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A Fixed Point Approach to the Stability of Quadratic Equations in Quasi Normed Spaces

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Abstract. We use the fixed alternative theorem to establish Hyers–Ulam–Rassias stability of the quadratic functional equation where functions map a linear space into a complete quasi p-normed space. Moreover, we will show that the continuity behavior of an approximately quadratic mapping, which is controlled by a suitable continuous function, implies the continuity of a unique quadratic function, which is a good approximation to the mapping. We also give a few applications of our results in some special cases.

1. Introduction and preliminaries

In 1940, S. M. Ulam [25] gave a wide ranging talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of important unsolved problems. Among those was the following question concerning the stability of homomorphisms:
Let $(G_1, *)$ be a group and $(G_2, \diamond, d)$ be a metric group with the metric d. Given $\varepsilon > 0$, does there exists a $\delta_\varepsilon > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality

$$d\big(h(x * y), h(x) \diamond h(y)\big) < \delta_\varepsilon \quad \forall x, y \in G_1,$$

then there is a mapping $H : G_1 \to G_2$ such that for each $x, y \in G_1$, $H(x * y) = H(x) \diamond H(y)$ and $d\big(h(x), H(x)\big) < \varepsilon$?

In the next year, D. H. Hyers [12] gave an affirmative answer to the question of Ulam. Hyers' theorem was generalized by T. Aoki [2] for additive mappings and by Th. M. Rassias [22] for linear mappings by considering an unbounded Cauchy difference. The concept of the Hyers–Ulam–Rassias stability was originated from Th. M. Rassias' paper [22] for the stability of the linear mappings and its importance in the proof of further results in functional equations. In 1994, a generalization of Th.M. Rassias' theorem was obtained by Găvruţă [11], who replaced the bound $\varepsilon(\|x\|^p + \|y\|^p)$ by a general control function $\varphi(x, y)$. During the last decades several stability problems for various functional equations have been investigated by many mathematicians; we refer the reader to [9], [13], [16], [17], [18], [23] and references

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therein.

The functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{1.1}$$

is called the *quadratic functional equation*, since the function $f(x)=x^2$ is a solution of the functional equation. Every solution of the quadratic functional equation is said to be a *quadratic mapping*. For example, in any Hilbert space $(H,< . , . >)$, $f(x)=< x,x >$ defines a quadratic mapping. The first stability theorem for the quadratic functional equation was proved F. Skof [24] for a mapping from a normed space X into a Banach space Y satisfying the inequality $\|f(x+y)+f(x-y)-2f(x)-2f(y)\|\leq\epsilon$ for some $\epsilon>0$. P. W. Cholewa [6] extended Skof 's theorem by replacing X by an abelian group G. This result was later generalized by S. Czerwik [7] in the spirit of Hyers–Ulam–Rassias. He also proved the stability of quadratic equation of Pexider type [8]. Recently, the stability problem of the quadratic equation has been investigated by a number of mathematicians, see [13], [14], [15], [19] and references therein.

Definition 1.1. The pair (X,d) is called a generalized complete metric space if X is a nonempty set and $d:X^2\to[0,\infty]$ satisfies the following conditions:

(a) $d(x,y)\geq 0$ and the equality holds if and only if $x=y$,

(b) $d(x,y)=d(y,x)$,

(c) $d(x,z)\leq d(x,y)+d(y,z)$,

(d) every d-Cauchy sequence in X is d-convergent.

Note that the distance between two points in a generalized metric space is permitted to be infinity.

Definition 1.2. Let (X,d) be a generalized complete metric space. A mapping $\Lambda:X\to X$ satisfies a Lipschitz condition with Lipschitz constant $L\geq 0$ if

$$d(\Lambda(x),\Lambda(y))\leq Ld(x,y)\quad (x,y\in X).$$

If $L<1$, then Λ is called a strictly contractive operator.

In 2003, Radu [20] employed the following result, due to Diaz and Margolis [10], to prove the stability of Cauchy additive functional equation. Using such an elegant idea, several authors applied the method to investigate the stability of some functional equations, see [4], [5], [14], [21].

Proposition 1.3 (The fixed point alternative principle)**.** *Suppose that a complete generalized metric space* $(\mathcal{E},d)$ (*i.e., one for which* d *may assume infinite values*) *and a strictly contractive mapping* $J:\mathcal{E}\to\mathcal{E}$ *with the Lipschitz constant* $0<L<1$ *are given. Then, for a given element* $x\in\mathcal{E}$*, exactly one of the following assertions is true: either*

(a) $d(J^n x, J^{n+1}x) = \infty$ *for all* $n \geq 0$ *or*
(b) *there exists some integer* k *such that* $d(J^n x, J^{n+1}x) < \infty$ *for all* $n \geq k$.
Actually, if (b) *holds, then the sequence* $\{J^n x\}$ *is convergent to a fixed point* x^* *of* J *and*
(b1) x^* *is the unique fixed point of* J *in* $\mathcal{F} := \{y \in \mathcal{E}, d(J^k x, y) < \infty\}$;
(b2) $d(y, x^*) \leq \dfrac{d(y, Jy)}{1-L}$ *for all* $y \in \mathcal{F}$.

Remark 1.4. The fixed point x^*, if it exists, is not necessarily unique in the whole space $\mathcal{E}$; it may depend on x. Actually, if (b) holds, then $(\mathcal{F}, d)$ is a complete metric space and $J(\mathcal{F}) \subset \mathcal{F}$. Therefore the properties (b1) and (b2) are follows from "The Banach fixed point Theorem".

Definition 1.5. A quasi-norm on a real vector space X is a function $x \rightarrowtail |||x|||$ from X to $[0, \infty)$ which satisfies

(i) $|||x||| > 0$ for every $x \neq 0$ in X,

(ii) $|||tx||| = |t|.|||x|||$ for every $t \in \mathbb{R}$ and $x \in X$,

(iii) there is a $k \geq 1$ such that $|||x + y||| \leq k(|||x||| + |||y|||)$ for every $x, y \in X$.

Aoki [1] (see also [3]) has shown that every quasi-normed space $(X, |||\ .\ |||)$ admits an equivalent quasi-norm $||\ .\ ||$ such that for some $0 < p \leq 1$,

$$||x + y||^p \leq ||x||^p + ||y||^p \quad (x, y \in X). \tag{1.2}$$

In this case, $(X, ||\ ||)$ is called a quasi p-normed space. In special case, when $p = 1$, $(X, |||\ .\ |||)$ turns into a normed linear space.

In the next section, we employ fixed point alternative theorem (Proposition 1.3) to establish Hyers–Ulam–Rassias stability of the quadratic functional equation (1.1) in the setting of quasi p-normed spaces. In fact, we will show that if a function f from a linear space X to a complete p-normed space Y satisfies the inequality

$$||f(x + y) + f(x - y) - 2f(x) - 2f(y)|| \leq \varphi(x, y) \quad (x, y \in X)$$

for suitable control function φ, then f can be suitably approximated by a unique quadratic function $Q : X \to Y$. In section 3, we will show that, for each $x \in X$, the continuity of $s \mapsto f(sx)$ and $s \mapsto \varphi(sx, sx)$ guarantee the continuity of $s \mapsto Q(sx)$. We also give some applications of our results in special cases.

2. Stability of quadratic functional equations

Throughout the remainder of this paper, unless otherwise stated, we will assume that $0 < p \leq 1$ and $q = \dfrac{1}{p}$, X is real vector spaces and Y is a complete quasi p-norm space. Let

$$Df(x, y) = f(x + y) + f(x - y) - 2f(x + y) - 2f(x - y). \tag{2.1}$$

We start with the following lemma.

Lemma 2.1. *Let $\psi : X \to [0,\infty)$ be a function. Let $\mathcal{E} = \{g : X \to Y\}$ and define*

$$d(g,h) = \inf\{a > 0 : ||g(x) - h(x)|| \le a^q \psi(x) \quad \forall x \in X\} \quad (g,h \in \mathcal{E}).$$

Then d is a generalized complete metric on $\mathcal{E}$.

Proof. Let $g,h,k \in \mathcal{E}$, $d(g,h) < a_1$ and $d(h,k) < a_2$. Then

$$||g(x) - h(x)|| \le a_1^q \psi(x) \quad \text{and} \quad ||h(x) - k(x)|| \le a_2^q \psi(x),$$

for each $x \in X$. It follows that

$$\begin{aligned} ||g(x) - k(x)||^p &\le ||g(x) - h(x)||^p + ||h(x) - k(x)||^p \\ &\le (a_1 + a_2)\Big(\psi(x)\Big)^p \quad (x \in X). \end{aligned}$$

Therefore $d(g,k) \le a_1 + a_2$. This proves the triangle inequality for d. The rest of the proof is similar to the proof of the main result of [20]. □

Theorem 2.2. *Let $\varphi : X \times X \to [0,\infty)$ and $f : X \to Y$ satisfy the inequality*

$$||Df(x,y)|| \le \varphi(x,y) \quad (x,y \in X). \tag{2.2}$$

If for some $\alpha < 4$,

$$\varphi(2x,2x) \le \alpha\varphi(x,x) \qquad (x \in X) \tag{2.3}$$

and $\lim_{n\to\infty} 2^{-2n}\varphi(2^n x, 2^n y) = 0$ for all x,y in X, then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$||Q(x) - f(x)|| \le \frac{\varphi(x,x)}{(4^p - \alpha^p)^q} \quad (x \in X). \tag{2.4}$$

Proof. Put $x = y$ in (2.2), then we have

$$||f(2x) - 2^2 f(x)|| \le \varphi(x,x) \quad (x \in X). \tag{2.5}$$

Let $\mathcal{E} = \{g : X \to Y\}$. By Lemma 2.1,

$$d(g,h) = \inf\{a > 0 : ||g(x) - h(x)|| \le a^q \varphi(x,x), \forall x \in X\} \quad (g,h \in \mathcal{E})$$

defines a complete generalized metric on $\mathcal{E}$. Define $J : \mathcal{E} \to \mathcal{E}$ by $J(g)(x) = 2^{-2}g(2x)$ for each $g \in \mathcal{E}$ and $x \in X$. Let $d(g,h) < a$, by the definition,

$$||g(x) - h(x)|| \le a^q \varphi(x,x) \quad (x \in X).$$

According to (2.3), for each $x \in X$,

$$\begin{aligned} ||J(g)(x) - J(h)(x)|| &= ||2^{-2}g(2x) - 2^{-2}h(2x)|| \\ &\le 2^{-2}a^q\varphi(2x, 2x) \\ &\le \left(\frac{\alpha}{4}\right)a^q\varphi(x, x). \end{aligned}$$

Hence, by the definition, $d(J(g), J(h)) \le \left(\frac{\alpha}{4}\right)^p a$. Therefore

$$d(J(g), J(h)) \le \left(\frac{\alpha}{4}\right)^p d(g, h) \quad (g, h \in \mathcal{E}).$$

This means that J is a contractive mapping with Lipschitz constant $L = (\frac{\alpha}{4})^p < 1$. By (2.5), $d(f, J(f)) \le (\frac{1}{2^2})^p$, therefore, by Proposition 1.3, J has a unique fixed point $Q : X \to Y$ in the set $\mathcal{F} = \{g \in \mathcal{E} : d(f, g) < \infty\}$, where Q is defined by

$$(2.6) \qquad Q(x) := \lim_{n\to\infty} J^n(f)(x) = \lim_{n\to\infty} 2^{-2n}f(2^n x) \qquad (x \in X).$$

Moreover,

$$d(f, Q) \le \frac{d(f, J(f))}{1 - L} \le \frac{4^{-p}}{1 - 4^{-p}\alpha^p} = \frac{1}{4^p - \alpha^p}.$$

This means that (2.4) holds. According to (2.6),

$$DQ(x, y) = \lim_{n\to\infty} 2^{-2n}Df(2^n x, 2^n y) \quad (x, y \in X).$$

Replace x, y by $2^n x, 2^n y$ respectively in (2.2) to get

$$||2^{-2n}Df(2^n x, 2^n y)|| \le 2^{-2n}\varphi(2^n, 2^n y) \qquad (x, y \in X).$$

By our assumption $\lim_{n\to\infty} 2^{-2n}\varphi(2^n x, 2^n y) = 0$, it follows that $DQ(x, y) = 0$ for all $x, y \in X$. Hence Q is a quadratic function. To prove the uniqueness assertion, let us assume that there exists a quadratic function $S : X \to Y$ which satisfies (2.4). Then S is a fixed point of J in $\mathcal{F}$. However, by Proposition 1.3, J has only one fixed point in $\mathcal{F}$, hence $S \equiv Q$. □

By a modification in the proof of Theorem 2.2, one can prove the following result:

Theorem 2.3. *Let $\varphi : X \times X \to [0, \infty)$ be a function. Let $f : X \to Y$ satisfy the inequality*

$$||Df(x, y)|| \le \varphi(x, y) \quad (x, y \in X).$$

If for some $\alpha > 4$,

$$\alpha\varphi(\frac{x}{2}, \frac{x}{2}) \le \alpha\varphi(x, x) \quad (x \in X)$$

$\lim_{n\to\infty} 2^{2n}\varphi(2^{-n}x, 2^{-n}y) = 0$ *for all* x, y *in* X*, then there exists a unique quadratic mapping* $Q : X \to Y$ *such that*

$$||Q(x) - f(x)|| \leq \frac{\varphi(x,x)}{(\alpha^p - 4^p)^q} \quad (x \in X).$$

3. Continuity behavior of quadratic mappings

In this section, we investigate continuity of quadratic mappings in quasi p-normed. In fact, we will show that under some conditions on Theorem 2.2 (or Theorem 2.3), the quadratic mapping $s \longmapsto Q(sx)$ is continuous. It follows that in such a case, $Q(rx) = r^2Q(x)$ for each $x \in X$ and $r \in \mathbb{R}$.
We need to the following result which can be easily proved by induction.

Lemma 3.1. *Let* $Q : X \to Y$ *be a quadratic function, then* $Q(rx) = r^2Q(x)$ *for each* $x \in X$ *and rational number* r*.*

Now, we are ready to mention the main result of this section.

Theorem 3.2. *Let the conditions of Theorem 2.2 hold. If for each* $x \in X$*, the functions*

$$\begin{array}{rcl} \mathbb{R} & \to & Y \\ s & \mapsto & f(sx) \end{array} \qquad \text{and} \qquad \begin{array}{l} \mathbb{R} \to [0,\infty) \\ s \mapsto \varphi(sx, sx) \end{array}$$

are continuous, then for each $x \in X$*,* $s \longmapsto Q(sx)$ *from* $\mathbb{R}$ *to* Y *is continuous and* $Q(sx) = s^2Q(x)$ *for each* $x \in X$ *and* $s \in \mathbb{R}$*.*

Proof. Fix $x \in X$ and $s_0 \in \mathbb{R}$. Take n_0 large enough so that

$$\left(\frac{\alpha}{4}\right)^{n_0} < \frac{\varepsilon}{6^q} \quad \text{and} \quad \left(\frac{\alpha}{4}\right)^{n_0} \frac{\varphi(s_0x, s_0x)}{\left(4^p - \alpha^p\right)^q} < \frac{\varepsilon}{6^q}. \tag{3.1}$$

Since $\alpha < 4$ such a choice is possible. By the continuity of maps $s \mapsto f(2^n sx)$ and $s \longmapsto \varphi(sx, sx)$ at s_0, we can find some $\delta > 0$ such that

$$0 < |s - s_0| < \delta \Rightarrow \begin{cases} ||f(2^{n_0}sx) - f(2^{n_0}s_0x)|| < \dfrac{\varepsilon}{3^q} \\ |\varphi(sx, sx) - \varphi(s_0x, s_0x)| < \left(4^p - \alpha^p\right)^q \end{cases} \tag{3.2}$$

Let $|s - s_0| < \delta$. Then

$$\begin{aligned} ||Q(sx) - Q(s_0x)||^p &= ||2^{-2n_0}Q(2^{n_0}sx) - 2^{-2n_0}Q(2^{n_0}s_0x)||^p \\ &\leq 2^{-2n_0p}\Big(||Q(2^{n_0}sx) - f(2^{n_0}sx)||^p + ||f(2^{n_0}sx) - f(2^{n_0}s_0x)||^p \\ &+ ||f(2^{n_0}s_0x) - Q(2^{n_0}s_0x)||^p\Big). \end{aligned} \tag{3.3}$$

By (3.2), the second term of the right hand side of (3.3) is less than $\frac{\varepsilon^p}{3}$. Since for all $s \in \mathbb{R}$,

$$(3.4)\quad 2^{-2n_0}||Q(2^{n_0}sx) - f(2^{n_0}sx)|| \leq \frac{1}{4^{n_0}}.\frac{\varphi(2^{n_0}sx, 2^{n_0}sx)}{\left(4^p - \alpha^p\right)^q} \leq \left(\frac{\alpha}{4}\right)^{n_0}.\frac{\varphi(sx, sx)}{\left(4^p - \alpha^p\right)^q},$$

by (3.1), the last term of the right hand side of the inequality (3.3) is less than $\frac{\varepsilon^p}{6}$. By (3.1), (3.2) and (3.4)

$$\begin{aligned} 2^{-2n_0}||Q(2^{n_0}sx) - f(2^{n_0}sx)|| &\leq \left(\frac{\alpha}{4}\right)^{n_0}\left\{\frac{|\varphi(sx, sx) - \varphi(s_0x, s_0x)|}{\left(4^p - \alpha^p\right)^q} + \frac{\varphi(s_0x, s_0x)}{\left(4^p - \alpha^p\right)^q}\right\} \\ &\leq \frac{\varepsilon}{6^q} + \frac{\varepsilon}{6^q} = \frac{2\varepsilon}{6^q}. \end{aligned}$$

Hence,

$$|s - s_0| < \delta \Rightarrow ||Q(sx) - Q(s_0x)||^p < \frac{2^p\ \varepsilon^p}{6} + \frac{\varepsilon^p}{3} + \frac{\varepsilon^p}{6} < \varepsilon^p.$$

Here we used the fact that $p \leq 1$. The above inequality proves continuity of $s \mapsto Q(sx)$. By Lemma 3.1, $Q(rx) = r^2Q(x)$ for every rational number r. Let s be a real number, then there exists a sequence $\{r_n\}$ of rational numbers such that $r_n \to s$. By the continuity of $t \mapsto Q(tx)$, for every $x \in X$,

$$Q(sx) = \lim_{n\to\infty} Q(r_nx) = \lim_{n\to\infty} r_n^2Q(x) = s^2Q(x).$$

This completes the proof of the Theorem. □

The proof of the following result is similar to that in Theorem 3.2, hence it is omitted.

Theorem 3.3. *Let conditions of Theorem 2.3 hold. If for each $x \in X$, the functions*

$$s \longmapsto f(sx) \quad \textit{and} \quad s \longmapsto \varphi(sx, sx)$$

are continuous, then the function $s \longmapsto Q(sx)$ is continuous for each $x \in X$ and $Q(sx) = s^2Q(x)$ for each $x \in X$ and $s \in \mathbb{R}$.

Corollary 3.4. *Let $\varphi : X \times X \to [0, \infty)$ be a mapping such that either*

(i) *for some $\alpha < 4$, $\varphi(2x, 2x) \leq \alpha\varphi(x, x)$ for all $x \in X$ and for each $x, y \in X$, $\lim_{n\to\infty} 2^{-2n}\varphi(2^nx, 2^ny) = 0$ or*

(ii) *for some $\alpha > 4$, $\alpha\varphi(x, x) \leq \varphi(2x, 2x)$ for all $x \in X$ and for each $x, y \in X$, $\lim_{n\to\infty} 2^{2n}\varphi(2^{-n}x, 2^{-n}y) = 0$.*

Let $f : X \to Y$ satisfy the inequality

$$(3.5)\qquad ||Df(x, y)|| \leq \varphi(x, y)$$

for each $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$||f(x) - Q(x)|| \le \frac{\varphi(x,x)}{|\alpha^p - 4^p|^q} \quad (x \in X). \tag{3.6}$$

Moreover, if the mappings $s \mapsto f(st)$ and $s \mapsto \varphi(sx, sx)$ are continuous, then $s \mapsto Q(sx)$ is continuous and $Q(sx) = s^2 Q(x)$ for each $s \in \mathbb{R}$ and $x \in X$.

Proof. If (i) holds, then conditions of Theorem 2.2 are fulfilled. If (ii) holds, then the hypotheses of Theorem 2.3 are satisfied. In either case, by the above mentioned theorems, we can find a quadratic mapping $Q : X \to Y$ such that (3.6) holds. If the mappings $s \mapsto f(st)$ and $s \mapsto \varphi(sx, sx)$ are continuous, then we apply Theorems 3.2 and 3.3 to get to the last assertions of the theorem. □

Corollary 3.5. *Let $\big(X, |||\,.\,|||\big)$ be a normed space. Let for some $\varepsilon > 0$ and positive real number $r \neq 2$, $f : X \to Y$ satisfy the inequality*

$$||Df(x,y)|| \le \varepsilon\Big(|||x|||^r + |||y|||^r\Big) \quad (x, y \in X).$$

Then there is a unique continuous quadratic mapping $Q : X \to Y$ such that

$$||f(x) - Q(x)|| \le \frac{2\ \varepsilon|||x|||^r}{|2^{rp} - 2^{2p}|^q}$$

and $Q(sx) = s^2 Q(x)$ for each $s \in \mathbb{R}$ and $x \in X$.

Proof. Apply Corollary 3.4 (i) for $\alpha = 2$ and $\varphi(x,y) = \varepsilon\Big(|||x|||^r + |||y|||^r\Big)$ for each $x, y \in X$. □

Corollary 3.6. *Let for some $\varepsilon > 0$, $f : X \to Y$ satisfy the inequality*

$$||Df(x,y)|| \le \varepsilon \quad (x, y \in X).$$

Then there is a unique continuous quadratic mapping $Q : X \to Y$ such that

$$||f(x) - Q(x)|| \le \frac{\varepsilon}{(2^{2p} - 1)^q}$$

and $Q(sx) = s^2 Q(x)$ for each $s \in \mathbb{R}$ and $x \in X$.

Proof. Apply Corollary 3.4 for $\varphi(x,y) = \varepsilon$ for each $x, y \in X$ and $\alpha = 1$. □

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