1) $\exists\, \alpha_i,\ \beta_i > 0$ *such that* $\phi_i(t) \leq \alpha_i t$ *for all* $t \geq \beta_i$, $1 \leq i \leq N$.

(C2) $\sum_{n=1}^\infty c_n^{(i)} < \infty$ *and* $\sum_{n=1}^\infty d_n^{(i)} < \infty$, $1 \leq i \leq N$;

Then $\{x_n\}$ converges strongly to a common fixed point of $\{T_i\}_{i=1}^N$ if and only if $\liminf_{n\to\infty} d(x_n, F_N) = 0$, where $d(z, A) := \inf_{a \in A} \|x - a\|$ for all $z \in C$ and $A \subset C$.

Remark 1.11. (i) $\phi_i(t) = t^s$, $0 < s \leq 1$, $1 \leq i \leq N$ enjoys the condition (C1); see Remark 15 of [3].

(ii) For any fixed $\beta_i > 0$ and $f \in \Gamma([0, \beta_i))$, $1 \leq i \leq N$, define a function ϕ_i by

$$\phi_i(t) = \begin{cases} \int_0^t f(s)\,ds & \text{if } 0 \leq t < \beta_i \\ \frac{1}{\beta_i}\phi(\beta_i -)t & \text{if } t \geq \beta_i \end{cases} \tag{1.14}$$

provided $\phi(\beta_i -) := \lim_{t \to \beta_i -} \phi(t)$ exists. Then, $\phi_i \in \Gamma(\mathbb{R}^+)$, $1 \leq i \leq N$ and it obviously satisfies the condition (C1) with $\alpha_i := \frac{1}{\beta_i}\phi(\beta_i -)$. Especially, note that if X is uniformly convex, taking $f := \delta_X$, *modulus of convexity* of X, we see that (1.14) holds with $\beta_i := 2$, $1 \leq i \leq N$, because $\delta_X : [0,2] \to [0,1]$ is continuous on $[0,2)$, $\delta_X(2) = 1$, and strictly increasing on $[0,2]$; see [5] or [9].

(iii) Note that (C1) is equivalent to $(\widetilde{C}1)$. In fact, for (C1) $\Rightarrow$ $(\widetilde{C}1)$, take $\alpha := \max\{\alpha_i : 1 \leq i \leq N\}$ and $\beta := \max\{\beta_i : 1 \leq i \leq N\}$. For the converse, take $\alpha_i = \alpha$ and $\beta_i = \beta$ for all $i = 1, 2, \cdots, N$.

In particular, taking $T_n^{(i)} = T_i^n$, $\alpha_n^{(i)} = \alpha_n$ in (1.13), by Example 1.7, the family $\mathcal{S} = \{S_n : C \to C\}$ defined by (1.12) is also continuous TAN on C. Therefore, the explicit algorithm (1.13) can be shortly rewritten as

$$x_{n+1} = S_n x_n, \qquad n \geq 1. \tag{1.15}$$

In this paper, motivated and stimulated by the result(see Theorem 1.10) by Chidume and Ofoedu [3], we shall give necessary and sufficient conditions for strong convergence of the algorithm (1.15) to a common fixed point for a *continuous* TAN family $\mathcal{S} = \{S_n : C \to C\}$ of non-Lipschitzian mappings, defined as in Definition 1.3. Also, some applications to a finite family of TAN self mappings are added.

2. Necessary and sufficient conditions for convergence

Theorem 2.1. *Let X be a real Banach space, C be a nonempty closed convex subset of X. Let a discrete family $\mathcal{S} = \{S_n : C \to C\}$ be continuous TAN on C with $F := \cap_{n=1}^{\infty} F(S_n) \neq \emptyset$. Assume that $\{c_n\}$, $\{d_n\}$ and ϕ satisfy the following two conditions:*

(A1) $\exists\, \alpha,\ \beta > 0$ *such that* $\phi(t) \leq \alpha t$ *for all* $t \geq \beta$.

(A2) $\sum_{n=1}^{\infty} c_n < \infty,\ \sum_{n=1}^{\infty} d_n < \infty$.

Then the sequence $\{x_n\}$ defined by the explicit iteration method (1.15) converges strongly to a common fixed point of $\mathcal{S}$ if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$.

For the proof of Theorem 2.1, we shall need the following subsequent lemmas.

Lemma 2.2([8], [10]). *Let $\{a_n\}$, $\{\tilde{c}_n\}$ and $\{\tilde{d}_n\}$be sequences of nonnegative real numbers such that*

$$a_{n+1} \leq (1 + \tilde{c}_n)a_n + \tilde{d}_n$$

for all $n \geq 1$. Suppose that $\sum_{n=1}^{\infty} \tilde{c}_n < \infty$ and $\sum_{n=1}^{\infty} \tilde{d}_n < \infty$. Then $\lim_{n\to\infty} a_n$ *exists. Moreover, if in addition,* $\liminf_{n\to\infty} a_n = 0$, *then* $\lim_{n\to\infty} a_n = 0$.

Lemma 2.3. *Under the same hypotheses as Theorem 2.1, there hold the following properties:*

(i) $\lim_{n\to\infty} \|x_n - p\|$ *exists for all* $p \in F$, *and hence* $\{x_n\}$ *is bounded.*

(ii) $\lim_{n\to\infty} d(x_n, F)$ *exists.*

Proof. First, to prove (i), let $p \in F$ and let $n \geq 1$ be arbitrarily given. Using (A1) and strict increasing of ϕ, we easily get

$$\phi(t) \leq \phi(\beta) + \alpha t, \quad t \geq 0. \tag{2.1}$$

Use (1.8) and (2.1) in turn to derive

$$\begin{aligned} \|x_{n+1} - p\| &= \|S_n x_n - S_n p\| \\ &\leq \|x_n - p\| + c_n \phi(\|x_n - p\|) + d_n \\ &\leq \|x_n - p\| + c_n[\phi(\beta) + \alpha\|x_n - p\|] + d_n \\ &= (1 + \alpha c_n)\|x_n - p\| + c_n \phi(\beta) + d_n. \end{aligned}$$

Putting $\tilde{c}_n := \alpha c_n$ and $\tilde{d}_n := c_n\phi(\beta) + d_n$, this implies that

$$\|x_{n+1} - p\| \leq (1 + \tilde{c}_n)\|x_n - p\| + \tilde{d}_n \tag{2.2}$$

and

$$\sum_{n=1}^{\infty} \tilde{c}_n < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \tilde{d}_n < \infty \tag{2.3}$$

by using (A2). So, by Lemma 2.2, the limit $\lim_{n\to\infty} \|x_n - p\|$ exists.

Now to show (ii), taking the infimum over all $p \in F$ on the both sides of inequality (2.2) we obtain

$$d(x_{n+1}, F) \leq (1 + \tilde{c}_n)d(x_n, F) + \tilde{d}_n.$$

Applying Lemma 2.2 again, (ii) is quickly obtained. □

Proof of Theorem 2.1. It suffices to show the sufficiency. Assume that

$$\liminf_{n\to\infty} d(x_n, F) = 0.$$

Then it follows from (ii) of Lemma 2.3 that $\lim_{n\to\infty} d(x_n, F) = 0$. Since $\sum \tilde{c}_n < \infty$ in the process proving Lemma 2.3, we observe that

$$1 \leq K := \prod (1 + \tilde{c}_n) \leq e^{\sum \tilde{c}_n} < \infty. \tag{2.4}$$

Given $\epsilon > 0$, since $\lim_{n\to\infty} d(x_n, F) = 0$ and $\sum \tilde{d}_n < \infty$, we can choose a positive integer n_0 sufficiently large so that

$$d(x_n, F) < \frac{\epsilon}{4K} \quad \text{and} \quad \sum_{i=n}^{\infty} \tilde{d}_i < \frac{\epsilon}{4K}, \quad n \geq n_0. \tag{2.5}$$

Let $n, m \geq n_0$ and $p \in F$. First, use the inequality (2.2) repeatedly together with (2.4) to derive

$$\begin{aligned}
&\|x_n - p\| \\
\leq\ & \prod_{i=n_0}^{n-1} (1 + \tilde{c}_i)\, \|x_{n_0} - p\| + \sum_{i=n_0}^{n-2} \tilde{d}_i \prod_{k=i+1}^{n-1} (1 + \tilde{c}_k) + \tilde{d}_{n-1} \\
\leq\ & K\Big[\|x_{n_0} - p\| + \sum_{i=n_0}^{n-1} \tilde{d}_i\Big],
\end{aligned}$$

which implies that

$$\begin{aligned} \|x_n - x_m\| &\le \|x_n - p\| + \|x_m - p\| \\ \le \quad & K\Big[\|x_{n_0} - p\| + \sum_{i=n_0}^{n-1} \tilde{d}_i\Big] + K\Big[\|x_{n_0} - p\| + \sum_{i=n_0}^{m-1} \tilde{d}_i\Big] \\ \le \quad & 2K\Big[\|x_{n_0} - p\| + \sum_{i=n_0}^{\infty} \tilde{d}_i\Big]. \end{aligned}$$

Taking the infimum over all $p \in F$ firstly on both sides and next using (2.5), we have

$$\begin{aligned} \|x_n - x_m\| &\le 2K\Big[d(x_{n_0}, F) + \sum_{i=n_0}^{\infty} \tilde{d}_i\Big]) \\ &\le 2K\Big(\frac{\epsilon}{4K} + \frac{\epsilon}{4K}\Big) = \epsilon, \quad n, m \ge n_0. \end{aligned}$$

This shows that $\{x_n\}$ is a Cauchy sequence in X. Say $x_n \to x^* \in X$. Finally, we claim that $x^* \in F$. In fact, note first that

$$\|x^* - p\| \le \|x^* - x_n\| + \|x_n - p\|$$

for all $p \in F$ and $n \ge 1$. Taking the infimum again over all $p \in F$ on both sides ensures that

$$d(x^*, F) \le \|x^* - x_n\| + d(x_n, F) \to 0$$

as $n \to \infty$. Since F is closed by continuity of $\mathcal{S}$, it follows that $x^* \in F$ and the proof is complete. □

Corollary 2.4. *Under the same hypotheses as Theorem 2.1, the sequence* $\{x_n\}$ *converges strongly to a common fixed point* $p \in F$ *if and only if there exists a subsequence* $\{x_{n_k}\}$ *of* $\{x_n\}$ *which converges strongly to* p*.*

Proof. Note that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges strongly to p if and only if $\lim_{k\to\infty} d(x_{n_k}, F) = 0$. Since

$$\liminf_{n\to\infty} d(x_n, F) \le \liminf_{k\to\infty} d(x_{n_k}, F) = \lim_{k\to\infty} d(x_{n_k}, F) = 0,$$

it follows that $\liminf_{n\to\infty} d(x_n, F) = 0$. Now apply Theorem 2.1 to complete the proof. □

3. Applications to a finite family of TAN self mappings

Let C be a nonempty closed convex subset of a real Banach space X and let $N \ge 1$ be fixed. Let $\{T_i\}_{i=1}^N$ be a finite family of N continuous TAN mappings defined on C, that is, for $i = 1, 2, \cdots, N$,

$$\|T_i^n x - T_i^n y\| \le \|x - y\| + c_n^{(i)} \phi_i(\|x - y\|) + d_i^{(i)} \tag{3.1}$$

for all $x, y \in C$, where $\phi_i \in \Gamma(\mathbb{R}^+)$, $\{c_n^{(i)}\}$ and $\{d_n^{(i)}\}$ are sequences of nonnegative real numbers such that $c_n^{(i)} \to 0,\ d_n^{(i)} \to 0$ as $n \to \infty$ $(1 \le i \le N)$.

In this section, as a special case, we recall the following explicit iteration algorithm (1.13) studied by Chidume and Ofoedu [3] for such a finite family $\{T_i\}_{i=1}^N$:

$$\begin{cases} x_1 \in C \text{ chosen arbitrarily,} \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T_1^n\, y_{1n}, \\ y_{1n} = (1-\alpha_n)x_n + \alpha_n T_2^n\, y_{2n}, \\ \quad \vdots \\ y_{(N-2)n} = (1-\alpha_n)x_n + \alpha_n T_{N-1}^n\, y_{(N-1)n}, \\ y_{(N-1)n} = (1-\alpha_n)x_n + \alpha_n T_N^n\, x_n, \quad n \ge 1 \end{cases}$$

Then the above algorithm (1.13) can be rewritten as a special form of (1.12) equipped with $T_n^{(i)} = T_i^n$, $\alpha_n^{(i)} \equiv \alpha_n$ for $1 \le i \le N$, namely,

$$x_{n+1} = S_n x_n, \quad n \ge 1, \tag{3.2}$$

where

$$\begin{cases} S_n = (1-\alpha_n)I + \alpha_n T_1^n\, U_n^{(1)}, \\ U_n^{(1)} = (1-\alpha_n)I + \alpha_n T_2^n\, U_n^{(2)}, \\ \quad \vdots \\ U_n^{(N-2)} = (1-\alpha_n)I + \alpha_n T_{N-1}^n\, U_n^{(N-1)}, \\ U_n^{(N-1)} = (1-\alpha_n)I + \alpha_n T_N^n, \quad n \ge 1, \end{cases} \tag{3.3}$$

$x_1 \in K$ is arbitrarily given, $\alpha_n \in [0,1]$ and $y_{in} = U_n^{(i)} x_n$ for $1 \le i \le N$.

By virtue of Example 1.7, $\mathcal{S} = \{S_n : C \to C\}$ is obviously continuous TAN on C, and $F_N = \cap_{i=1}^N F(T_i) \subset F = \cap_{n=1}^\infty F(S_n)$.

Corollary 3.1([3]). *Let C be a nonempty closed convex subset of a real Banach space X, $N \ge 1$ a positive integer, and let $\{T_i\}_{i=1}^N$ be a finite family of TAN mappings from C into itself with $F_N = \cap_{i=1}^N F(T_i) \ne \emptyset$. Let $\{x_n\}$ be the sequence defined explicitly by* (1.13)*. Assume that $\{c_n^{(i)}\}$, $\{d_n^{(i)}\}$ and ϕ_i, $1 \le i \le N$, satisfy properties* (C1) *and* (C2)*. Then $\{x_n\}$ converges strongly to a common fixed point of $\{T_i\}_{i=1}^N$ if and only if* $\liminf_{n\to\infty} d(x_n, F_N) = 0$.

Proof. By Example 1.7, since the family $\mathcal{S} = \{S_n : C \to C\}$ defined recursively by (3.2) is continuous TAN on C and $F_N \subset F$, and $(C1) \Leftrightarrow (\widetilde{C}1) \Leftrightarrow (\widetilde{C}1)' \Rightarrow (A1)$ and $(C2) \Leftrightarrow (\widetilde{C}2) \Rightarrow (\widetilde{C}2)' \Leftrightarrow (A2)$, all the assumptions in Theorem 2.1 are therefore fulfilled. Now to prove the sufficiency, assume that $\liminf_{n\to\infty} d(x_n, F_N) = 0$. Since $F_N \subset F$, (2.2) still remains true for all $p \in F_N$ and so $\lim_{n\to\infty} d(x_n, F_N)$ exists. Hence $\lim_{n\to\infty} d(x_n, F_N) = 0$. Now similarly mimicking the proof of Theorem 2.1 with F_N instead F, we conclude that $\{x_n\}$ strongly converges to a common fixed point of $\{T_i\}_{i=1}^N$. □

Remark 3.2. (a) We still don't know whether $F \subset F_N$ under the hypotheses of Corollary 3.1 or not.

(b) Note that our proofs in Theorem 2.1 and Corollary 3.1 are simpler than the one given by Chidume and Ofoedu [3].

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