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# Weakly Hyponormal Composition Operators and Embry Condition

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Abstract. We investigate the gaps among classes of weakly hyponormal composition operators induced by Embry characterization for the subnormality. The relationship between subnormality and weak hyponormality will be discussed in a version of composition operator induced by a non-singular measurable transformation.

## 1. Introduction and preliminaries

Let $\mathcal{H}$ be a separable infinite dimensional complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. An operator $A$ in $\mathcal{L}(\mathcal{H})$ is *normal* if $A^*A = AA^*$. An operator $A$ is *subnormal* if $A$ is the restriction of a normal operator to an invariant subspace. In [5], the Bram-Halmos criterion states that an operator $A$ is subnormal if and only if $\sum_{i,j=0} \left\langle A^i f_j, A^j f_i \right\rangle \geq 0$ for all $\{f_i\}_{i=0}^n$ in $\mathcal{H}$ and any $n \in \mathbb{N}$. Another well-known condition for the subnormality is Embry criterion which states that an operator $A$ is subnormal if and only if $\sum_{i,j=0}^n \left\langle A^{i+j} f_i, A^{i+j} f_j \right\rangle \geq 0$ for all $\{f_i\}_{i=0}^n$ in $\mathcal{H}$ and any $n \in \mathbb{N}$ ([6]). Recall that $A$ is *$n$-hyponormal* if $\sum_{i,j=0}^n \left\langle A^i f_j, A^j f_i \right\rangle \geq 0$ for all $\{f_i\}_{i=1}^n$ in $\mathcal{H}$ ([5], [8], [9], [10]). Recall that an operator $A$ is *$E(n)$-hyponormal* if $\sum_{i,j=0}^n \left\langle A^{i+j} f_i, A^{i+j} f_j \right\rangle \geq 0$ for any $f_0, f_1, \cdots, f_n$ in $\mathcal{H}$([7]). Note that $E(n)$-hyponormality is weaker than $n$-hyponormality. In [7], $E(n)$-hyponormality was discussed as a bridge between subnormality and weak hyponormalities in $\mathcal{L}(\mathcal{H})$.

In this note, we discuss $E(n)$-hyponormality for composition operators induced by a non-singular measurable transformation which is applied to being distinct the classes of $E(n)$-hyponormality. In Section 2, we show that the subnormality and

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$E(n)$-hyponormality are equivalent under the composition operators. In Section 3, we consider some examples which distinct the classes of $E(n)$-hyponormal operators for each positive integer $n$. Some of calculations in Section 3 are obtained throughout computer experiments using software tool Mathematica [11].

## 2. Relationship between subnormality and $E(n)$-hyponormality

We now introduce definitions and well-known facts in reference [5] and [3] which provide good materials for our work.

**Basic Properties(BP)** (i) Put an $2\times 2$-operator matrix of $\widetilde{A} := \begin{pmatrix} A & \mathbf{b} \\ \mathbf{b}^* & c \end{pmatrix}$, where $A \in M_n(\mathbb{C})$, $\mathbf{b} \in \mathbb{C}^n$ and $c \in \mathbb{C}$. If $A \geq 0$ and rank $\widetilde{A} =$ rank $A$, then $\widetilde{A} \geq 0$.

(ii) Let $A = (a_{ij})_{i,j=0}^{\infty}$ be an infinite Hermitian matrix and let $A_k$ be the truncation of $A$ to the first $(k+1)$ rows and columns. Assume that $A \geq 0$ and $\det A_k = 0$ for some $k$. Then $\det A_l = 0$ for all $l \geq k$.

(iii) For $\widetilde{A} \in M_{n+1}(\mathbb{C})$ and $1 \leq k \leq n$, let $\widetilde{A}_k \in M_k(\mathbb{C})$ be the truncation of $\widetilde{A}$. If $\det(\widetilde{A}_k) > 0$ for $1 \leq k \leq n$ and $\det(\widetilde{A}) \geq 0$, then $\widetilde{A} \geq 0$. (This is called the Nested Determinants Test.)

(iv) Let $(X, \mathcal{F}, \mu)$ be a $\sigma$-finite measure space and let $T$ be a non-singular measurable transformation $T : X \to X$ (i.e., $\mu \circ T^{-1} \ll \mu$, $T^{-1}\mathcal{F} \subset \mathcal{F}$). Then there exist the (first) Radon-Nikodym derivative $h = \dfrac{d\mu \circ T^{-1}}{d\mu}$ and the $n$-th Radon-Nikodym derivative, $h_n \equiv \dfrac{d\mu \circ T^{-n}}{d\mu}$ $(n \geq 1)$. And it holds that $\int_{T^{-1}A} f \circ T \, d\mu = \int_A h \cdot f \, d\mu$.

(v) The composition operator $C_T : L^2(X, \mathcal{F}, \mu) \to L^2(X, \mathcal{F}, \mu)$ is defined by $C_T f = f \circ T$ for all $f \in L^2(X, \mathcal{F}, \mu)$. We assume that $C_T$ is continuous (i.e., $\|C_T\| = \|h\|_\infty^{1/2} < \infty$).

Let $\mathcal{F}$ be the $\sigma$-algebra by all subsets of $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $l \in \mathbb{N}$, we consider a point mass measure $\mu_l$ on $\mathbb{N}_0$ determined as follows:

$$\underbrace{1, 1, \cdots, 1}_{(l+1)}, c_1, c_2, \cdots, c_l, (c_1)^2, (c_2)^2, \cdots, (c_l)^2, (c_1)^3, \cdots, (c_l)^3, (c_1)^4, \cdots$$

with $c_i > 0$ $(i = 1, \cdots, l)$. Let $(\mathbb{N}_0, \mathcal{F}, \mu_l)$ be the $\sigma$-finite measure space as above. Define a measurable non-singular transformation $T_l$ on $\mathbb{N}_0$ by $T_l(k) = 0$ for $k = 0, 1, 2, \cdots, l$ and $T_l(k) = k - l$ for $k \geq l + 1$.

**Proposition 2.1.** *For a fixed number $l \in \mathbb{N}$, let transformation $T_l$ and measure $\mu_l$ be defined as above. Then the $n$-th Radon-Nikodym derivatives $h_n(k)$ with $h_0(k) \equiv 1$,*

$n \geq 1$, $k \in \mathbb{N}_0$ *are expressed by the followings;*

$$h_n(0) = 1 + \sum_{1 \leq j \leq l} \frac{c_j^n - 1}{c_j - 1}, h_n(k) = (c_r)^n \text{ for } k = l \cdot q + r,\ q \geq 0 \text{ and } r = 1, \cdots, l.$$

*Proof.* For each $n \geq 1$, we show that the $\sigma$-algebra $T^{-n}\mathcal{F}$ is generated by the sets $\{0, 1, 2, \cdots, nl\}$, $\{nl+1\}$, $\{nl+2\}, \cdots$. It follows from the definition of $n$-th Radon-Nikodym derivatives $h_n(k)$ that

$$h_n(0) = \frac{\mu \circ T^{-n}(0)}{\mu(0)} = \mu(\{0, 1, 2, \cdots, nl\}) = 1 + \sum_{1 \leq j \leq l} \frac{c_j^n - 1}{c_j - 1}.$$

On the other hand, for $k \neq 0$, we write $k = lq + r$ for $q \geq 0$ and $r = 1, 2, \cdots, l$. So $T^{-n}(k) = nl + k$ and $\mu \circ T^{-n}(k) = \mu(\{nl + k\}) = c_r^{n+q}$. Hence

$$h_n(k) = \frac{\mu \circ T^{-n}(k)}{\mu(k)} = \frac{c_r^{n+q}}{c_r^q} = c_r^n$$

for all $n,\ k \geq 1$. Hence the proof is complete. □

For positive integers $m$ and $n$, we set

$$J_n^{(m)} = \{(j_1, \cdots, j_n) : 1 \leq j_1 < j_2 < \cdots < j_n \leq m,\ j_i \in \mathbb{N}\}$$

with $J_n^{(m)} = \emptyset$ for $n > m$. We denote for $(j_1, \cdots, j_n) \in J_n^{(m)}$ and $n \geq 1$, $c_{j_1, \cdots, j_n} \equiv \prod_{i=1}^n c_{j_i}$.

**Lemma 2.2.** *For* $l \in \mathbb{N}$, *let* $d_n = 1 + \sum_{1 \leq j \leq l} \dfrac{c_j^n - 1}{c_j - 1}$ $(n \geq 1)$ *with* $d_0 = 1$. *Then* $\{\mathbf{v}_i, \mathbf{v}_{i+1}, \cdots, \mathbf{v}_{i+l+1}\}$ *is linearly dependent for all* $i \in \mathbb{N}_0$ *where* $\mathbf{v}_i = (d_i, d_{i+1}, \cdots, d_{i+l+1}) \in \mathbb{C}^{l+2}$ *for all* $i \in \mathbb{N}_0$. *In particular, the infinite matrix with row vectors* $\mathbf{v}_i, \mathbf{v}_{i+1}, \cdots, \mathbf{v}_{i+l+1}$ $(i \geq 0)$ *has rank* $l + 1$.

*Proof.* For simple notations, we write $J_i := J_i^{(l)}$ for all $i = 2, 3, \cdots, l-1$. Put

$$a_0 = (-1)^l \prod_{1 \leq j \leq l} c_j,\ a_1 = (-1)^{l-1}\Bigg(\prod_{1 \leq j \leq l} c_j + \sum_{(j_1, \cdots, j_{l-1}) \in J_{l-1}} c_{j_1, \cdots, j_{l-1}}\Bigg),$$

$$a_2 = (-1)^{l-2}\Bigg(\sum_{(j_1, \cdots, j_{l-1}) \in J_{l-1}} c_{j_1, \cdots, j_{l-1}} + \sum_{(j_1, \cdots, j_{l-2}) \in J_{l-2}} c_{j_1, \cdots, j_{l-2}}\Bigg), \cdots,$$

$$a_{l-1} = (-1)^1\Bigg(\sum_{(j_1, j_2) \in J_2} c_{j_1, j_2} + \sum_{1 \leq j \leq l} c_j\Bigg),\ a_l = \sum_{1 \leq j \leq l} c_j + 1.$$

For simple calculations, we can obtain that $\sum_{0 \leq j \leq l} a_j d_{j+i} = d_{i+l+1}$ for all $i \in \mathbb{N}_0$. Hence the set $\{\mathbf{v}_i, \mathbf{v}_{i+1}, \cdots, \mathbf{v}_{i+l+1}\}$ is linearly dependent for all $i \in \mathbb{N}_0$. □

For a $\sigma$-finite measure space $(X, \mathcal{F}, \mu)$, it follows from [7] that the composition operator $C_T$ on the space $L^2(X, \mathcal{F}, \mu)$ is $E(n)$-hyponormal for a positive integer $n$ if and only if the $(n+1)\times(n+1)$ matrix $(h_{i+j}(x))_{i,j=0}^n \geq 0$ for almost all $x \in X$ with respect to $\mu$, where $h_n(x)$ is the $n$-th Radon-Nikodym derivative with $h_0(x) \equiv 1$. Then we obtain the following theorem.

**Theorem 2.3.** *For $l \in \mathbb{N}$, let $C_{T_l}$ be a composition operator on the space $L^2(\mathbb{N}_0, \mathcal{F}, \mu_l)$. Then $C_{T_l}$ is $E(l)$-hyponormal if and only if $C_{T_l}$ is subnormal.*

*Proof.* Let $l \in \mathbb{N}$. According to the remark above this theorem, we obtain that the composition operator $C_{T_l}$ is $E(l)$-hyponormal if and only if the $(l+1)\times(l+1)$ matrix $(h_{i+j}(k))_{i,j=0}^l \geq 0$ for almost all $k$, where $h_n(k)$ is $n$-th Radon-Nikodym derivatives. For the case $k \neq 0$, using the Proposition 2.1, we see that each column vectors of the infinite matrix $(h_{i+j}(k))_{i,j=0}^\infty$ is linearly dependent and its rank is 1. So from BP(i), we have that the infinite matrix $(h_{i+j}(k))_{i,j=0}^\infty \geq 0$. Hence $C_{T_l}$ is subnormal.

Finally we only show the result for the case $k = 0$. For brevity, we write $h_n := h_n(0)$ for all $n \geq 1$ and $h_0 = 1$. By Proposition 2.1 and Lemma 2.2, we see that the $(l+1)\times(l+1)$ matrix $(h_{i+j})_{i,j=0}^l$ has rank $l+1$. And by BP(i), rank $(h_{i+j})_{i,j=0}^l = l+1 =$ rank $(h_{i+j})_{i,j=0}^n$ for all $n \geq l+1$. Also, from the condition $(h_{i+j})_{i,j=0}^l \geq 0$, we can obtain that $(h_{i+j})_{i,j=0}^n \geq 0$ for all $n \geq 1$. Hence the composition operator $C_{T_l}$ is subnormal. The converse implication is obvious. □

**Corollary 2.4.** *For $l \in \mathbb{N}$, let $C_{T_l}$ be a composition operator on the space $L^2(\mathbb{N}_0, \mathcal{F}, \mu_l)$. Then $C_{T_l}$ is $E(l)$-hyponormal if and only if $C_{T_l}$ is $l$-hyponormal.*

*Proof.* We note that $n$-hyponormality implies $E(n)$-hyponormality for each $n \in \mathbb{N}$. From Theorem 2.3, we can have the assertion. □

In addition, we show formulae of determinants for the matrix $(h_{i+j})_{i,j=0}^n$ $(n \geq 1)$ in the following proposition.

**Proposition 2.5.** *For $l \in \mathbb{N}$, we have that*

$$\det(h_{i+j}(0))_{i,j=0}^n = \begin{cases} \prod_{(j_1,j_2)\in J_2^{(l)}} (c_{j_1} - c_{j_2})^2 \cdot D_l & \text{for } n = l, \\ 0 & \text{for } n \geq l+1, \end{cases}$$

*where*

$$D_l = \sum_{r=0}^{l} (-1)^{l-r}(l+1-r) \sum_{(i_1,\cdots,i_r)\in J_r^{(l)}} c_{i_1,\cdots,i_r}.$$

*In particular,* $\det(h_{i+j}(k))_{i,j=0}^n = 0$ *for all $k \neq 0$ and $n \geq 1$.*

*Proof.* From the Proposition 2.1 and Lemma 2.2, we can obtain the result. □

**Remark 2.6.** From Theorem 2.3 and Proposition 2.5, we can see that the matrix $(h_{i+j}(k))_{i,j=0}^n \geq 0$ for all $k \in \mathbb{N}$ and $n \geq 1$. i.e., the composition operator $C_{T_l}$ is always subnormal.

## 3. Distinctions of $E(n)$-hyponormalities

In our constructed model, we want to show the distinctions of $E(n)$-hyponormalities for each $n \in \mathbb{N}$. Owing to Theorem 2.3, we can see that disjointness of $E(n)$-hyponormal operators comes from only cases $n = 1, 2, \cdots, l$ for the given positive integer number $l$. So we show that the gaps between $E(n)$-hyponormal operators step by step for given number $n$.

**3.1.** $E(1)$**-hyponormal but not** $E(2)$**-hyponormal.** For $k \in \mathbb{N}_0$ and $n = 1, 2$, we set

$$RE(2, n) = \{(c_1, c_2) : C_{T_2} \text{ is } E(n)\text{-hyponormal}\}$$

and

$$RD(2, n) = \{(c_1, c_2) : \det \Delta_i > 0 \ (i = 1, \cdots, n-1) \text{ and } \det \Delta_n \geq 0\},$$

where $\Delta_l = (h_{i+j}(0))_{i,j=0}^{l}$ for $l = 1, 2, \cdots$. Then we can obtain that $RE(2, n) = RD(2, n)$, $n = 1, 2$. In fact, from BP(iii), we have that $RD(2, n) \subset RE(2, n)$. To show the reverse implication, let $(c_1, c_2) \in RE(2, n)$, i.e., $\Delta_n \geq 0$ for all $k \in \mathbb{N}_0$ and $n = 1, 2, \cdots$. Suppose that there exists $(\alpha_1, \alpha_2)$ such that $\det \Delta_1 = c_1 + c_2 - 6 = 0$ for $c_1 > 0$ and $c_2 > 0$. Since $\det \Delta_2 = (c_1 - c_2)^2(3 - 2c_1 - 2c_2 + c_1 c_2)$, if we put $f(c_1, c_2) := 3 - 2c_1 - 2c_2 + c_1 c_2$, then we can have that $f(\alpha_1, \alpha_2) < 0$, which is contradicts to $\Delta_2 \geq 0$. Hence we have the following assertions;

$$C_{T_2} \text{ is } E(1)\text{-hyponormal} \Longleftrightarrow c_1 + c_2 - 6 \geq 0 \text{ for } c_1 > 0, c_2 > 0$$

and

$$C_{T_2} \text{ is } E(2)\text{-hyponormal} \Longleftrightarrow 3 - 2c_1 - 2c_2 + c_1 c_2 \geq 0 \text{ for } c_1 > 0, c_2 > 0.$$

**Remark 3.1.** More specially, to see the gaps between $E(n)$-hyponormalities for $n = 1, 2$, in $\mathbb{R}^1$, we restrict $d = 2c$ with the positive number $c$. Put

$$I_i = \{c > 0 : C_{T_2} = E(i)\text{-hyponormal}\}$$

for $i = 1$, $2$. Then we have two intervals, $I_2 = [\alpha, \infty) \subsetneq I_1 = [2, \infty)$, where $\alpha = \frac{3+\sqrt{3}}{2}$.

**3.2.** $E(2)$**-hyponormal but not** $E(3)$**-hyponormal.** From now on, because of conveniences of calculations, we will look for the gaps in $\mathbb{R}^1$ about the classes of $E(n)$-hyponormal composition operators for each positive integer $n$. Put each point mass $c_j = j \cdot c$ for $j = 1, 2, \cdots, l$ for a positive number $c$. For $k \in \mathbb{N}_0$ and $n = 1, 2, 3$, we set $RE(3, n) = \{c > 0 : \Delta_n \geq 0\}$ and

$$RD(3, n) = \{c > 0 : \det \Delta_i > 0 \ (i = 1, \cdots, n-1) \text{ and } \det \Delta_n \geq 0\},$$

where $\Delta_n = (h_{i+j}(0))_{i,j=0}^{n}$. Then we can obtain that $RE(3, n) = RD(3, n)$ for $n = 1, 2, 3$. Indeed, from simple calculations, $\det \Delta_1 = 6(c - 2)$ and $\det \Delta_2 = 4c^2(5c^2 -$

$15c+6) = 0$ for $c > 0$. Suppose that there exists $\alpha_0 \geq 2$ such that $5c^2 - 15c + 6 = 0$. Since $\det \Delta_3 = 8c^6(-2 + 9c - 11c^2 + 3c^3)$, if we put $f(c) := -2 + 9c - 11c^2 + 3c^3$, then we can have that $f(\alpha_0) = -\frac{3\alpha_0}{5} + \frac{2}{5} < 0$ (because $\alpha_0 \geq 2$), which contradicts to $\Delta_3 \geq 0$. If we denote an interval $I_n = \{c > 0 : C_{T_3}$ is $E(n)$-hyponormal$\}$ for $n = 1, 2, 3$, then we have the following relationships for $E(n)$-hyponormalities,

$$I_3 = [\alpha_3, \infty) \subsetneq I_2 = [\alpha_2, \infty) \subsetneq I_1 = [2, \infty),$$

where $\alpha_2 \approx 2.525$, $\alpha_3 \approx 2.618$.

**3.3. Algorithm.** Throughout previous examples, we provide the following algorithm giving the distinctions of $E(n)$-hyponormalities for a fixed integer $l \geq 3$ and a constant $c > 0$.

**I**. Set a matrix $\Omega = (h_{i+j})_{i,j=0}^{\infty}$, where each $h_m := h_m(0)$ is the same as in Proposition 2.1.

**II**. Compute the determinants of matrices $\Omega_k$ for $k = 1, 2, \cdots, l$. Put $d_k(c) = \det \Omega_k$ for $k = 1, 2, \cdots, l$. Then $d_1(c) = \frac{l(l+1)}{2}(c - 2)$. So we take $\alpha_1(\equiv c) > 2$.

**III**. Find polynomial remainder $R_k(c)$ of $d_k(c)$,

$$d_k(c) = (\sum_{1 \leq j \leq l} j^{2k-1})c^{2k-1} d_{k-1}(c) + R_k(c), \ 2 \leq k \leq l.$$

**IV**. For each $\alpha_{k-1} > 2$, $2 \leq k \leq l$, check $R_k(\alpha_{k-1}) < 0$, where $\alpha_{k-1}$ is the greatest root of $d_{k-1}(c) = 0$.

**V**. Find $E(l, n) = \{c > 0 : d_k > 0, \ d_n \geq 0, \ 1 \leq k \leq n - 1\}$ for $n = 1, 2, \cdots, l$.

**3.4. Some estimations.** Using Algorithm, we can obtain mutually disjoint values $\alpha_1, \alpha_2, \cdots, \alpha_l$ satisfying $\alpha_n \in E(l, n)$ $(n = 1, \cdots, l)$ for some low numbers permitted by computer estimations, which means that the classes of $E(n)$-hyponormal operators are distinct, i.e., $E(l, n-1) \backslash E(l, n) = [\alpha_{n-1}, \alpha_n)$ for such low numbers.

For examples, we give the numerical values $\alpha_{l-1}$ and $\alpha_l$ in the Table 3.1 which show the distinct classes of $E(n)$-hyponormal operators for $1 \leq n \leq l$, $2 \leq l \leq 10$, where the values of $\alpha_i$ are approximated ones.

| | $l=2$ | $l=3$ | $l=4$ | $l=5$ | $l=6$ | $l=7$ | $l=8$ | $l=9$ | $l=10$ |
|---|---|---|---|---|---|---|---|---|---|
| $\alpha_{l-1}$ | 2 | 2.525 | 2.789 | 2.965 | 3.105 | 3.2226 | 3.3264 | 3.419313 | 3.5035871 |
| $\alpha_l$ | 2.366 | 2.618 | 2.812 | 2.971 | 3.106 | 3.2229 | 3.3265 | 3.419336 | 3.5035923 |

Table 3.1

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